This year, every issue of delta-K *will devote a section to the NCTM Standards. In this issue, the focus will be on the* Curriculum and Evaluation Standards for School Mathematics *(1989) with particular emphasis on mathematical connections.*

The Curriculum Standards for school mathematics for K--Grade 12 include mathematical connections as one of the important standards to be addressed. At the primary level, students should study mathematics in ways that includes opportunities to make connections with experiences in their daily lives, with other curriculum areas and with different topics in mathematics. Our failure to do that will inevitably result in students continuing to see mathematics as discrete and unrelated strands.

As children enter school, their learning to this point has not been segregated into separate, unconnected school subjects. It is therefore important that teachers build on the wholeness of their perspective of the world through an emphasis on connecting, relating and linking various representations of concepts or procedures to one another within and across subject areas and to the world outside the classroom. This approach will also dispel the incorrect notion that mathematics is a collection of isolated topics and computation only. In addition, the children will come to see that mathematics is of considerable relevance and usefulness and is present in their daily lives outside school.

This emphasis on mathematical connections should naturally continue at the middle and high school levels, so that students solidify their view of mathematics as an integrated whole. At this level, mathematics should include exploration of problems, the application of mathematical thinking and modeling to solve problems that arise in mathematics, but also in other disciplines. Students at the high school level should be exposed to investigations of the connections and the interplay among various mathematical topics and their applications. These experiences will ultimately lead students to being able to apply and translate among different representations of the same problem or concept and to have a deeper appreciation and understanding of mathematics.

The three articles that follow provide excellent examples of mathematical connections.

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Promoting Mathematics Connections with Concept Mapping

Bobbye Hoffman Bartels

Recent documents calling for reform in mathematical education stress promoting connections during instruction (NCTM 1989; National Research Council 1989, 1990). Mathematical connections are important because they link mathematical concepts to each other and to the real world. When teachers emphasize these connections during instruction, mathematics becomes less compartmentalized and more cohesive and has relevance to real life. In addition, stressing mathematical connections during instruction helps students develop better and deeper understandings of mathematics.

Typically, mathematical connections are implicit in instruction. That is, teachers use instruction that is connected but do not make the connections explicit for students. Implicit connections can result in difficulties with the way students make connections when learning mathematics; students are less likely to make the connections themselves or to make correct connections.

Concept mapping is a useful tool for explicitly stressing mathematical connections. This tool helps teachers and their students perceive and make connections among key ideas in mathematics. In addition, concept mapping is an alternative method of assessment useful for evaluating students' understanding. For an example of using a form of concept mapping in college algebra and trigonometry classes, see Entrekin (1992).

Concept Mapping

A concept map is an instrument for explicitly describing concepts and the relationship among them. The mathematical ideas on a concept map are placed in a hierarchical position, and lines connect the mathematical concepts to form propositions. Thus, a concept map is a finite graph in which the nodes are the mathematical concepts and the edges are the connections between them. The nodes are arranged hierarchically with general concepts placed at higher positions on the map than specific concepts. Thus, movement down the map leads to more specific concepts. The edges are labeled with linking words indicating the relationship between the concepts or key ideas, thus forming propositions. Arrows are placed on the linking lines to indicate the direction of the relationship between concepts. Whenever arrows are not used, the direction is assumed to be top-down.

A student-constructed concept map for whole-number arithmetic is shown in Figure 1. In the figure, the concepts are placed in ellipses—any closed figure is appropriate—with connecting lines between the concepts. The lines are labeled with linking words to specify the relationship between the concepts. *Whole numbers* is the most general mathematical idea and is placed at the highest point in the map hierarchy. Other concepts, such as *operations, addition* and *difference*, are more specific and are placed at lower levels.

Concept-Map Construction

Students can construct maps in several ways. The appropriate method depends on the intended use of the map, the age and ability of the student, and the student's previous concept-mapping experiences. The concept-mapping methods given subsequently range from simple to challenging. In all situations, students should be instructed to identify the most general mathematical ideas first and place them at the top of the map. Next, more specific concepts are added below. The following two methods work well for students' initial experiences with concept mapping.

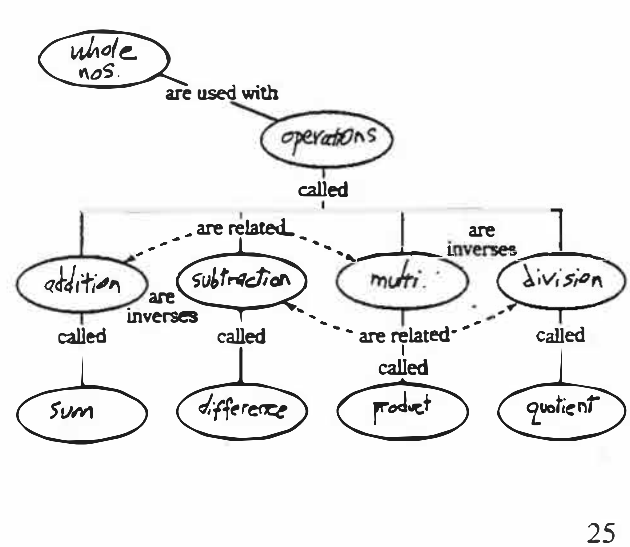
This simplest method is to furnish students with a list of concepts and an incomplete map—one without the concepts but with the connections and linking words (see Figure 2). For this activity, students are given the list of concepts and the incomplete map with the connections drawn between the empty ellipses representing the missing concepts. Students write the concepts from the list in the appropriate ellipses.

The three activities in Figure 2 range from simple to increasingly more complex. The vocabulary on the concept map for whole numbers (Figure 2a) should be familiar to middle school students. In addition, the map contains only four levels of hierarchy. This simplification should make the map easier for students to complete than the next two maps. Figure 1 shows a completed map for the activity in Figure 2a.

The concept map in Figure 2b, which involves solving linear equations, is more complicated. The vocabulary is much more difficult for students because many terms, such as *additive inverse* and *equivalent equations*, will be new. In addition, this map has five levels of hierarchy.

The concept map for measurement in Figure 2c is the most extensive of the three maps, with six levels of hierarchy and many concepts and examples that the students must place on the map to make the correct corrections. Another aspect of this map that makes it more extensive are the connections that are indicated from one major branch to another. These cross-connections require students to have a higher level of understanding than connections made vertically on the same branch of the map.

Figure 1
Student-Constructed Concept Map for Whole-Number Arithmetic



The second method of introducing students to this cognitive tool is to give students a list of concepts but no map—a more difficult approach than the first method. Students construct their maps from the list of concepts, deciding how to represent them hierarchically, determining the connections among them and determining the linking words that show the relationships among the concepts. The following example of such an activity is suitable for middle school students.

Figure 2a

Complete this concept map by using the following concepts: addition, difference, division, multiplication, operations, products, quotient, subtraction, sum and whole numbers.

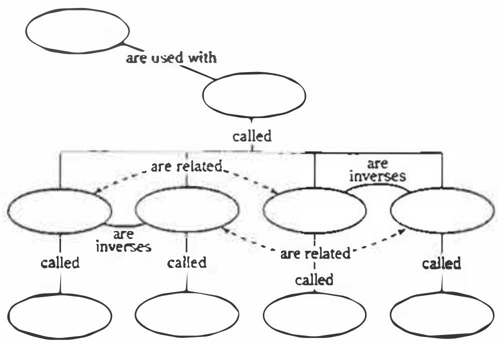


Figure 2b

Complete this concept map by using the following concepts: additive, distributive, equivalent equations, identity, inverses, multiplicative, opposite, properties, reciprocal, solving linear equations in x, and $x = \#$. Examples include 3 and –3, 4 and 1/4, $x = 4$, and $5x + 3x - 4 = 28$.

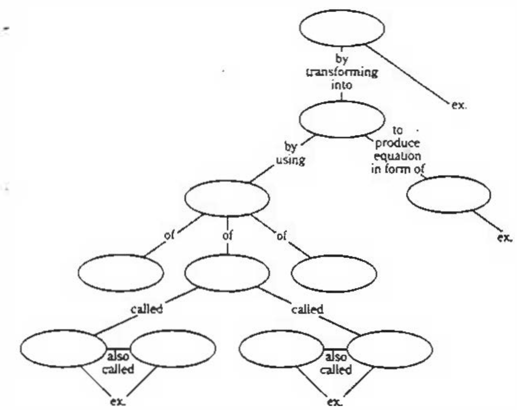


Figure 2c

Complete this concept map by using the following concepts: capacity, English/customary, length, measurement, metric, most of world, nonstandard, stable, standard, systems, temperature, time, unit of measure, unstable, USA and weight. Examples include 3:15 p.m., 4 litres, 0° Celsius, 1,515 hours, 3 paper clips, 2 pints, 3 kilograms, 32° Fahrenheit, 7 metres, 4 pounds and 8 miles.

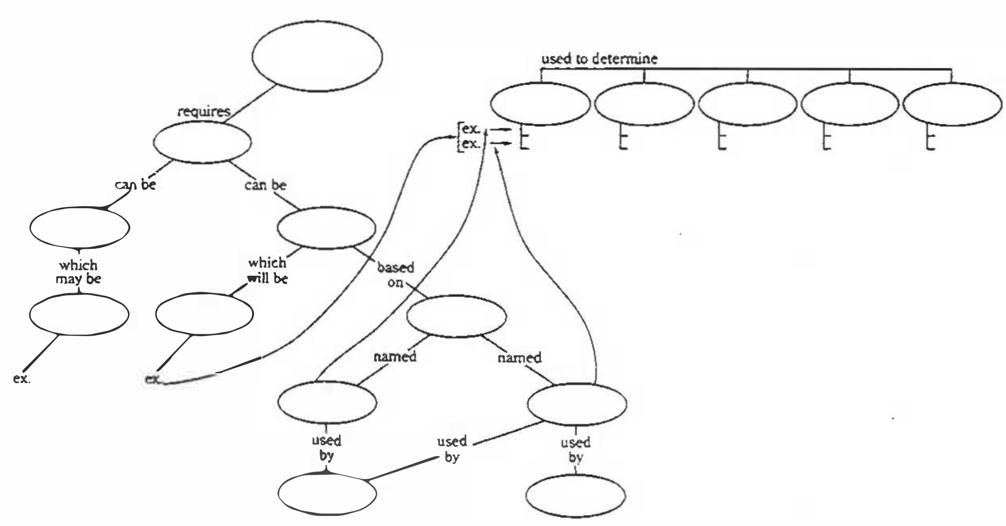
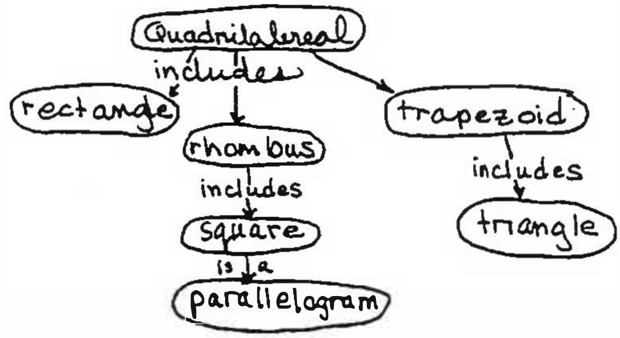


Figure 3

A Student-Constructed Concept Map Relating Kinds of Quadrilaterals



Make a map showing the connections among the following concepts: parallelogram, quadrilateral, rectangle, rhombus, square, trapezoid and triangle.

A variation on this method is to include on the list one or two unrelated concepts to verify the students' understanding of relationships among them. The concept map in Figure 3 was constructed using the foregoing list of concepts. This student's work is subsequently considered in greater depth.

A more challenging method of concept mapping is to have the students themselves determine the key concepts and construct a map using them. This method is more challenging because students supply

Figure 4

Concept Maps for Ratios Constructed by Two Groups of Students

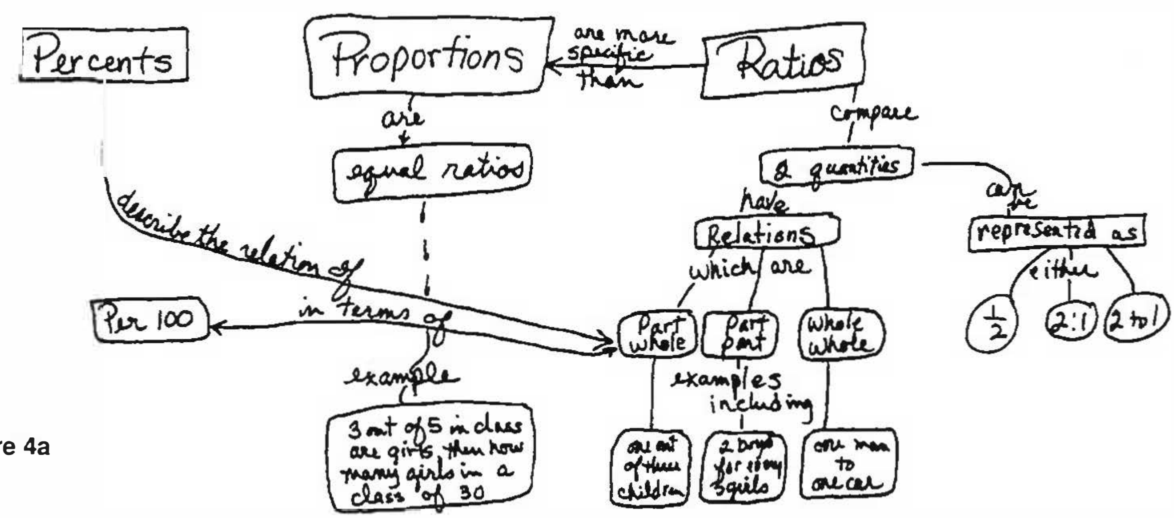


Figure 4a

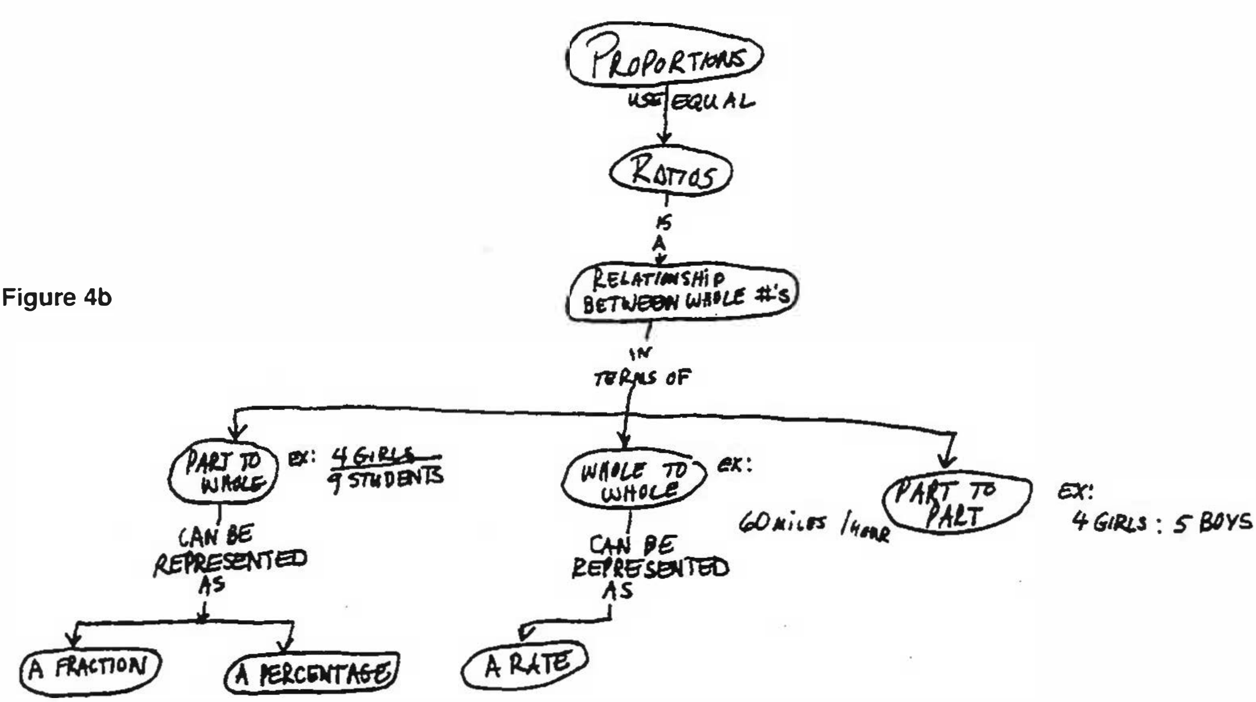


Figure 4b

their own concepts and, therefore, are required to analyze the content of the lesson, topic or unit to determine the key components. The concept maps in Figure 4 were constructed by students using this method.

In addition to mapping the connections among concepts, the concept map is valuable for showing mathematical connections to the real world and to other disciplines. Real-world connections are shown by adding examples to the map, as demonstrated by Figure 4. Examples are given for using proportions and for part-whole, part-part and whole-whole ratios. Examples are usually placed at the bottom of the concept-map hierarchy to separate them from the concepts and, therefore, to keep the organization of the map intact. But as Figure 4b indicates, placing examples next to concepts is acceptable.

When constructing concept maps, always begin with a simple one, such as the map in Figure 2a. From the instructional unit, choose concepts for one or two days, and using any one of the methods, make a concept map. Make simple maps every one or two days, and at the end of the unit make a comprehensive concept map using the simpler maps developed during the unit.

If concept maps for each unit are saved, they can be posted on the wall or bulletin boards. When a concept or unit has connections to one previously taught, the appropriate map can be accessed and the connections discussed. In addition, old maps are useful models. For example, the simple concept map for whole number arithmetic (see Figure 1) is useful for developing a simple map for integer or rational-number arithmetic. The basic difference among the three maps will be the general concept at the top of the map.

Concept Mapping as an Instructional Tool

Concept mapping is useful as an instructional tool during whole-group or small-group instruction or for individual students. During whole-group instruction, concept mapping is initiated and monitored by the teacher, whereas when it is used for an assignment by small groups or individuals, the students work more independently.

Whole-Group Use

During whole-group instruction, the teacher may use concept mapping in three ways. For the first way, the teacher presents a completed map at the beginning of a lesson as an advance organizer or at the

Table 1
Scoring Rubric for Concept Maps

Concepts and Terminology	
3 points	Shows an understanding of the topic's mathematical concepts and principles and uses appropriate mathematical terminology and notations
2 points	Makes some mistakes in mathematical terminology or shows a few misunderstandings of mathematical concepts
1 point	Makes many mistakes in terminology and shows a lack of understanding of many mathematical concepts
0 points	Shows no understanding of the topic's mathematical concepts and principles
Knowledge of the Relationships Among Concepts	
3 points	Identifies all the important concepts and shows an understanding of the relationships among them
2 points	Identifies important concepts but makes some incorrect connections
1 point	Makes many incorrect connections
0 points	Fails to use any appropriate concepts or appropriate connections
Ability to Communicate Through Concept Maps	
3 points	Constructs an appropriate and complete map and includes examples; places concepts in an appropriate hierarchy and places linking words on all connections; produces a concept map that is easy to interpret
2 points	Places almost all concepts in an appropriate hierarchy and assigns linking words to most connections; produces a concept map that is easy to interpret
1 point	Places only a few concepts in an appropriate hierarchy or uses only a few linking words; produces a concept map that is difficult to interpret
0 points	Produces a final product that is not a concept map

end of a lesson and connects them to show the relationships to other ideas from that lesson and from other lessons.

A second way to use concept mapping is somewhat different because it promotes reflection and discussion about the key ideas that are mapped and the connections among them. With this method, the teacher displays a map that may have inaccurate connections among concepts or uses inappropriate concepts. Students orally analyze the map to verify the use of the concepts and the connections among them.

For a third way, the students and teacher collectively construct a map as the lesson progresses. Through discussion, students give input into this map by supplying the key ideas and the connections.

Small-Group or Individual Use

The alternative to whole-group instruction is to have the students themselves construct maps that represent the concepts and connections that were emphasized during instruction. With this method, students construct the maps while working in small collaborative groups or individually.

In addition to promoting mathematical connections, constructing concept maps in small groups also promotes communication as students discuss the key ideas and their relationships. Whether students work as individuals or in groups, students use mathematical reasoning while constructing their maps.

Concept Mapping as an Assessment Tool

Concept maps are also useful for assessing students' understanding. As an assessment tool, the concepts on the map and their connections are evaluated. Any of the foregoing methods for concept mapping can be used for the assessment. The maps are elevated on an informal or formal level.

Informal evaluation is invaluable as an ongoing assessment of the students' learning. Through an informal evaluation, the teacher looks at the maps constructed by the students, noting any misunderstandings about connections among concepts or a lack of connections. For example, when a map in Figure 3 is informally evaluated, the teacher should note that the student connected *triangle* to *trapezoid* through the linking word *includes*. This student appears to think that a triangle is a trapezoid and a quadrilateral. Further evaluation might show the misconception that the *parallelogram* is more specific than the *rhombus* or the *square*. An informal evaluation of the map in Figure 4a indicates a relatively clear understanding of the relationships between ratio concepts. The teacher, however, might make a mental note to investigate further to see if the student has a misconception that ½ is equivalent to "2:1" and "2 to 1." Through an informal evaluation, the teacher generalizes students' understanding of the

Table 2
Using the Scoring Rubric

	Rubric	*Points*	*Comments*
Figure 3			
	Concepts and terminology	1.0	Triangle is not a trapezoid and should not be on map because it is not a quadrilateral.
	Relationships	2.5	Square is a rectangle but is not shown as such.
	Communication	1.0	Hierarchy is incorrect.
Total points		4.5	
Figure 4a			
	Concepts and terminology	3.0	Mathematical terminology and concepts are correct.
	Relationship	2.5	Percent should be connected to per 100.
	Communication	1.5	Hierarchy for a few concepts is incorrect. The map is not easy to interpret.
Total points		7.0	
Figure 4b			
	Concepts and terminology	2.5	The relationship to whole numbers is incorrect.
	Relationships	3.0	Connections are correct.
	Communication	3.0	Hierarchy is appropriate, linking words are used and the map is easy to interpret.
Total points		8.5	

connections among concepts. This informal evaluation is useful for remedying students' erroneous conclusions, whether individually or as a group.

A formal evaluation of concept maps might adopt a scoring rubric. The rubric found in Table 1 presents the opportunity to assess a student's knowledge of mathematical concepts and the connections among concepts, the ability to construct a map and the ability to communicate through a concept map. These three areas are given equal weight here, but the rubric can be adjusted to assign more weight for one or two areas or to eliminate an area, depending on the objectives for the assignment. The scores in each category range from 3 to 0 points, but the teacher can use scores between the whole-number ratings. In all situations, the scoring must be based on material that was covered in class. One way to use the rating is to subtract 0.5 points for each mistake; however, the amount subtracted must reflect the importance of the mistake in relation to the entire map and the objectives for the assessment.

Table 2 shows a way to score the maps in Figures 3 and 4 using the scoring rubric from Table 1. Following each rating are comments that describe the reasons for the rating.

When the map in Figure 3 is scored using the rubric, points are subtracted for incorrect mathematical knowledge (for example, a triangle is a trapezoid and a quadrilateral) and for misconceptions about the relationships among concepts (for example, the relationship between the rectangle and the square is not indicated). In addition, the concepts *parallelogram* and *rhombus* are not placed in an appropriate hierarchy. *Parallelogram* is more than *square*, and *rhombus* is more specific than *parallelogram*. The student who constructed this map shows a poor understanding of the relationships among kinds of quadrilaterals, as indicated by his map score of 4.5.

The maps in Figure 4 were constructed by students working in small groups using identical instructions:

> Construct a map that shows the relationships among concepts associated with the unit on rations, proportions and percents.

It is interesting to note—and to be expected—that the maps are not identical. The concept map in Figure 4a received a score of 7.0. The map shows an understanding of the topic's key ideas and principles and correct terminology is used. The connection between *percents* and *per 100* is not drawn, so 0.5 was subtracted from the relationship score. All other connections are appropriate. This group of students lost the most credit in the communication category. The hierarchy on their map is incorrect, since ratios should be higher that proportions and percents are a kind of part-whole ration and should be below ratio. In addition, their map is difficult to follow.

The concept map in Figure 4b is much easier to follow. This group of students lost the most credit in the area of concepts and terminology. Their map indicates that a ratio can be a relationship between whole numbers only and could not be a relationship between a whole number and a decimal and so on. This inaccurate mathematical knowledge resulted in a deduction of 0.5 points in the concepts-and-terminology category. This group received a score of 8.5 on their map.

The Value of Concept Mapping

The real value in constructing a concept map is the visual representation of the mathematical connections that is produced. With this tool, the connections are explicitly depicted and are visible to the person constructing the map as well as to anyone who observes the map. Without the concept map, connections are assumed; however, they may not actually exist or may be incorrect. A student-constructed map is the "hard copy" of the connections made by the student.

Since the connections are visually depicted with the concept map, the person constructing the map has the opportunity to evaluate perceived connections. This evaluation allows the map creator to deal with the effectiveness and meaning of the connections. In addition, the creator can modify connections while constructing the map.

Concept mapping by students leads to rich discussions among students. As the students explore the connections among key ideas, they can clear up misconceptions and develop new meaning for concepts. Additionally, this discussion promotes mathematical reasoning and mathematical communication among students, two goals emphasized in the *Curriculum and Evaluation Standards for School Mathematics* (NCTM 1989). The social context of mapping concepts in groups reinforces mathematical learning in the same way that using cooperative groups reinforces learning.

The evidence of communication and reasoning is concrete. While students are constructing maps in their groups, they will be heard discussing mathematics, not just doing mathematics. They will ask the group to clarify terminology and relationships, such as the different kinds of ratios in Figure 4. The concept-mapping activity affords an opportunity for students to better understand mathematical relationships.

Making connections among concepts also helps students solidify their mathematical understanding.

When constructing a map for geometric figures in two dimensions, students investigate the connections among squares and rectangles. By discussing the placement of these figures on their map, students are helped to see that all squares are rectangles but that not all rectangles are squares. This relationship was missed by the student who constructed the map in Figure 3. If the student had worked with a group, a discussion about the relationship between these concepts might have helped this student have a better understanding.

To be an effective instructional tool, concept mapping should be routinely used. In addition to increasing students' efficiency and proficiency with concept mapping, routine use enhances the connections among units of instruction. For example, the concept map for surface area and volume will have connections to the map for perimeter and area.

The concept map is not meant to replace instructional tools currently found in mathematics classes, but it can augment those tools. For example, Venn diagrams are excellent tools for investigating logic and relationships, but at times, a concept map can better show the relationships. This advantage is especially true when showing relationships among different topics. In addition, the linking words used by students on a concept map furnish information about the understanding of those students that is not evident when Venn diagrams are used. For students who are experienced with concept mapping, the Venn diagram can be changed into a branch on the concept map with little difficulty.

Summary

Concept maps are a valuable instructional tool to help students reflect on, and make connections among, concepts in mathematics. Because the maps are an articulation of the perceived connections among mathematical ideas, students constructing concept maps can visualize how they are connecting concepts and can adjust the connections, if necessary. As a visualization of the connections among concepts, this form of mapping both promotes and assesses understanding in mathematics.

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Flashing Lights

One light flashes every 2 minutes and another light flashes every 7 minutes. If both lights flash at 1 p.m., what is the first time after 3 p.m. that same day that both lights flash together?