Building Mathematical Models of Simple Harmonic and Damped Motion

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Given the recent public mania over bungee jumping, stimulating students' interest in a model of that situation should be an easy "leap." Students should investigate the connections among various mathematical representations and their relationships to applications in the real world, asserts the *Curriculum and Evaluation Standards for School Mathematics* (NCTM 1989). Mathematical modeling of real-world problems can make such connections more natural for students, the standards document further indicates. Moreover, explorations of periodic real-world phenomena by all students, as well as the modeling of such phenomena by college-intending students, is called for by Standard 9: Trigonometry.

What follows is an activity that the author has successfully used with Grades 11 and 12 students in a precalculus course in which daily use was made of graphing calculators. In addition to meeting the explicit recommendations previously noted, the activity presents an application of trigonometric functions in a nongeometric setting, giving students an opportunity to apply such functions to a real-world situation.

In *Precalculus: A Graphing Approach*, Demana and Waits (1989, 526–27) present a series of problems aimed at students' development of mathematical models of harmonic motion followed by damped motion. Instead of just using "made up" data to build the model, the decision was made to bring the physical situation into the classroom. The hope was that asking students to attempt to model something that they could actually see would make the problem more vivid for them.

This activity can be completed in one or two class periods. The materials required are a spring, weight sufficient to stretch the spring, some means of suspending the spring and attaching the weight to the spring, a stopwatch and a graphing utility. A screen-door spring with eight to twelve ounces of weight has proved a satisfactory combination. One's physics colleagues might also be a good resource.

In this activity, the goal is for students to produce a mathematical model of the motion that results when

1. the weight is attached to the spring,
2. the spring-weight combination is suspended so as to allow the weight to hang freely,
3. the spring is stretched by pulling down on the weight and
4. the weight is released, beginning an oscillatory motion.

In a sense, mathematical modeling is a process of successive approximation: a number of models are built, each imitating more of the properties of the situation than the one that came before. Throughout the modeling activity, it is important to convey to students the notion that mathematical models are best thought of not as "right" or "wrong" but as better or poorer representations of the problem situation. The interested reader is encouraged to see Davis and Hersh (1981, 70, 77–79) for a further discussion of the nature of mathematical models.

Building the First Model

The first thing that must be one is to establish an equilibrium point for the weight. If the weight is suspended near the chalkboard, for example, the equilibrium point can easily be marked on the chalkboard behind the weight. Next, the spring can be stretched to begin the oscillation, as in Figure 1. As the weight is oscillating, the teacher can begin to pose questions to engage students' thinking about the situation and elicit from them a verbal description of what they are seeing. Students will frequently say things like, "Well, you stretched the spring, let go, and the weight started bouncing up and down." From such a beginning, the teacher might ask, "Do we know any mathematical functions that do just that?"

If students have difficulty associating a sinusoid with this physical situation, the teacher might suggest the possibility of measuring the "bounce." Here

Figure 1
Beginning the Motion

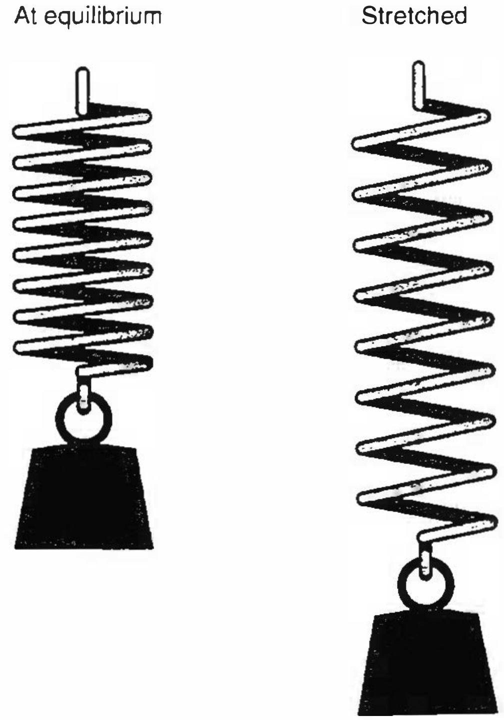
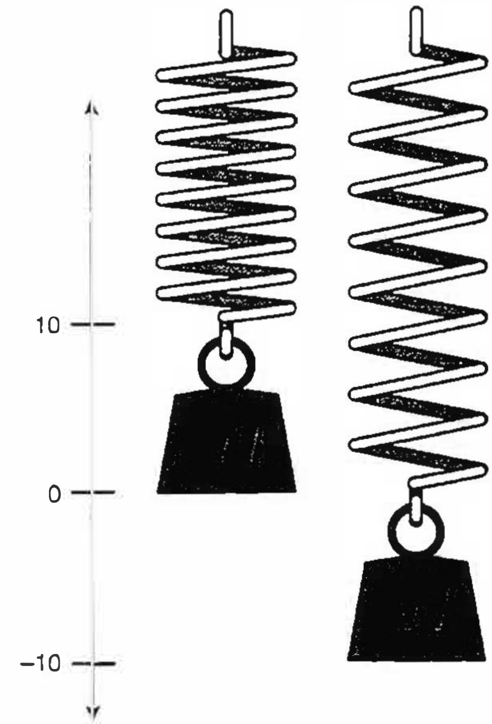


Figure 2
Quantifying the Motion



the question, "Measure from where?" is sure to arise. Restarting the oscillation, the teacher might ask where an appropriate "zero point" would be. For convenience, negotiating the equilibrium point as the zero point is fairly easy, and measuring the "deflection," or amount of stretch, should seem reasonable. One possibility is illustrated in Figure 2. The situation has been quantified, and the function sought can be described numerically. For example, we are looking for a mathematical function that has value −10, then 0 then 10, 0, −10, 0, 10, ... It is hoped that the notion of a sine or cosine function will follow. In the author's experience, it always has!

The teacher will also need to negotiate with students an appropriate sinusoid for this problem. In so doing, a second critical quantity, time, should enter the discussion. In deciding which sinusoid to use in the model, students will need to focus on the known ordered pair at the start of the oscillations: When the time is 0, the displacement of the weight is −10. What should emerge form the discussion is a tentative model: $y = A \cos Bx$.

Once a tentative model has been elicited from the students, the remaining task is to associate the constants A and B with the measurable physical quantities present in the problem. Students have had no trouble connecting the deflection of the weight with the amplitude of the graph of the cosine function and, hence, with A. Thus, if the original deflection of the weight was, say, 10 cm, the tentative model can be adjusted to $y = -10 \cos Bx$.

What has often caused students more difficulty was connecting the constant B with something. This something is, of course, related to the period of the graph of the cosine function, but how is the period of a cosine graph related to the present situation?

Here students are being asked to make a connection between the period of a cosine graph and the period of an oscillation. Having made this connection, students will usually see that the period of oscillation is really a period of *time* and, hence, that the independent variable in this situation is time. However, the really tricky part remains: can the periods of oscillation be measured with some degree of accuracy, and how is that period related to the constant B?

Students usually devise some effective means to measure the period of oscillation. Most often, they have suggested measuring the time required for a certain number of oscillations, say, 5, and then dividing by 5. A more sophisticated group might suggest taking several measurements and averaging them. This pursuit might lead to a discussion of "outliers" and their possible causes, as well as their resolution!

Having a measure of the period of oscillation, students then need to connect that number with the constant B in their model. The teacher might ask them to recall the relationship of B to the period of a cosine graph:

$$\text{period} = \frac{2\pi}{B},$$

from which it follows that

$$B\,(\text{period}) = 2\pi$$

and

$$B = \frac{2\pi}{\text{period}}.$$

For example, if the period of oscillation was 1.6 seconds, we would have

$$B = \frac{2\pi}{1.6},$$

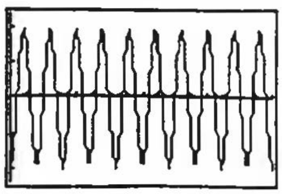
or

$$B = \frac{\pi}{0.8}.$$

Thus, our tentative model can be further adjusted to yield

$$y = -10 \cos\left[\left(\frac{\pi}{0.8}\right)t\right].$$

Figure 3
The Graph of $y = -10 \cos[(\pi/0.8)t]$ for $0 \le t \le 16$



At this point, students benefit from examining the graph of this function with the aid of a graphing utility. It seems preferable that students do so using their own graphing calculator, but such activities have also been successfully conducted using a graphing utility projected on the overhead projector. Whichever means is used, students must determine whether the graph of this function indeed models the physical situation. Some discussions of an appropriate "viewing rectangle" should precede the graphing activity.

Here a word of caution is in order. When using a graphing utility to graph periodic functions, one must think carefully about the size of the viewing rectangle with respect to the period of the function. In the present situation, for example, a period of oscillation of 1.6 seconds has been assumed. What would happen if an attempt was made to graph this model of $0 \le x \le 150$? Since the weight might continue oscillating for several minutes, it might, in fact, seem quite reasonable to use such a domain for x, as it represents only a 2.5-minute span.

However, the graph that many utilities would produce in such a viewing rectangle is very misleading. See Hansen (1994) for a discussion of graphical misrepresentations that occur when the domain divided by the period of a function is a multiple of the number of pixels in the width of the screen of the graphing utility. I have found that 12–15 cycles of a periodic function are the maximum that can be conveniently displayed with reasonable accuracy using a graphing utility such as the TI-81. To require more than that is to push the technology beyond its limits.

Figure 3 shows a graph of the first model. At this point, students are usually quite pleased with themselves for having produced this model. They are quite unprepared for the next question, which the reader may already have guessed, "How could we improve this model?"

Building a Better Model

Once the existing model has been suggested as problematic, on reflection, students will see that they have modeled a "perpetual motion machine." This notion should begin the search for a better model, one that accounts for the damping of the motion. Once again, students will need to make a connection, this time between the coefficients A and B and the physical reality that the motion is "slowing down."

Asking students the question, "What physical quantity have we been treating as a constant, although it is not really a constant?" helps them to associate A, or the amplitude, with the damping effect. Students can then be encouraged to try out various variable expressions in place of the constant -10 in their model. For example, Demana and Waits (1989) suggest the equivalent of $-10 + t$ in the problem set cited earlier. Figure 4 shows a second model, using $A = -10 + t$.

While furnishing a model of the damping effect, this function has the undesirable property that it seems to show the motion starting up again after stopping! This shortcoming leads to a part of the situation that is difficult to model: an amplitude is sought that will approach zero, then *equal* zero for some value of t and *all larger* values of t.

Figure 4
The Graph of $y = (-10 + t) \cos[(\pi/0.8)t]$ for $0 \le t \le 16$

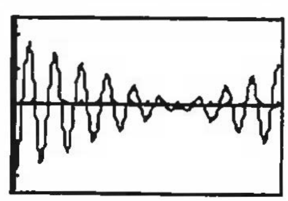
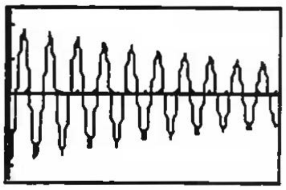
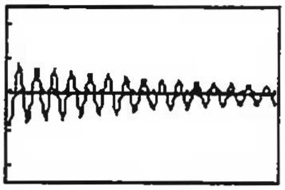


Figure 5

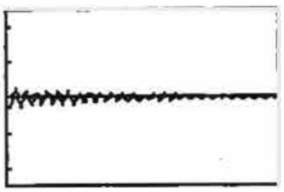
(a) The graph of $y = -10\, e^{-0.05t} \cos[(\pi/0.8)t]$ "going strong" in a (0, 16) by (-12, 12) viewing rectangle.



(b) The graph of $y = -10\, e^{-0.05t} \cos[(\pi/0.8)t]$ "still going" in a (16, 40) by (-12, 12) viewing rectangle



(c) The graph of $y = -10\, e^{-0.05t} \cos[(\pi/0.8)t]$ "coming to rest" in a (40, 80) by (-12, 12) viewing rectangle



Searching for the Best Model

Although such a function might be piecewise defined, the model that physicists have suggested uses a function that is asymptotic with $y = 0$. Students who have had some experience with the graphs of exponential functions should be able to make a connection here. If they are familiar with the graphs of $y = e^x$ and $y = e^{-x}$, then the customary model of damped motion can be constructed. If not, then making a connection with $y = 2^x$ may do.

In any event, the connection that needs to be made is that multiplying a function that is asymptotic with zero by a constant such as -10 produces a function that remains asymptotic with zero. Of course, students must also recognize that a function is sought that asymptotic with the *positive* x-axis. Thus, our model could be adjusted to

$$y = -10e^{kt} \cos\left[\left(\frac{\pi}{0.8}\right) t\right].$$

With the aid of the graphing utility, students can explore the effect of various values of k on the model. Students might be encouraged to find the value of k that best models their situation. This task could be accomplished by measuring the time it takes for the weight to come to rest and searching for the value of k whose graph best depicts that aspect of the situation. Figures 5a, b and c depict such a model graphed in different viewing rectangles to show the "coming to rest" process. Note that in this example, one graph is clearly not sufficient.

Even this exponential model, which is the one usually used in physics, is not a perfect descriptor of the physical situation. After all, we would probably agree that the weight does indeed eventually come to rest, but y does not equal zero for any value of x in the domain of these models. What makes this the best model, in fact, what make any model a better model, is that it mimics more of the physical aspects of the situation than do other models.

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