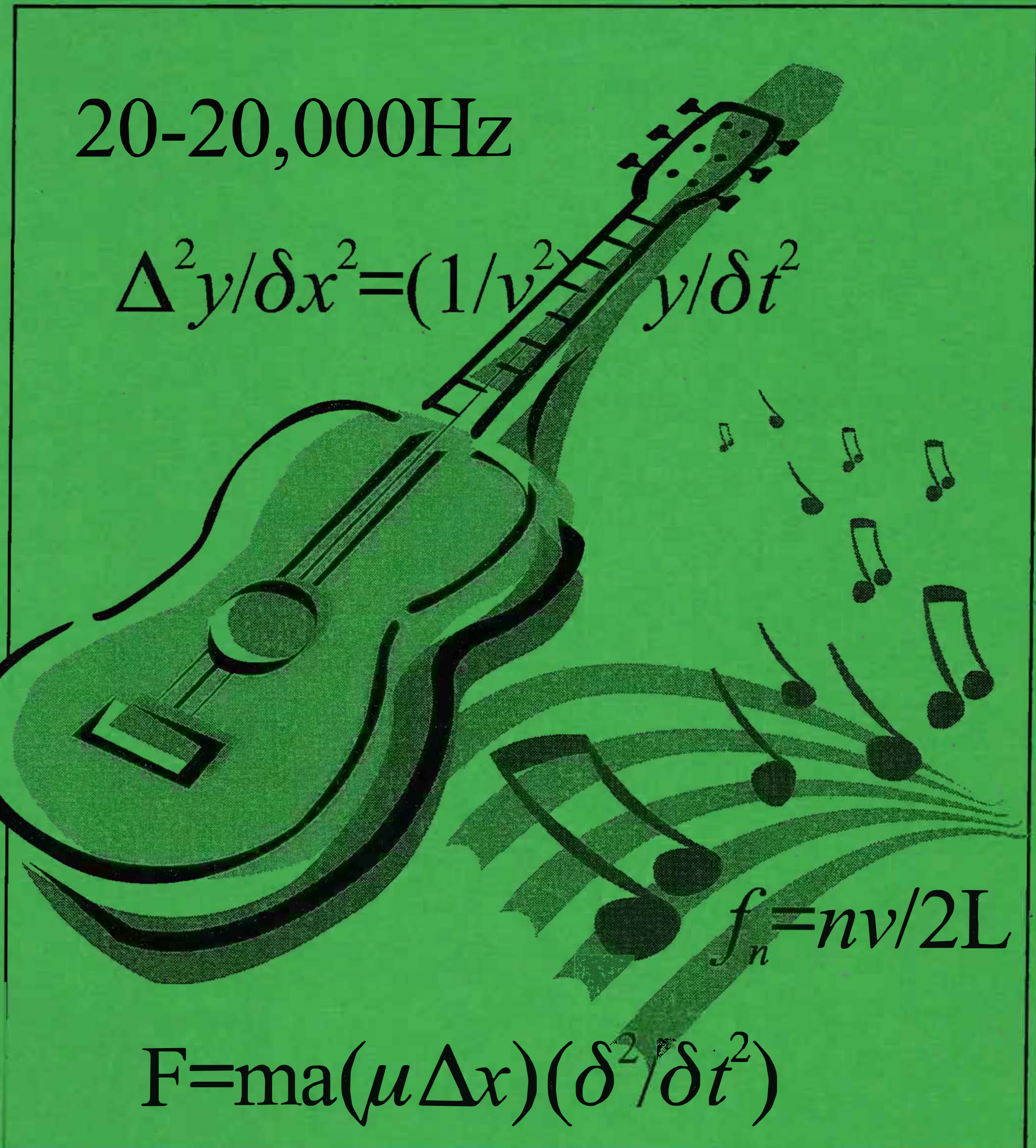
delta-k

JOURNAL OF THE
MATHEMATICS COUNCIL
OF THE ALBERTA
TEACHERS' ASSOCIATION

Volume 35, Number 2 May 1998



GUIDELINES FOR MANUSCRIPTS

delta-K is a professional journal for mathematics teachers in Alberta. It is published to

- promote the professional development of mathematics educators, and
- stimulate thinking, explore new ideas and offer various viewpoints.

Submissions are requested that have a classroom as well as a scholarly focus. They may include

- personal explorations of significant classroom experiences;
- descriptions of innovative classroom and school practices;
- reviews or evaluations of instructional and curricular methods, programs or materials;
- discussions of trends, issues or policies;
- a specific focus on technology in the classroom; and
- a focus on the curriculum, professional and assessment standards of the NCTM.

Manuscript Guidelines

1. All manuscripts should be typewritten, double-spaced and properly referenced.
2. Preference will be given to manuscripts submitted on 3.5-inch disks using WordPerfect 5.1 or 6.0 or a generic ASCII file. Microsoft Word and AmiPro are also acceptable formats.
3. Pictures or illustrations should be clearly labeled and placed where you want them to appear in the article. A caption and photo credit should accompany each photograph.
4. If any student sample work is included, please provide a release letter from the student's parent allowing publication in the journal.
5. Limit your manuscripts to no more than eight pages double-spaced.
6. Letters to the editor or reviews of curriculum materials are welcome.
7. *delta-K* is not refereed. Contributions are reviewed by the editor(s) who reserve the right to edit for clarity and space. **The editor shall have the final decision to publish any article.** Send manuscripts to Klaus Puhlmann, Editor, PO Box 6482, Edson, Alberta T7E 1T9; fax 723-2414, e-mail klaupuhl@gyrd.ab.ca.

Submission Deadlines

delta-K is published twice a year. Submissions must be received by August 31 for the fall issue and December 15 for the spring issue.

MCATA Mission Statement

Providing leadership to encourage the continuing enhancement of teaching, learning and understanding mathematics.

delta-K

Volume 35, Number 2

May 1998

CONTENTS

 delta-K is published by the ATA for the Mathematics Council (MCATA). EDITOR: Klaus Puhlmann, PO Box 6482, Edson, Alberta T7E 1T9. EDITORIAL AND PRODUCTION SERVICES: Central Word Services staff, ATA. Opinions expressed herein are not necessarily those of MCATA or of the ATA. Address correspondence regarding this publication to the editor. *delta-K* is indexed in the Canadian Education Index. ISSN 0319-8367

COMMENTS ON CONTRIBUTORS

Bobbye Hoffman Bartels teaches in the mathematics department at Christopher Newport University, Newport News, Virginia.

Ryan Cassidy wrote "The Physics of a Classical Guitar" while in Grade 12 at St. Mary's High School in Calgary, Alberta. He is currently studying engineering at the University of Calgary.

Mary Chan wrote "Ode to π" while in Grade 11 at St. Mary's High School in Calgary, Alberta. She currently attends the University of Calgary.

David R. Duncan is a professor of mathematics at the University of Northern Iowa, Cedar Falls, Iowa.

Jo Dodd teaches mathematics at Paul Kane High School, St. Albert, Alberta.

Thomas Edwards teaches at Wayne State University, Detroit, Michigan.

Jane Ehrenfeld teaches at Glassmanor Elementary School in Oxon Hill, Maryland.

Marjorie Gann is a Nova Scotian elementary teacher and freelance writer, now living in Sackville, New Brunswick.

Claire Groden is an elementary mathematics specialist in Watertown, Massachusetts.

Brenda Healing teaches at Bentley High School, Bentley, Alberta.

Katherine Heinrich is president of the Canadian Mathematical Society (1996–98), Ottawa, Ontario.

Alvin Johnston is the principal of École Pine Grove Elementary School in Edson, Alberta.

Arthur Jorgensen is a retired junior high school principal from Edson, Alberta, and a longtime MCATA member.

Thomas E. Kieren is a professor of education (emeritus) in the Faculty of Education, Department of Secondary Education, University of Alberta, Edmonton.

Murray Lauber is a professor in the Division of Mathematical Sciences at Augustana University College in Camrose, Alberta.

Shirley LeMoine is the district coordinator of gifted and talented education with Garfield Re-2 School District, Rifle, Colorado.

Bonnie H. Litwiller is a professor of mathematics at the University of Northern Iowa, Cedar Falls, Iowa.

Jan Lockley teaches at Jerry Potts Elementary School in Calgary, Alberta.

Cheryl Matern is a teacher at Joel E. Ferris High School in Spokane, Washington.

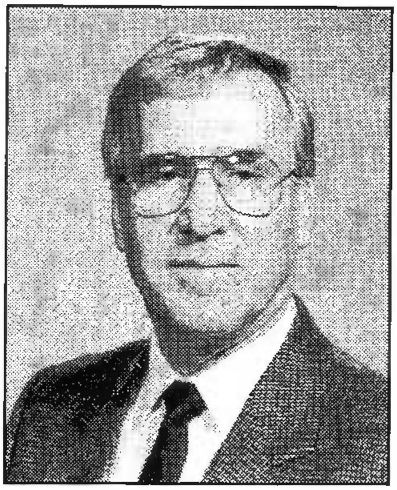
Terri-Lynn McLeod teaches at Calling Lake High School, Calling Lake, Alberta.

Laurie Pattison-Gordon is interested in developing and implementing visual technologies to support inquiry learning and critical observation of classroom environments.

Sol E. Sigurdson is a professor of education (emeritus) in the Faculty of Education, Department of Secondary Education, University of Alberta, Edmonton.

Connie Williams is in the B.Ed. After Degree program at the University of Alberta, Edmonton, majoring in biological sciences and minoring in mathematics.

EDITORIAL



With the 1997 MCATA annual conference behind us, we turn our attention to another major professional development event—the Canadian Regional MCATA/NCTM Conference that will be held in Calgary, October 23–24, 1998.

All annual conferences organized by MCATA, while appearing to be major single events, are not intended to be one-shot deals. The purpose is to explore topics that are timely as well as on the cutting edge of mathematics reform. These conferences are also about sharing and networking with others from within and outside the province. They provide an excellent springboard for further professional development at school, district and regional levels.

We have produced excellent mathematics programs in this province—the Western Canadian Protocol is one example—however, unless teachers are helped to use the program and materials effectively, the efforts of these excellent programs will not be felt in classrooms. Realizing the massive nature of implementing a new program, there is a need to explore better ways to make staff development continuous, focused and effective. This calls for our unqualified commitment to and support of teachers to overcome the obstacles of lack of time and resources. Teachers need opportunities to meet with others so that they can talk about the *context* of the new program, not just the *content*, as well as the delivery systems that are most appropriate and models for effective staff development. Many models of professional development are in use; among these are numerous self-help activities related to the use of professional journals, videotapes, critical reviews (with or without colleagues), CD-ROMs, the Internet, technology—to name a few. All require an investment of time.

However, the point is that if the implementation of a new program is important, and if it is to have a positive effect on the learners, then teachers must be able to engage in continuous, focused and effective professional development activities.

I hope that the articles contained in this issue are not just interesting, challenging and useful within your own classroom—although this is an important and major goal for me as editor—but that each issue of *delta-K* also contributes to your professional development. As much as you may benefit from the contributions made by each author, I hope that you, too, will find the time to share your ideas of teaching and learning with our readers. Your ideas and those of your students are needed and very important.

Klaus Puhlmann

From the President's Pen

Can you believe that the end of the school year is approaching? As I indicated in the last issue of *delta-K*, my president's messages this year focus on sharing my thoughts about MCATA's mission statement: "Providing leadership to encourage the continuing enhancement of teaching, learning, and understanding mathematics."

In my previous message, I reflected on what it meant to be a teacher and how I believe the role of the teacher is changing and evolving: in no way is the definition of "teacher" static. In this issue, I share my thoughts on what it might mean to "provide leadership to encourage the continuing enhancement of learning mathematics."

Close your eyes and remember what it meant to learn mathematics for you. My story is as follows: my first memory of learning mathematics is from Grade 1. I remember bundling popsicle sticks in groups of 5 and then in groups of 10. I don't remember how long we did this, but I do have a memory of the actions of bundling and counting. I don't have much memory about learning mathematics until Grade 7 when we were introduced to the "new math"; I do remember practising the drawing of the "squiggly brackets" and Venn diagrams and using set language such as *union* and *intersection*. Then I jump ahead to Grade 9 when I was on an individual learning program, where we did questions from a book at our own pace. Then came Mathematics 10, 20, 30 and 31—I remember memorizing proofs for geometry and solving linear and quadratic equations, solving systems of linear equations, factoring quadratic expressions and memorizing the product, quotient and chain rules for finding the derivative of an expression. I have not elaborated on the experiences; however, they cause me to consider how my own thinking about learning mathematics has been affected, which forms a concept I examine often as I teach and think about teaching.

I believe that we are in an exciting time in mathematics education because we have learned more about how children learn—we can attribute this to the research in cognition over the last 30 years. It is an exciting time because we are learning that children need to be involved in constructing their own meaning of mathematics and that they inherently know a lot of mathematics. This means not only that students must be involved in practice but also that *they* be involved in using mathematics to describe their world. My four-year-old niece Valisa is already a number-and-logic person. She loves to count and she counts everything. I watched her play with her dolls the other day as she sorted and classified shoes. She loves logic puzzles that involve some sort of manipulative. As I watch and talk to her, I am constantly reminded that context is important. Valisa is very good at these things because they are important to her. When I think about my teaching, I think about how I can make the mathematics important to my students. I think that this is our challenge as mathematics teachers. We all learn best when what we are learning is important to us. Think about learning to use a computer. I certainly learned about the capabilities of a word processing program, for instance, when I wanted to complete a particular task.

What steps do we take as mathematics teachers to help mathematics become important to our students? How we answer this question will be affected by our own experiences of learning mathematics. Think about your experiences. How have they affected your beliefs about learning mathematics? I think that answering these questions is recursive, but as we continue to answer and learn more about learning, we provide "leadership to encourage the continuing enhancement of learning mathematics."

Florence Glanfield

Your MCATA Executive at Work

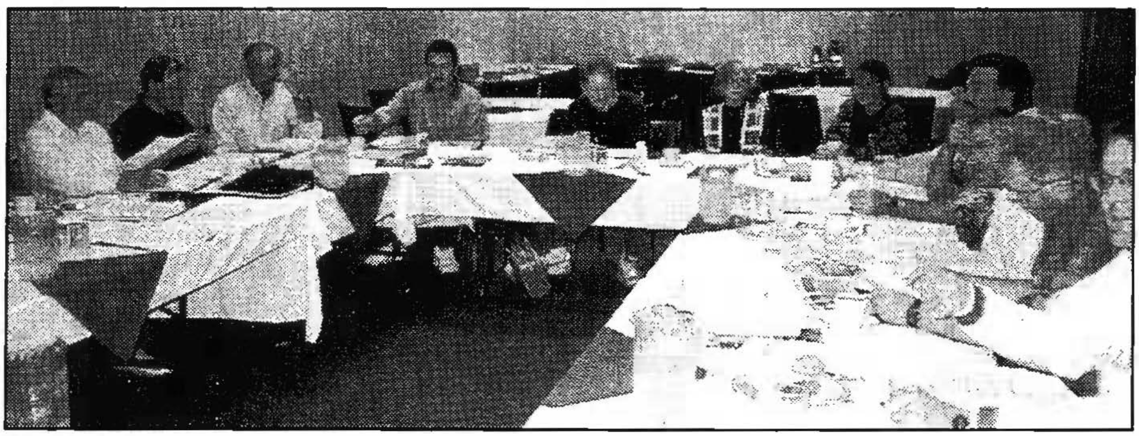
The MCATA executive meets several times yearly. These meetings are held on Friday evening and all-day Saturday. The agendas are always filled with many issues, proposals and action items that keep the executive focused not only on its immediate tasks but also on its overall mission, which is providing leadership to encourage the continuing enhancement of teaching, learning and understanding mathematics. The following pictures were taken at the executive meeting September 5–6, 1997.

Note: The following MCATA executive members are not shown in the pictures: Dale Burnett, Faculty of Education representative; Bob Michie, 1998 conference chair; and Klaus Puhlmann, delta-K *editor.*

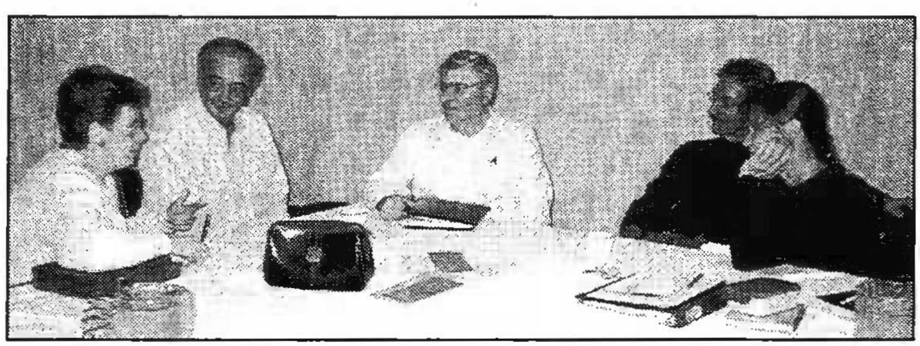
(l–r): Florence Glanfield, president; Donna Chanasyk, secretary; Rick Johnson, director; Marge Marika, 1997 conference chair; Art Jorgensen, newsletter editor; Carol Henderson, PEC liaison; and Dick Pawloff, Webmaster.

Three new members join the executive! (l–r): Rick Johnson, Edmonton, teacher at Sifton Elementary School; Elaine Manzer, Peace River, teacher at Peace River High School; and Dick Pawloff, Red Deer, district coordinator of technology.

(l–r): Florence Glanfield, president; Marge Marika, 1997 conference chair; Richard Kopan, director; Ivan Johnson, NCTM Regional Services Committee; Carol Henderson, PEC liaison; and Betty Morris, vice president.



(l–r): Dick Pawloff, Webmaster; Graham Keogh, director; David Jeary, ATA staff advisor; Richard Kopan, director; George Ditto, past president; Cynthia Ballheim, vice president; Sandra Unrau, director; Elaine Manzer, director; Doug Weisbeck, treasurer; Michael Stone, mathematics representative; Ivan Johnson, NCTM Regional Services Committee; and Daryl Chichak, Alberta Education representative.



Executive in small groups discussing the Interim Position Paper on Calculator Use in Provincial Assessment Programs. (l–r): Donna Chansyk, secretary; David Jeary, ATA staff advisor; Dick Pawloff, Webmaster; George Ditto, past president; and Sandra Unrau, director.

From the 1997 Conference Chair

"Doing Mathematics"

Another successful MCATA annual conference was held November 1–2, 1997, at the Edmonton Inn. On Friday, October 31, more than 120 delegates attended a preconference mathematics symposium. They discussed current issues in mathematics education and heard from speakers representing Alberta Education and school jurisdictions. By Friday evening, conference delegates were registering for the more than 45 weekend sessions and renewing acquaintances with friends and colleagues in the hospitality room. Our registrars saw that nearly 300 delegates got their name tags, portfolios and materials so that they would be well prepared for two days of "Doing Mathematics."

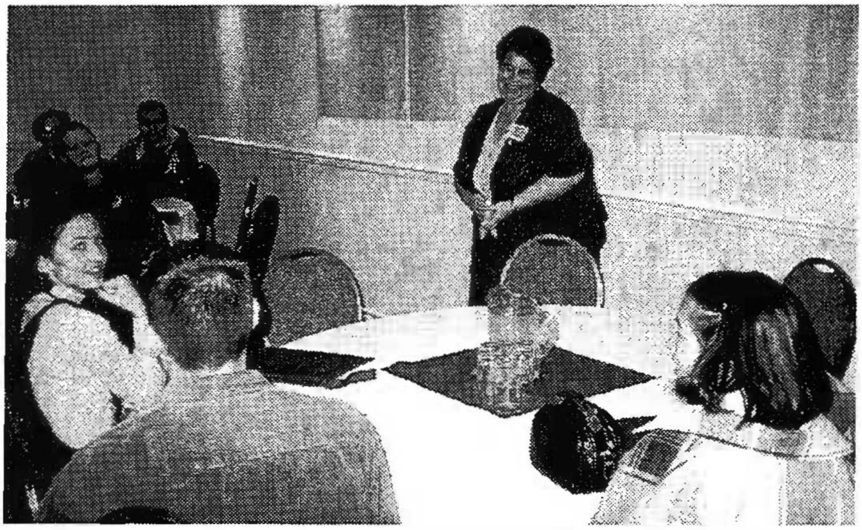
Saturday morning began with the dynamic and entertaining observations of our keynote speaker, Don Fraser, from the University of Toronto. He helped us to "Take the 'Numb' Out of 'Number' " and set the stage for the many sessions, roundtables and workshops that followed. Throughout Saturday, delegates interacted with ideas and materials that help to bring the processes and objectives of the Western Canadian Protocol to life. Methods for integrating technology, classroom assessment and problem-solving into professional practice were highlighted in many practical sessions. Many teachers found it valuable to have input into the implementation and development of resources for the new high school courses of study. A room devoted to displays of commercial and NCTM products had a great deal of traffic throughout the conference. Many delegates enjoyed the Saturday box lunch and the chance it provided for a flexible schedule and informal interaction with colleagues.

Saturday evening offered a chance to catch up on Edmonton activities, and many teachers, geared up by a day full of mathematical endeavors, were keen to head for the "tranquil" (!?) environment of West Edmonton Mall. Sunday sessions, including roundtable mini-sessions, workshops and demonstrations saw a group of hardy and interested attendees. Hugh Sanders from Alberta Education provided a thoughtful closing address to the conference on "Doing Mathematics: Where Are We and Where Are We Going?" A selection of door prizes provided a pleasant conclusion and warm send-off to all delegates. Comments from participating teachers, presenters and those who displayed indicated that all felt the conference will help them to better explore, understand and enjoy the important work of "Doing Mathematics."

Marge Marika

1997 Conference Photos

Ron Flaig is reviewing classroom assessment materials for secondary mathematics.



Florence Glanfield, MCATA president, facilitates conversation among teachers who are teaching mathematics for the first time.

Registration team: (r–l) Daryl Chichak, Patricia Chichak, Denise Lucyshyn, Katherine Kostyniuk and Elizabeth Kostyniuk.

Jack LeSage talking to Tanya LaRocque and Kari Webb from High Level Public School about using manipulatives in Grades 7–9 measurement and geometry.

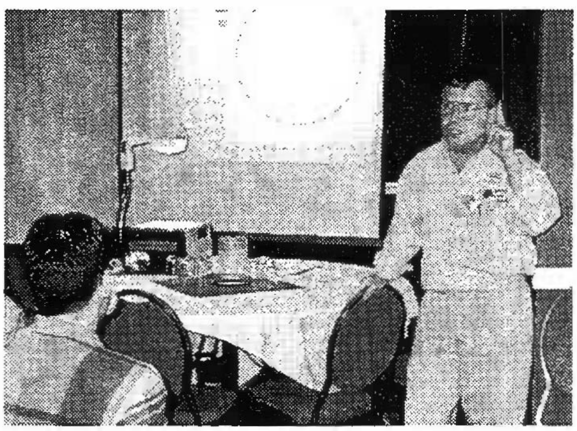
Marilynn Reid works with a group of teachers on patterns and relationship building to operations.

Kay Melville makes a presentation on performance-based assessment in Grade 6.

Bob Hart demonstrates scatter plots on the TI-83.

Linda Williams is "piecing it together with fractions" as she uses games to teach fractions and other related concepts.

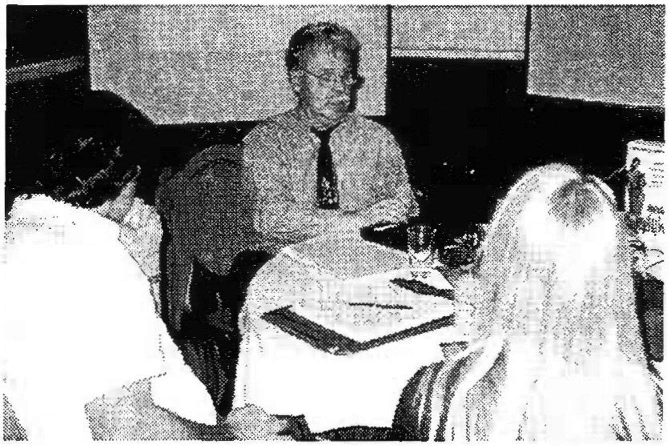
Sandra Carl Townsend discusses the planning of a balanced assessment program at the K–Grade 6 levels.



Harry Morrison engages teachers in a hands-on approach to enhance the understanding of integer operations using manipulatives.

Peggy Morrow demonstrates how to provide an enjoyable learning context for students by using games.

Marie Hauk focuses on paper folding and cutting activities that relate to specific learner outcomes at the junior high school level.

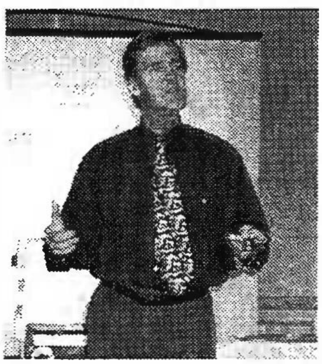


Dave Walker leads teachers through a variety of new software that makes "doing math" effective and active.

Sandra Unrau talks about planning for the math processes as well as skills and knowledge.

Enzo Timoteo shows off Alge-Tiles by focusing on "Modes of Learning: Concrete-Pictorial-Symbolic."

Bryan Quinn conducts a probability experiment with surprising results during his presentation.



Don Fraser, keynote speaker: "Taking the 'Numb' Out of 'Number'"

Cynthia Ballheim, busy at the MCATA desk in the display area

1997 Outstanding Mathematics Educator Awards

For the first time, MCATA presented *three* Alberta Mathematics Educator Awards. The following awards were presented at the 1997 MCATA annual conference in Edmonton to those who have demonstrated leadership in encouraging the continuing enhancement of teaching, learning and understanding mathematics in the province of Alberta:

- K–Grade 6 classroom teacher
- Grades 7–12 classroom teacher
- Those who have made exceptional contributions to the professional development of teachers at the school, local, provincial or national levels

Elementary Mathematics Educator Award: Lynwen Hart

Jan Lockley and Lynwen Hart

Lynwen Hart is a Grade 6 teacher at Capitol Hill School in northwest Calgary. Teaching Grades 4, 5 and 6 in the last three years has made her very familiar with the Division II curriculum, especially mathematics.

I met Lynwen in winter 1993, when the Calgary Board of Education began setting up study groups, using interested teachers as facilitators to examine the new math curriculum. We have been working together since. Last spring, I approached Lynwen's principal, Lynn Flaig, and her teaching colleague, Laurie Los, in my quest to see Lynwen nominated for this award. They were only too pleased to support my nomination. I would like to share some of their thoughts, as well as some other reasons as to why we believe Lynwen was a good choice for this award.

Laurie's son, Tyler, was in Lynwen's class for both Grades 5 and 6. In Tyler's words, "Mrs. Hart made me look at math in a different way." Laurie went on to say that Lynwen "has opened up a whole new world that is more than crunching numbers. She has challenged Tyler to think critically and taught him to approach math with a positive attitude."

Another parent of a learning disabled student in Lynwen's class wrote to affirm Lynwen's ability to reach students of all ability levels. She stated, "Mrs. Hart has gone the extra mile for Hilary in a number of ways. She has been willing to accommodate Hilary's special learning style so that she can learn to the best of her ability and can enjoy the successes we always knew she had in her."

Clearly Lynwen motivates and inspires her students. It isn't mere coincidence that her provincial math test results placed her school second in the city of Calgary, with 100 percent of her students achieving the acceptable level. Good teaching practice does equate to successful student learning.

Lynwen also believes that parents are partners in their children's learning. To help parents become familiar and more comfortable with the focus of our program of studies, Lynwen has provided evening curriculum meetings at her school to bridge the gap between home and school.

Furthermore, Lynwen helps teachers at her school and other teachers in the system by running Math Implementation Network meetings at her school to examine in detail and discuss the philosophy behind our new program of studies. Lynn Flaig believes that "Lynwen's success lies in her ability to combine theory and practice. She combines her knowledge of teaching theories with her curriculum knowledge to present workshops that cause teachers to reflect on their own teaching practice. They become increasingly confident in their understanding of teaching and learning mathematics." The number of teachers involved in the network is steadily growing. I'd even venture to say that Lynwen has a "following."

Lynwen continues to reach out to teachers by sharing her expertise at teachers' conventions, MCATA regional conferences, and through her involvement with Canadian Teachers Teaching with Technology. This past summer I had the pleasure of attending a calculator workshop that Lynwen conducted through Athabasca University. I learned how to use calculators to develop math concepts and improve problem-solving skills in a practical hands-on approach.

In summary, let me say that any child who has the privilege to be in Lynwen's class is fortunate, indeed. Similarly, any teacher who attends a session presented by Lynwen will learn to excite and challenge his or her students. Everyone should be so lucky to have a Lynwen in his or her life. She is my colleague, my friend and truly an inspiration.

Jan Lockley

Secondary Mathematics Educator Award: Donna Chanasyk

Donna Chanasyk and Jo Dodd

In searching through some materials for my leadership class the other day, I came across a wonderful quotation which I believe expresses and reflects many of the qualities that Donna Chanasyk demonstrates in her role as mathematics department head at Paul Kane High School. The quotation comes from an unusual source, Hollywood actor Meryl Streep, who says:

> Integrate what you believe into every single area of your life. Take your heart to work and ask the most and best of everybody else. Don't let your special character and values, the secret that you know and no one else does, the truth—don't let that get swallowed up by the great chewing complacency.

Donna does integrate her beliefs into what she does with her students. She believes in hard work, in paying amazing attention to detail, in being dedicated to one's purpose and to always being organized and prepared. She takes her heart to work with her. She truly believes that every student should be given every opportunity to be successful.

Last year at Paul Kane, Donna gathered a group of some 35 students who were supposed to graduate but had not yet fulfilled their two compulsory math class requirements. She took them under her wing and gave them materials for the Math 14 and 24 courses. Although they worked on their own time, she spent many hours of her own time tracking them down, encouraging them to complete their work and phoning parents to discuss their progress.

In the present era of educational cutbacks, increased class sizes and reductions in funding, it is difficult for one not to become complacent in one's view of schools. Not Donna. She organizes our school's Awards Night and Color Night; she is on our grad committee and our district Professional Development Committee; she is in the Edmonton Regional, the provincial AP Council and is the MCATA secretary. All this—and she has a son in Grade 10.

What is her secret? It is obvious that under that quiet, controlled, organized, knowledgeable and professional exterior, we have a woman who loves teaching, who is very dedicated to her teaching of mathematics and whom students can rely on. She is also honest and quick to share times when her classes have not gone as planned, students have gone wild on a Friday and material had to be taught again. She can laugh at her mistakes. I love her honesty and her ability to make other teachers feel wonderful about something they have done.

The greatest people are often the most humble. Donna is one of them. Congratulations, Donna, we are all honored to present you with this Secondary Mathematics Educator Award.

Jo Dodd

Non-Classroom Category Mathematics Educator Award: Klaus Puhlmann

Klaus Puhlmann and Art Jorgensen

Dr. Klaus Puhlmann, superintendent, Grande Yellowhead Regional Division No. 35, was awarded the Non-Classroom Category Mathematics Educator Award. Dr. Art Jorgensen, long-time friend and professional colleague, made the award presentation.

Dr. Jorgensen said that "he was honored to be asked to make the presentation to an individual whom he considers to be a credit to mathematics education in Alberta and in Canada." Klaus Pulhmann has dedicated his professional career to the development and betterment of mathematics education. His academic background—including a B.Sc. from the University of Berlin, and a B.Ed., M.Ed. and Ph.D. from the University of Alberta—has been directed toward mathematics curriculum development, improvement of instruction in mathematics and making mathematics alive, meaningful and relevant in the lives of teachers and students.

Klaus is committed to the enhancement of student understanding and the meaningful application of mathematical concepts. His personal involvement has included committee work and special projects with Alberta Education (member of the Examinations Program Committee, Examinations Review Committee, High School and Secondary Math Review Committee, Diagnostic Math Committee and Diploma Exam Advisory Committee: Math 30 and 33) and the Alberta School Board Association (Computers in School Task Force member).

Klaus unselfishly gives of his time and effort in assisting and serving professional colleagues. Because of his extensive background and expertise, many educational professional development organizations have invited him to make presentations and provide workshops in mathematics. He has been a regular presenter at The Alberta Teachers' Association teachers' conventions and National Council of Teachers of Mathematics (NCTM) conferences, as well as presenting at the Canadian Educational Association 100th Anniversary Conference, the Alberta Diagnostic Math Program and Grande Yellowhead Regional Division No. 35 Annual Teachers' Institutes. Academic institutions, such as the University of Alberta and Grande Prairie College, have invited Dr. Puhlmann to provide professional seminars and workshops.

Dr. Puhlmann has continued an ongoing personal professional development program throughout his career. He is a regular delegate at MCATA annual conferences, NCTM national and international conferences, as well as symposia for math teachers in Alberta; he also maintains involvement with the Canadian Mathematical Society. He has been a member of MCATA for 26 years, served as an executive director (1979–83), and has been a member of the MCATA executive since 1995, and as editor of *delta-K*.

Klaus Puhlmann is not only a recognized leader in mathematics education but also a valuable professional associate and fellow teacher. Sharon Armstrong, president, Evergreen Local No. 11, ATA, describes Dr. Puhlmann as "a leader, not only in words, but in your actions to the teachers of Grande Yellowhead Regional Division No. 35 as demonstrated by your initiatives in the mathematics area . . . to be the recipient of the Alberta Mathematics Educator Award is indeed an honor, and one you richly deserve."

Alvin Johnston

1997 Friends of MCATA

During the 1996–97 school year, the MCATA executive wanted to recognize those individuals who have given MCATA faithful and dedicated service in a variety of ways throughout the years and to say "thank you." We know that there are many ways that individuals contribute to the success of an organization. These individuals have given generously of their time, effort and expertise.

As a Friend of MCATA, an individual will receive a certificate, a "Friend of MCATA" pin and acknowledgment of his or her involvement with the Council in the program of the annual conference, the newsletter and on the MCATA Web site.

Other recipients not shown in the picture are Linda Brost, Calgary; Pat Cavanaugh, Red Deer; Leo Chauvet, Red Deer; Archie Coderre, Red Deer; Carol Davis, Red Deer; Judy Dobson, Red Deer; Doug Girvan, Red Deer; Ellen Hilton, Red Deer; Bill Hoppins, Edmonton; Karen Kenny, Red Deer; Pat Maschio, Red Deer; Bob Mitchell, Edmonton; Allan Quinn, Red Deer; Jim Schlachter, Red Deer; and Henry Taschuk, Edmonton.

Thank you for your strong support of our mission which is "Providing leadership to encourage the continuing enhancement of teaching, learning and understanding mathematics."

(l–r) Monique Gratix, Ponoka; Bruce Kabaroff, Edmonton; Graham Keogh, Red Deer; Cindy Meagher, Grande Prairie; Wendy Richards, Edmonton; and Colleen Williamson, Red Deer.

In this section, we will share your points of view on teaching mathematics and your responses to anything contained in this journal. We appreciate your interest and value the views of those who write.

The Importance of Numeracy in a Technological Society

Murray Lauber

Popular mathematician John Allen Paulos is an advocate for numeracy, claiming that it must take its place alongside literacy as an essential component of education in a complex society. He argues that numeracy will become even more crucial as we confront the mountains of data that will accompany the evolution of sophisticated information technology.

Paulos' conviction about numeracy reminds me of an experience a few years ago in one of my precalculus classes, when I observed that I had experienced the best of both worlds as far as computing technology is concerned: I had grown up and been forced to compute without it, but now had the privilege of using it. "Yeah, yeah . . . ," a student at the back of the classroom remarked, "and I bet you walked five miles to school every day, too. You sound like my Dad."

I doubt that the student was intending to be complimentary. But I think that I managed to turn that incident to advantage. I sometimes relate this story as I perform a computation on the chalkboard and, as the students reach for their calculators, write the appropriate answer on the board while remarking that for those of us who walked five miles to school every day, such an estimation is a piece of cake.

Now, I do not expect anyone to be persuaded that walking exercises the estimation cells in the brain. Nor do I anticipate a great deal of nostalgia for the good old days of log tables and slide rules. But there was in the use of those methods of computation the necessity to think consciously about the sizes of the numbers involved. "Is that in the order of hundreds, or thousands, or tenths, or hundredths?" A significant computation could not be performed without confronting and answering this question several times.

The development of facility with numbers, of which skill at estimation is but one facet, is not a passive enterprise. It requires conscious effort. Memorizing a host of formulas or techniques does not in and of itself develop numeracy. Engaging in repeated drills does not do it either. Nor does the exposure to calculators and computers. The development of numerical facility requires concentrated effort, often more demanding than that of walking five miles to school every day. It is the effort of understanding the logic behind the solution to a problem. That aspect of numeracy has not changed, nor is it about to, even with the development of smarter and smarter computers.

Thinking About Making Connections in Mathematics: One Student Teacher's Story

Connie Williams

The students at the school where I taught my Phase 2 practicum were eager to use anything they were taught. During the course work I often overheard them saying "When am I ever going to use this in the real world?" I believe it is important for students to see the applicability in the real world of what they are learning in math class. These same students were very excited when their social studies teacher talked about a formula to calculate interest. He also related this topic to world banks and global economies. I think that we teachers should strive to make subjects in school more connected.

A colleague in a secondary education course told me that some teachers with whom she was working thought that the material presented in the new Grade 9 textbook was too abstract. This teacher decided not to use the new text. I, however, think the text is great. The text tries to make connections between math and other subjects, for example, between card games and algebra, or exponents and distances between planets. While I was doing my practicum I saw some old textbooks that I used for high school math. The texts were mostly black and white. To jazz up the pages a bit, the authors put different colors on the top of the pages (usually to indicate a new topic). Graphs made up the entire art selection in these texts. Why not make these text more fun and interesting to look at? Maybe if a student picks up the text to look at the pictures then he or she might also look at the words. Why should a math text be boring? I think we should try to encourage students to learn math every way we can.

There are many negative feelings associated with math. In our secondary education course, a guest speaker spoke about some aspects of teaching high school math. She told us that math is a subject that people are proud to say they "suck" at. This attitude was apparent at parent-teacher interviews. Parents would tell their children that they had difficulties with math, too, suggesting to the child that it was okay for them to "suck" at math. People have a preconceived notion that math is hard and therefore they cannot do it. Maybe if we could show them that they use math every day in their lives to solve problems and evaluate situations, then they would have fewer problems understanding concepts.

So how do I see working with students to make connections? One possible way to start a conversation in math is to ask students, "When is the first time you use any aspect of math during your day?" Most will probably respond, "In math class." Ask them the following questions:

- ✓ "How many times do you let your alarm clock ring in the morning before you get up?"
- ✓ "Do you eat breakfast and then brush your teeth or the other way around?"
- ✓ "When you brush your teeth how many times do you brush each tooth?" "Do you count each stroke or do you estimate?"
- ✓ "How do you decide how much milk you will put on your cereal?" "Do you estimate or measure?"
- ✓ "How long do you let your bread toast?" "Do you estimate or have an exact time?"
- ✓ "When you get dressed, how do you determine what you are going to wear?" "Do you figure out each combination and choose one of your many combinations?"

Most students may look at you as if you are weird, but they will probably get your point. This conversation might encourage students to think about ways they use math in their lives.

Math is used in social studies. Determining interest, global economy and trade requires math. There is a long history associated with math; perhaps this can be incorporated into a social studies program. If we could incorporate math into areas that students enjoy, it might help them enjoy math as well. Rhythm in poetry and steps in a dance could not be accomplished without basic math. Math plays an important role in engineering, chemistry, physics and biology.

In science, math is used all the time. Do you use math when driving your car? (The engineers who built it definitely did.) Most things that are built use math in some way or another. Math is also used when looking at the distances between planets. Did you ever wonder how scientists know how far apart the planets were when no one has ever traveled between them? Rockets and other spacecraft use forms of math to take off and land. Without math how would the pilot of an airplane know where he or she was flying, especially if a gust of wind came up? The pilot uses vectors and coordinates to get back on track. In chemistry, it is important to be able to balance reaction equations. If you did not know how to solve mathematical equations, how would it be possible to solve chemical equations? If you could not measure and calculate the proper mixture of chemicals, we would not have batteries and many other chemicals that we use daily in our homes.

In biology, math is involved in many cellular processes. When I was working on my M.Sc., I choose to look at the uptake of nutrients (specifically Ca^{2+}) into cells. The methods of uptake are diffusion, facilitated transport and active transport. My study involved the uptake of Ca^{2+} into *Zea mays* L. sperm cells. Ca^{2+} uptake occurred in a time-dependent manner. The K_m and V_{max} values of 213 μM and 3.759 nmol/μl cell H_20 respectively, which suggests the presence of an active transporter in maize sperm cells. By the use of a few mathematical equations, we discovered the presence of a Ca^{2+} transporter on the maize sperm cell.

Math teachers fight against a lot of misconceptions surrounding math. I believe that if we can help students to make connections and to help them see how math is used every day in life, it will make the subject more interesting.

Triangles

There are 3*n* points (*n* is a natural number) in the plane, of which no three points are collinear. Using the points as vertices, is it possible to draw *n* triangles, which neither touch nor overlap each other?

Mathematically Incorrect?

Marjorie Gann

This article is a slightly revised version of the original published in Quill and Quire *(October 1997), p. 42. Reprinted with permission. Minor changes have been made to spelling and punctuation to fit ATA style.*

Schoolteachers may make good parents, but they're often not the best teachers for their own offspring. As my two daughters frequently complain, Mom has her own ways of doing things that often don't coincide with their teachers' expectations. Mom also has her own views on education, and when these contradict the curriculum or the textbook—watch out!

The conflict between Mom's views and the school's came to a head last year, when I discovered that—despite her As in math—my 12-year-old daughter could not multiply or divide one decimal by another, add or subtract fractions with unlike denominators, or operate with percentages to calculate sale prices. So the summer was spent with a Grade 6 math text, catching my daughter up to where I—though not the New Brunswick Department of Education—believed she should be.

I suspect more parents are about to discover deficits in their children's math skills, thanks to a sea change in math instruction. Dismayed by poor results on international measures, the influential National Council of Teachers of Mathematics (NCTM) in the U.S. has radically rethought mathematics instruction. The result is the NCTM's *Curriculum and Evaluation Standards for School Mathematics* (1989), now adopted by many states and provinces.

Rooted in child-centred educational theory, the NCTM approach favors active learning: using "manipulatives" such as counters or base-10 blocks and working in small groups ("cooperative learning" in today's jargon). It stresses problem-solving (a weak area on international comparisons) over arithmetical operations. Calculators are permitted, teaching students to divide by two-digit divisors is discouraged, and estimating, which enhances the ability to judge whether calculator results make sense, is emphasized. Math problems are realistic, embedding the math classroom in the "real world."

The New New Math

NCTM math is the object of considerable controversy in the United States, where it is variously tagged the New New Math, visual math, fuzzy math or—to highlight parallels with the highly controversial whole language approach—whole math. Critics point to a decline on standardized test scores in Oregon and Idaho following NCTM implementation. A powerful California lobby of highly educated parents, including a molecular biologist, geophysicist and statistician, has set up a Web site called "Mathematically Correct" (http://ourworld.compuserve.com/homepages/mathman) to counter NCTM claims.

In Atlantic Canada, NCTM math is on the ascendant. The NCTM Standards provide the philosophical underpinning for the new Foundation for the Atlantic Canada Mathematics Curriculum document. Two new math textbook series—Prentice Hall–Ginn's *Interactions* and Addison-Wesley's *Quest 2000*—have been developed to align with the NCTM guidelines. My daughter's school used *Interactions* for the first time this year, with the results I noted.

Without an extensive grasp of the research, it is hard for a parent or a teacher to determine whether the NCTM is on the right track. Does repeated practice in rudimentary arithmetic operations enhance a feel for numbers or do children arrive at arithmetical understanding better by playing with geoboards and centicubes? Do students learn to estimate more skillfully by doing accurate pen-and-pencil calculations, or does this skill develop by estimating answers to real-life problems? Have calculators made dividing by two-digit numbers obsolete, or is there inherent value in learning to do difficult division operations? The NCTM's lofty ideals—"To solve a problem is to find a way where no way is known offhand, to find a way out of a difficulty, to find a way around an obstacle" (G. Polya quoted by Krulik and Reys in the NCTM Standards)—certainly have a ring to them, but so did the elevated principles of whole language, with dismal results.

What is clear is that the execution of these ideals in *Interactions* and *Quest 2000* is ineffective. Lavishly printed in large format with brilliant graphics and full-color photographs of cheerful young mathematicians, the texts are poorly organized and frequently confusing. In traditional math texts, chapters are organized according the mathematical principles—measurement, fractions, ratio-rate-percent. To make math relevant, the organizing principle behind some chapters in *Interactions* is thematic. The Investigating Transportation unit, for example, lurches from volume and mass problems to time-rate-distance problems to money problems, time problems and map reading. There are no model problems for students to imitate; each is one-of-a-kind, requiring a great deal of teacher assistance. When I helped my daughter with a problem on population densities, for example, I had to explain rounding, ratio and equivalent fractions—all in one night! When concepts are dropped as quickly as they are introduced, nothing is practised, assessed and mastered.

To ensure understanding, both texts encourage students to invent their own arithmetic algorithms (operations) or to choose the ones they're most comfortable with. The Professional Handbook (Intermediate) to *Quest 2000* encourages students to "chunk" numbers into smaller parts for operations, so 35 × 4 may be computed as 30 × 4 = 120, 5 × 4 = 20, 120 + 20 = 140 or as (30 + 30 + 30 + 30) + (5 + 5 + 5 + 5). Such steps are useful in the initial stages of teaching operations—you'll find them in traditional texts as well—but is it really necessary to understand what you're doing every time you add, subtract, multiply or divide? Weren't algorithms invented to avoid cumbersome calculations? And why confuse children, as *Interactions* does, with five options for multiplying 24 × 25?

Interestingly, consumer pressure has forced both publishers to rethink the value of practice and drill. In its first edition, *Quest 2000* had very few practice exercises; the revised edition has short Practise Your Skills boxes—though the practice is often limited to a mere handful of exercises and a booklet of "Extra Practice and Testing Masters" that provide exercises hardly distinguishable from the practice pages in a traditional math text. *Interactions* now offers blackline masters for practice. Like the textbook, though, these sometimes assumed more knowledge than my daughter had (she didn't automatically know that 4 × 16 = 64, which was required to convert 25 × 64 to 100 × 16 in her head).

Problems With Groups

In compliance with the NCTM emphasis on collaborative problem-solving, *Interactions* tags certain problems for Work in a Group, and the promotional video for *Quest 2000* shows students working in twos and fours on problems or math games. In small doses, cooperative learning can be an effective teaching technique, but it is rarely well used. Activities must be carefully designed to avoid group reliance on the quickest and brightest worker. Monitoring what goes on—especially for a teacher with a class of 30 or more—is exceedingly difficult, and the noise level can be punishing. (Interestingly, the *Quest 2000* video never stays with any group long enough to track individual contributions to a solution.) In some classes, children simply don't work well together; they socialize or quarrel.

When I taught elementary math, I was often stymied by the enormous range in my class. At the bottom were children who didn't know their number facts and could barely read word problems. At the top were kids who got every mechanical operation right and were wasting their time on multiplication and division drills. With their challenging word problems, *Interactions* and *Quest 2000* might have been just what these top students needed—but only to supplement a solid, sequential core math program.

Marjorie Gann is a Nova Scotian elementary teacher and freelance writer, now living in Sackville, New Brunswick.

STUDENT CORNER

Mathematics as communication is an important Curriculum Standard and hence the mathematics curriculum emphasizes the continued development of language and symbolism to communicate mathematical ideas. Communication includes regular opportunities to discuss mathematical ideas and explain strategies and solutions using words, mathematical symbols, diagrams and graphs. While all students need extensive experience to express mathematical ideas orally and in writing, some students may have the desire—or should be encouraged by teachers—to publish their work in journals.

delta-K *invites students to share their work with others beyond their classroom. Such submissions could include, for example, papers on a particular mathematical topic, an elegant solution to a mathematical problem, posing interesting problems, an interesting discovery, a mathematical proof, a mathematical challenge, an alternative solution to a familiar problem, poetry about mathematics or anything that is deemed to be of mathematical interest.*

Teachers are encouraged to review students' work prior to submission. Please attach a dated statement that permission is granted to the Mathematics Council of The Alberta Teachers' Association to publish [insert title] in one of its issues of delta-K. *The student author must sign this statement (or the parents in the case of a student being under 18 years of age), indicate the student's grade level, and provide an address and telephone number.*

The following work, entitled "The Physics of a Classical Guitar," was written by Ryan Cassidy while he was a Grade 12 student at St. Mary's High School in Calgary. Ryan's paper is an excerpt from an extended essay that he wrote as part of his International Baccalaureate program.

Mary Chan wrote "The Ode to π" while in Grade 11 at St. Mary's High School in Calgary. This poem was written on the occasion of the annual π-day (March 14) celebration at St. Mary's High School.

The Physics of a Classical Guitar

Ryan Cassidy

I first picked up a guitar about four years ago and was captivated immediately by its complex sound and delicate tone. Since then, I have endeavored to study the instrument not only musically but also scientifically. The following is a brief synopsis of the results of one such endeavor.

The origins of the instrument are found in Egypt over 3,000 years ago, and though it became popular in all of Europe during the Middle Ages, it received most acclaim in Spain. As it evolved over the years, a number of characteristics became common.

A classical guitar is a fretted instrument composed of a wooden body over which six nylon (originally gut) strings are suspended to produce sound (the lower three strings are wrapped with a steel winding). Though different materials are used for different types of guitars (an acoustic guitar has steel strings, for example), classical strings are consistently constructed of nylon. The strings are tuned to the notes E_2 (82.41 Hz), A_2 (110.0 Hz), D_3 (146.8 Hz), G_3 (196.0 Hz), B_3 (246.9 Hz) and E_4 (329.6 Hz) from lowest to highest. They are suspended by means of tuning pegs found on the machine head.

Partial differentiation can be used to derive a formula for the speed of a transverse wave in a string. For a string with mass density μ, tension T, vertical displacement y, and horizontal displacement x over some time t, the force acting on a segment of string is given by $F = ma = (\mu \Delta x)(\partial^2 y/\partial t^2)$. As a wave does not transfer matter, only energy, the string particles do not move from side to side and are in horizontal equilibrium. The vertical component of the force acting on the string particles at x and $x + \Delta x$ (this force being the tension in the string) is given by the tension T times the slope of the string at these points: $F_{y1} = (T)(\partial y/\partial x\ (x + \Delta x))$; $F_{y2} = -(T)(\partial y/\partial x(x))$; because the string is in tension, the force at x will pull down on the segment (hence the negative sign on F_{y2}), while the force at $x + \Delta x$ will pull upwards. The total force on the segment, then, is given by

$(T)(\partial y/\partial x\ (x + \Delta x) - \partial y/\partial x(x))$, which is equal to $(\mu \Delta x)(\partial^2 y/\partial t^2)$. Dividing the left side of this equation by T, and the right side by Δx yields the following equation: $(\partial y/\partial x(x + \Delta x) - \partial y/\partial x(x))/\Delta x = (\mu T)(\partial^2 y/\partial t^2)$. Taking the limit of the left side of the equation as Δx approaches zero yields $\partial^2 y/\partial x^2$. Now recall the fundamental wave equation, $\partial^2 y/\partial x^2 = (1/v^2)\partial^2 y/\partial t^2$ (in other words, all wave functions y of x and t must satisfy the above partial differential equation, where v is the wave velocity down the string). Thus, the constant (μ/T) must be equal to $(1/v^2)$, and wave velocity is given by $v = \sqrt{\frac{T}{\mu}}$.

Modes of vibration of classical guitar strings are given by resonance formulas for fixed ends, where the length L is equal to $n\lambda/2$, where n is an integer greater than zero. The formula for the frequency of these modes is thus $f_n = nv/2L$, where $v = \sqrt{\frac{T}{\mu}}$. When a string is plucked, the resultant string vibration is a combination of a number of different modes. These modes can be isolated as harmonics, by placing a finger at select points on the neck and just "touching" the string. The first harmonic, for instance, can be enhanced by placing the finger over the 12th fret (half the string distance of 65.0 cm). As the string is now unable to vibrate in its fundamental mode (the finger prevents this from happening), the second harmonic is accentuated. Guitars can also be tuned using this method of harmonics at different frets.

The original intent of the investigation was also to apply Fourier Analysis to guitar waveforms. The Fourier Theorem states that any periodic function can be expressed as a sum of sines and cosines. It should, therefore, be possible to find a mathematical function for the guitar waveform. However, a number of factors complicated this objective. I was initially limited by the equipment available, as the sampling rate of the digital oscilloscope that I had at my disposal was too slow. The Nyquist Theorem states that in order to sample data unambiguously, the sampling rate must be twice the maximum possible frequency. The range of human hearing is about 20–20,000 Hz, so a sampling rate of at least 40,000 Hz is required to collect data, whereas the instruments available could only sample at 8,000 Hz. The waveform is also constantly changing after a string is plucked, as certain frequencies are dampened and amplified at different rates, and at different times. Taking all these limitations into account, it is easy to see why most synthesizers replicate such sounds through digitization and playback.

Undoubtedly, my future study of the guitar will be much more enjoyable as a result of the work that has engaged me during this project. To realize that the physics behind the instrument can be just as fascinating as the sound itself was illuminating.

Irrational Root

Prove that the positive root of the equation $x^2 + x = 10$ is irrational.

The Ode to π

Mary Chan

Oh, symbol in math class and letter of Greek,
Close to our hearts, your value we keep,
Your irrationality helps us to see
The kinship twixt circles and their properties.

Oh, where would we be without π in our lives?
There would be no pizza, no blueberry pies,
No straw to drink juice from, no lamp shade to hide
The glare of a light bulb from our squinting eyes.

There would be no wine glass, no round tops to spin,
No glass bulbs for Christmas, no baseball to win,
And if not for π, much to his chagrin,
King Arthur's Round Table would never had been!

No chocolate chip cookies, no wheels on a car,
No compass to tell us if we're near or far,
No CDs to play, no bright distant stars,
No donuts, no cupcakes, no round candy jars.

No happyface stickers, no tennis to play,
No puddles to jump in on gray rainy days,
No glasses to see with, no chocolate-cream cake,
No reinforcements, no "twonies" to break.

No flashlight batteries to see in the dark,
No retracting rooftop at Sky Dome park,
No 100 percents when we get a mark,
No giant redwoods with ancient old bark.

We'd never go places if there were no wheels,
And would not go fishing without rod and reel,
There would be no halos, no aspirin to heal,
Or mysterious circles, in farmer's fields.

Oh, π, what a glorious letter of Greek,
Your irrationality helps us to see
That without your helpful and glad company,
Much of this world would cease to be.

This year, every issue of delta-K *will devote a section to the NCTM Standards. In this issue, the focus will be on the* Curriculum and Evaluation Standards for School Mathematics *(1989) with particular emphasis on mathematical connections.*

The Curriculum Standards for school mathematics for K–Grade 12 include mathematical connections as one of the important standards to be addressed. At the primary level, students should study mathematics in ways that includes opportunities to make connections with experiences in their daily lives, with other curriculum areas and with different topics in mathematics. Our failure to do that will inevitably result in students continuing to see mathematics as discrete and unrelated strands.

As children enter school, their learning to this point has not been segregated into separate, unconnected school subjects. It is therefore important that teachers build on the wholeness of their perspective of the world through an emphasis on connecting, relating and linking various representations of concepts or procedures to one another within and across subject areas and to the world outside the classroom. This approach will also dispel the incorrect notion that mathematics is a collection of isolated topics and computation only. In addition, the children will come to see that mathematics is of considerable relevance and usefulness and is present in their daily lives outside school.

This emphasis on mathematical connections should naturally continue at the middle and high school levels, so that students solidify their view of mathematics as an integrated whole. At this level, mathematics should include exploration of problems, the application of mathematical thinking and modeling to solve problems that arise in mathematics, but also in other disciplines. Students at the high school level should be exposed to investigations of the connections and the interplay among various mathematical topics and their applications. These experiences will ultimately lead students to being able to apply and translate among different representations of the same problem or concept and to have a deeper appreciation and understanding of mathematics.

The three articles that follow provide excellent examples of mathematical connections.

References

Alberta Education. *The Common Curriculum Framework for K–12 Mathematics*. Edmonton: Author, 1995.

National Council of Teachers of Mathematics. *Curriculum and Evaluation Standards for School Mathematics*. Reston, Va: Author, 1989.

Promoting Mathematics Connections with Concept Mapping

Bobbye Hoffman Bartels

Recent documents calling for reform in mathematical education stress promoting connections during instruction (NCTM 1989; National Research Council 1989, 1990). Mathematical connections are important because they link mathematical concepts to each other and to the real world. When teachers emphasize these connections during instruction, mathematics becomes less compartmentalized and more cohesive and has relevance to real life. In addition, stressing mathematical connections during instruction helps students develop better and deeper understandings of mathematics.

Typically, mathematical connections are implicit in instruction. That is, teachers use instruction that is connected but do not make the connections explicit for students. Implicit connections can result in difficulties with the way students make connections when learning mathematics; students are less likely to make the connections themselves or to make correct connections.

Concept mapping is a useful tool for explicitly stressing mathematical connections. This tool helps teachers and their students perceive and make connections among key ideas in mathematics. In addition, concept mapping is an alternative method of assessment useful for evaluating students' understanding. For an example of using a form of concept mapping in college algebra and trigonometry classes, see Entrekin (1992).

Concept Mapping

A concept map is an instrument for explicitly describing concepts and the relationship among them. The mathematical ideas on a concept map are placed in a hierarchical position, and lines connect the mathematical concepts to form propositions. Thus, a concept map is a finite graph in which the nodes are the mathematical concepts and the edges are the connections between them. The nodes are arranged hierarchically with general concepts placed at higher positions on the map than specific concepts. Thus, movement down the map leads to more specific concepts. The edges are labeled with linking words indicating the relationship between the concepts or key ideas, thus forming propositions. Arrows are placed on the linking lines to indicate the direction of the relationship between concepts. Whenever arrows are not used, the direction is assumed to be top-down.

A student-constructed concept map for whole-number arithmetic is shown in Figure 1. In the figure, the concepts are placed in ellipses—any closed figure is appropriate—with connecting lines between the concepts. The lines are labeled with linking words to specify the relationship between the concepts. *Whole numbers* is the most general mathematical idea and is placed at the highest point in the map hierarchy. Other concepts, such as *operations, addition* and *difference*, are more specific and are placed at lower levels.

Concept-Map Construction

Students can construct maps in several ways. The appropriate method depends on the intended use of the map, the age and ability of the student, and the student's previous concept-mapping experiences. The concept-mapping methods given subsequently range from simple to challenging. In all situations, students should be instructed to identify the most general mathematical ideas first and place them at the top of the map. Next, more specific concepts are added below. The following two methods work well for students' initial experiences with concept mapping.

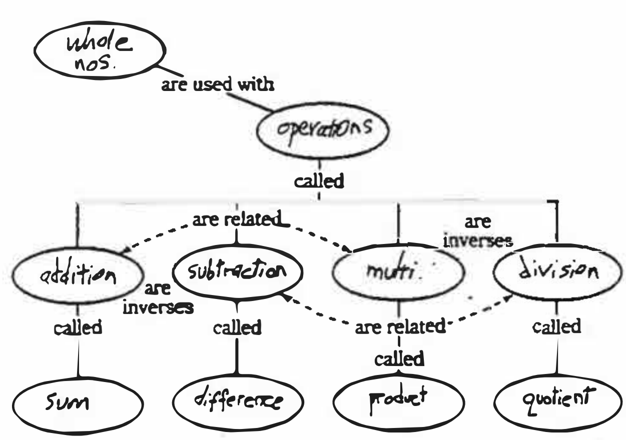
This simplest method is to furnish students with a list of concepts and an incomplete map—one without the concepts but with the connections and linking words (see Figure 2). For this activity, students are given the list of concepts and the incomplete map with the connections drawn between the empty ellipses representing the missing concepts. Students write the concepts from the list in the appropriate ellipses.

The three activities in Figure 2 range from simple to increasingly more complex. The vocabulary on the concept map for whole numbers (Figure 2a) should be familiar to middle school students. In addition, the map contains only four levels of hierarchy. This simplification should make the map easier for students to complete than the next two maps. Figure 1 shows a completed map for the activity in Figure 2a.

The concept map in Figure 2b, which involves solving linear equations, is more complicated. The vocabulary is much more difficult for students because many terms, such as *additive inverse* and *equivalent equations*, will be new. In addition, this map has five levels of hierarchy.

The concept map for measurement in Figure 2c is the most extensive of the three maps, with six levels of hierarchy and many concepts and examples that the students must place on the map to make the correct corrections. Another aspect of this map that makes it more extensive are the connections that are indicated from one major branch to another. These cross-connections require students to have a higher level of understanding than connections made vertically on the same branch of the map.

Figure 1
Student-Constructed Concept Map for Whole-Number Arithmetic



The second method of introducing students to this cognitive tool is to give students a list of concepts but no map—a more difficult approach than the first method. Students construct their maps from the list of concepts, deciding how to represent them hierarchically, determining the connections among them and determining the linking words that show the relationships among the concepts. The following example of such an activity is suitable for middle school students.

Figure 2a

Complete this concept map by using the following concepts: addition, difference, division, multiplication, operations, products, quotient, subtraction, sum and whole numbers.

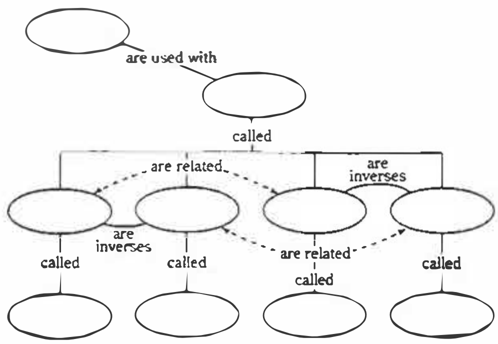


Figure 2b

Complete this concept map by using the following concepts: additive, distributive, equivalent equations, identity, inverses, multiplicative, opposite, properties, reciprocal, solving linear equations in x, and $x = \#$. Examples include 3 and –3, 4 and 1/4, $x = 4$, and $5x + 3x - 4 = 28$.

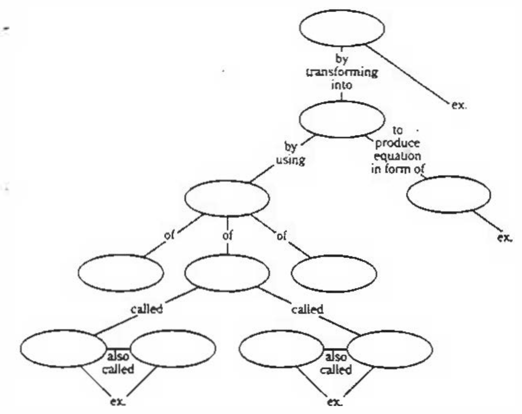


Figure 2c

Complete this concept map by using the following concepts: capacity, English/customary, length, measurement, metric, most of world, nonstandard, stable, standard, systems, temperature, time, unit of measure, unstable, USA and weight. Examples include 3:15 p.m., 4 litres, 0° Celsius, 1,515 hours, 3 paper clips, 2 pints, 3 kilograms, 32° Fahrenheit, 7 metres, 4 pounds and 8 miles.

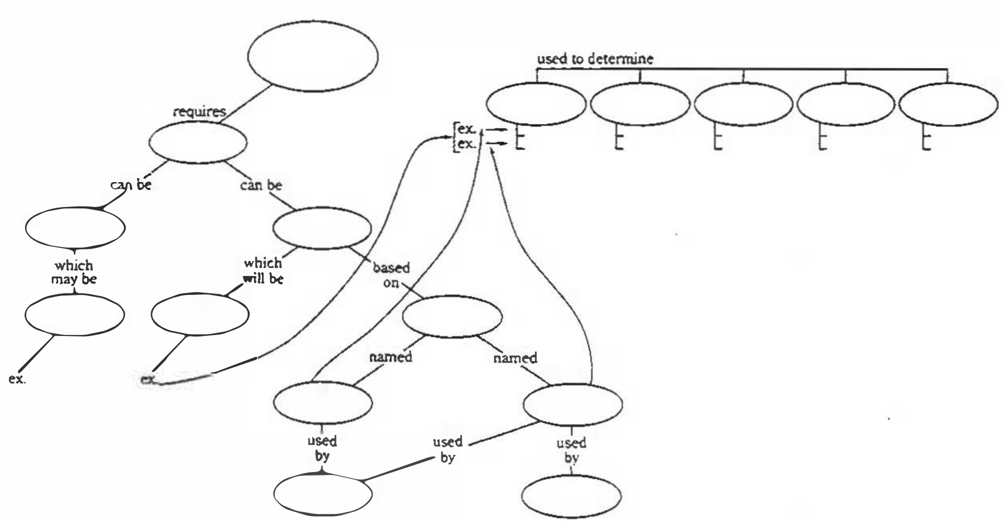
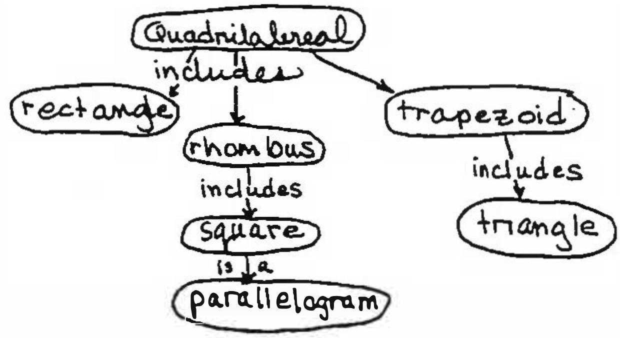


Figure 3

A Student-Constructed Concept Map Relating Kinds of Quadrilaterals



Make a map showing the connections among the following concepts: parallelogram, quadrilateral, rectangle, rhombus, square, trapezoid and triangle.

A variation on this method is to include on the list one or two unrelated concepts to verify the students' understanding of relationships among them. The concept map in Figure 3 was constructed using the foregoing list of concepts. This student's work is subsequently considered in greater depth.

A more challenging method of concept mapping is to have the students themselves determine the key concepts and construct a map using them. This method is more challenging because students supply

Figure 4

Concept Maps for Ratios Constructed by Two Groups of Students

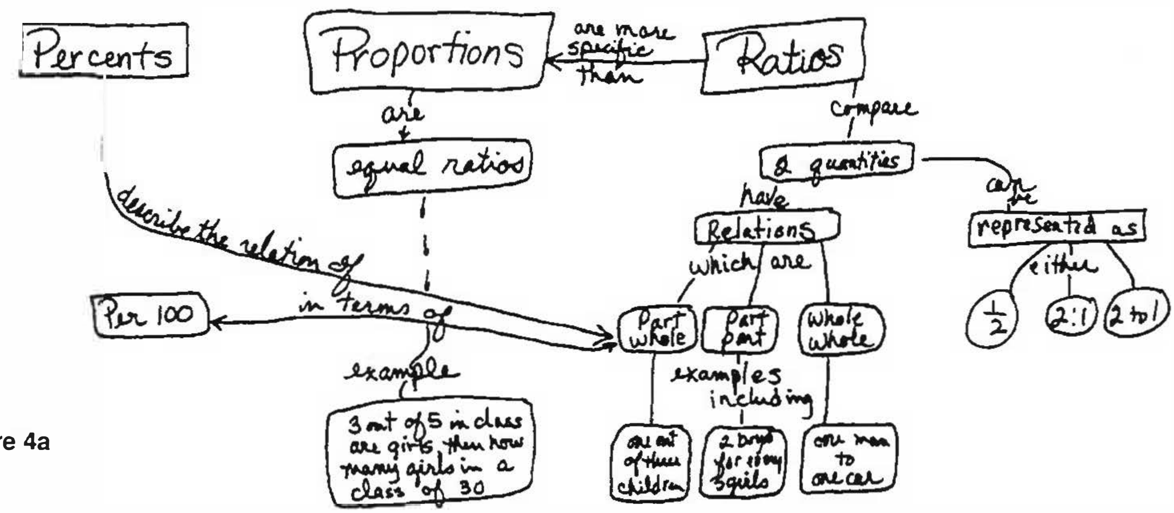


Figure 4a

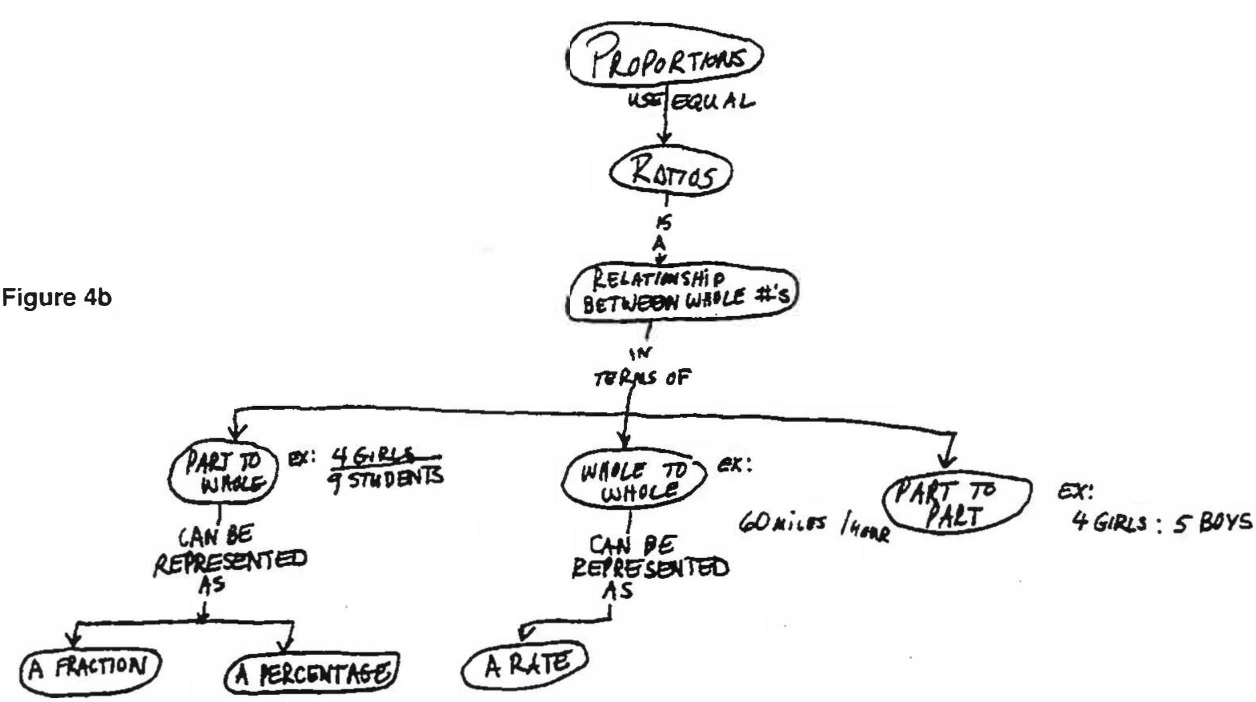


Figure 4b

their own concepts and, therefore, are required to analyze the content of the lesson, topic or unit to determine the key components. The concept maps in Figure 4 were constructed by students using this method.

In addition to mapping the connections among concepts, the concept map is valuable for showing mathematical connections to the real world and to other disciplines. Real-world connections are shown by adding examples to the map, as demonstrated by Figure 4. Examples are given for using proportions and for part-whole, part-part and whole-whole ratios. Examples are usually placed at the bottom of the concept-map hierarchy to separate them from the concepts and, therefore, to keep the organization of the map intact. But as Figure 4b indicates, placing examples next to concepts is acceptable.

When constructing concept maps, always begin with a simple one, such as the map in Figure 2a. From the instructional unit, choose concepts for one or two days, and using any one of the methods, make a concept map. Make simple maps every one or two days, and at the end of the unit make a comprehensive concept map using the simpler maps developed during the unit.

If concept maps for each unit are saved, they can be posted on the wall or bulletin boards. When a concept or unit has connections to one previously taught, the appropriate map can be accessed and the connections discussed. In addition, old maps are useful models. For example, the simple concept map for whole number arithmetic (see Figure 1) is useful for developing a simple map for integer or rational-number arithmetic. The basic difference among the three maps will be the general concept at the top of the map.

Concept Mapping as an Instructional Tool

Concept mapping is useful as an instructional tool during whole-group or small-group instruction or for individual students. During whole-group instruction, concept mapping is initiated and monitored by the teacher, whereas when it is used for an assignment by small groups or individuals, the students work more independently.

Whole-Group Use

During whole-group instruction, the teacher may use concept mapping in three ways. For the first way, the teacher presents a completed map at the beginning of a lesson as an advance organizer or at the

Table 1
Scoring Rubric for Concept Maps

Concepts and Terminology	
3 points	Shows an understanding of the topic's mathematical concepts and principles and uses appropriate mathematical terminology and notations
2 points	Makes some mistakes in mathematical terminology or shows a few misunderstandings of mathematical concepts
1 point	Makes many mistakes in terminology and shows a lack of understanding of many mathematical concepts
0 points	Shows no understanding of the topic's mathematical concepts and principles
Knowledge of the Relationships Among Concepts	
3 points	Identifies all the important concepts and shows an understanding of the relationships among them
2 points	Identifies important concepts but makes some incorrect connections
1 point	Makes many incorrect connections
0 points	Fails to use any appropriate concepts or appropriate connections
Ability to Communicate Through Concept Maps	
3 points	Constructs an appropriate and complete map and includes examples; places concepts in an appropriate hierarchy and places linking words on all connections; produces a concept map that is easy to interpret
2 points	Places almost all concepts in an appropriate hierarchy and assigns linking words to most connections; produces a concept map that is easy to interpret
1 point	Places only a few concepts in an appropriate hierarchy or uses only a few linking words; produces a concept map that is difficult to interpret
0 points	Produces a final product that is not a concept map

end of a lesson and connects them to show the relationships to other ideas from that lesson and from other lessons.

A second way to use concept mapping is somewhat different because it promotes reflection and discussion about the key ideas that are mapped and the connections among them. With this method, the teacher displays a map that may have inaccurate connections among concepts or uses inappropriate concepts. Students orally analyze the map to verify the use of the concepts and the connections among them.

For a third way, the students and teacher collectively construct a map as the lesson progresses. Through discussion, students give input into this map by supplying the key ideas and the connections.

Small-Group or Individual Use

The alternative to whole-group instruction is to have the students themselves construct maps that represent the concepts and connections that were emphasized during instruction. With this method, students construct the maps while working in small collaborative groups or individually.

In addition to promoting mathematical connections, constructing concept maps in small groups also promotes communication as students discuss the key ideas and their relationships. Whether students work as individuals or in groups, students use mathematical reasoning while constructing their maps.

Concept Mapping as an Assessment Tool

Concept maps are also useful for assessing students' understanding. As an assessment tool, the concepts on the map and their connections are evaluated. Any of the foregoing methods for concept mapping can be used for the assessment. The maps are elevated on an informal or formal level.

Informal evaluation is invaluable as an ongoing assessment of the students' learning. Through an informal evaluation, the teacher looks at the maps constructed by the students, noting any misunderstandings about connections among concepts or a lack of connections. For example, when a map in Figure 3 is informally evaluated, the teacher should note that the student connected *triangle* to *trapezoid* through the linking word *includes*. This student appears to think that a triangle is a trapezoid and a quadrilateral. Further evaluation might show the misconception that the *parallelogram* is more specific than the *rhombus* or the *square*. An informal evaluation of the map in Figure 4a indicates a relatively clear understanding of the relationships between ratio concepts. The teacher, however, might make a mental note to investigate further to see if the student has a misconception that ½ is equivalent to "2:1" and "2 to 1." Through an informal evaluation, the teacher generalizes students' understanding of the

Table 2
Using the Scoring Rubric

Rubric	*Points*	*Comments*
Figure 3		
Concepts and terminology	1.0	Triangle is not a trapezoid and should not be on map because it is not a quadrilateral.
Relationships	2.5	Square is a rectangle but is not shown as such.
Communication	1.0	Hierarchy is incorrect.
Total points	4.5	
Figure 4a		
Concepts and terminology	3.0	Mathematical terminology and concepts are correct.
Relationship	2.5	Percent should be connected to per 100.
Communication	1.5	Hierarchy for a few concepts is incorrect. The map is not easy to interpret.
Total points	7.0	
Figure 4b		
Concepts and terminology	2.5	The relationship to whole numbers is incorrect.
Relationships	3.0	Connections are correct.
Communication	3.0	Hierarchy is appropriate, linking words are used and the map is easy to interpret.
Total points	8.5	

connections among concepts. This informal evaluation is useful for remedying students' erroneous conclusions, whether individually or as a group.

A formal evaluation of concept maps might adopt a scoring rubric. The rubric found in Table 1 presents the opportunity to assess a student's knowledge of mathematical concepts and the connections among concepts, the ability to construct a map and the ability to communicate through a concept map. These three areas are given equal weight here, but the rubric can be adjusted to assign more weight for one or two areas or to eliminate an area, depending on the objectives for the assignment. The scores in each category range from 3 to 0 points, but the teacher can use scores between the whole-number ratings. In all situations, the scoring must be based on material that was covered in class. One way to use the rating is to subtract 0.5 points for each mistake; however, the amount subtracted must reflect the importance of the mistake in relation to the entire map and the objectives for the assessment.

Table 2 shows a way to score the maps in Figures 3 and 4 using the scoring rubric from Table 1. Following each rating are comments that describe the reasons for the rating.

When the map in Figure 3 is scored using the rubric, points are subtracted for incorrect mathematical knowledge (for example, a triangle is a trapezoid and a quadrilateral) and for misconceptions about the relationships among concepts (for example, the relationship between the rectangle and the square is not indicated). In addition, the concepts *parallelogram* and *rhombus* are not placed in an appropriate hierarchy. *Parallelogram* is more than *square*, and *rhombus* is more specific than *parallelogram*. The student who constructed this map shows a poor understanding of the relationships among kinds of quadrilaterals, as indicated by his map score of 4.5.

The maps in Figure 4 were constructed by students working in small groups using identical instructions:

> Construct a map that shows the relationships among concepts associated with the unit on rations, proportions and percents.

It is interesting to note—and to be expected—that the maps are not identical. The concept map in Figure 4a received a score of 7.0. The map shows an understanding of the topic's key ideas and principles and correct terminology is used. The connection between *percents* and *per 100* is not drawn, so 0.5 was subtracted from the relationship score. All other connections are appropriate. This group of students lost the most credit in the communication category. The hierarchy on their map is incorrect, since ratios should be higher that proportions and percents are a kind of part-whole ration and should be below ratio. In addition, their map is difficult to follow.

The concept map in Figure 4b is much easier to follow. This group of students lost the most credit in the area of concepts and terminology. Their map indicates that a ratio can be a relationship between whole numbers only and could not be a relationship between a whole number and a decimal and so on. This inaccurate mathematical knowledge resulted in a deduction of 0.5 points in the concepts-and-terminology category. This group received a score of 8.5 on their map.

The Value of Concept Mapping

The real value in constructing a concept map is the visual representation of the mathematical connections that is produced. With this tool, the connections are explicitly depicted and are visible to the person constructing the map as well as to anyone who observes the map. Without the concept map, connections are assumed; however, they may not actually exist or may be incorrect. A student-constructed map is the "hard copy" of the connections made by the student.

Since the connections are visually depicted with the concept map, the person constructing the map has the opportunity to evaluate perceived connections. This evaluation allows the map creator to deal with the effectiveness and meaning of the connections. In addition, the creator can modify connections while constructing the map.

Concept mapping by students leads to rich discussions among students. As the students explore the connections among key ideas, they can clear up misconceptions and develop new meaning for concepts. Additionally, this discussion promotes mathematical reasoning and mathematical communication among students, two goals emphasized in the *Curriculum and Evaluation Standards for School Mathematics* (NCTM 1989). The social context of mapping concepts in groups reinforces mathematical learning in the same way that using cooperative groups reinforces learning.

The evidence of communication and reasoning is concrete. While students are constructing maps in their groups, they will be heard discussing mathematics, not just doing mathematics. They will ask the group to clarify terminology and relationships, such as the different kinds of ratios in Figure 4. The concept-mapping activity affords an opportunity for students to better understand mathematical relationships.

Making connections among concepts also helps students solidify their mathematical understanding.

When constructing a map for geometric figures in two dimensions, students investigate the connections among squares and rectangles. By discussing the placement of these figures on their map, students are helped to see that all squares are rectangles but that not all rectangles are squares. This relationship was missed by the student who constructed the map in Figure 3. If the student had worked with a group, a discussion about the relationship between these concepts might have helped this student have a better understanding.

To be an effective instructional tool, concept mapping should be routinely used. In addition to increasing students' efficiency and proficiency with concept mapping, routine use enhances the connections among units of instruction. For example, the concept map for surface area and volume will have connections to the map for perimeter and area.

The concept map is not meant to replace instructional tools currently found in mathematics classes, but it can augment those tools. For example, Venn diagrams are excellent tools for investigating logic and relationships, but at times, a concept map can better show the relationships. This advantage is especially true when showing relationships among different topics. In addition, the linking words used by students on a concept map furnish information about the understanding of those students that is not evident when Venn diagrams are used. For students who are experienced with concept mapping, the Venn diagram can be changed into a branch on the concept map with little difficulty.

Summary

Concept maps are a valuable instructional tool to help students reflect on, and make connections among, concepts in mathematics. Because the maps are an articulation of the perceived connections among mathematical ideas, students constructing concept maps can visualize how they are connecting concepts and can adjust the connections, if necessary. As a visualization of the connections among concepts, this form of mapping both promotes and assesses understanding in mathematics.

References

Entrekin, V. S. "Sharing Teaching Ideas: Mathematical Mind Mapping." *Mathematics Teacher* 85 (September 1992): 444–45.

National Council of Teachers of Mathematics (NCTM). *Curriculum and Evaluation Standards for School Mathematics.* Reston, Va.: Author, 1989.

National Research Council. *Everybody Counts: A Report to the Nation on the Future of Mathematics Education.* Washington, D.C.: National Academy Press, 1989.

——. *Reshaping School Mathematics: A Philosophy and Framework for Curriculum.* Washington D.C.: National Academy Press, 1990.

Reprinted with permission from the NCTM publication Mathematics Teaching in the Middle School, *Volume 1, no. 7 (November–December 1995) pages 542–49. Minor changes have been made to spelling and punctuation to fit ATA style.*

Flashing Lights

One light flashes every 2 minutes and another light flashes every 7 minutes. If both lights flash at 1 p.m., what is the first time after 3 p.m. that same day that both lights flash together?

Building Mathematical Models of Simple Harmonic and Damped Motion

Thomas Edwards

Given the recent public mania over bungee jumping, stimulating students' interest in a model of that situation should be an easy "leap." Students should investigate the connections among various mathematical representations and their relationships to applications in the real world, asserts the *Curriculum and Evaluation Standards for School Mathematics* (NCTM 1989). Mathematical modeling of real-world problems can make such connections more natural for students, the standards document further indicates. Moreover, explorations of periodic real-world phenomena by all students, as well as the modeling of such phenomena by college-intending students, is called for by Standard 9: Trigonometry.

What follows is an activity that the author has successfully used with Grades 11 and 12 students in a precalculus course in which daily use was made of graphing calculators. In addition to meeting the explicit recommendations previously noted, the activity presents an application of trigonometric functions in a nongeometric setting, giving students an opportunity to apply such functions to a real-world situation.

In *Precalculus: A Graphing Approach*, Demana and Waits (1989, 526–27) present a series of problems aimed at students' development of mathematical models of harmonic motion followed by damped motion. Instead of just using "made up" data to build the model, the decision was made to bring the physical situation into the classroom. The hope was that asking students to attempt to model something that they could actually see would make the problem more vivid for them.

This activity can be completed in one or two class periods. The materials required are a spring, weight sufficient to stretch the spring, some means of suspending the spring and attaching the weight to the spring, a stopwatch and a graphing utility. A screen-door spring with eight to twelve ounces of weight has proved a satisfactory combination. One's physics colleagues might also be a good resource.

In this activity, the goal is for students to produce a mathematical model of the motion that results when

1. the weight is attached to the spring,
2. the spring-weight combination is suspended so as to allow the weight to hang freely,
3. the spring is stretched by pulling down on the weight and
4. the weight is released, beginning an oscillatory motion.

In a sense, mathematical modeling is a process of successive approximation: a number of models are built, each imitating more of the properties of the situation than the one that came before. Throughout the modeling activity, it is important to convey to students the notion that mathematical models are best thought of not as "right" or "wrong" but as better or poorer representations of the problem situation. The interested reader is encouraged to see Davis and Hersh (1981, 70, 77–79) for a further discussion of the nature of mathematical models.

Building the First Model

The first thing that must be one is to establish an equilibrium point for the weight. If the weight is suspended near the chalkboard, for example, the equilibrium point can easily be marked on the chalkboard behind the weight. Next, the spring can be stretched to begin the oscillation, as in Figure 1. As the weight is oscillating, the teacher can begin to pose questions to engage students' thinking about the situation and elicit from them a verbal description of what they are seeing. Students will frequently say things like, "Well, you stretched the spring, let go, and the weight started bouncing up and down." From such a beginning, the teacher might ask, "Do we know any mathematical functions that do just that?"

If students have difficulty associating a sinusoid with this physical situation, the teacher might suggest the possibility of measuring the "bounce." Here

Figure 1
Beginning the Motion

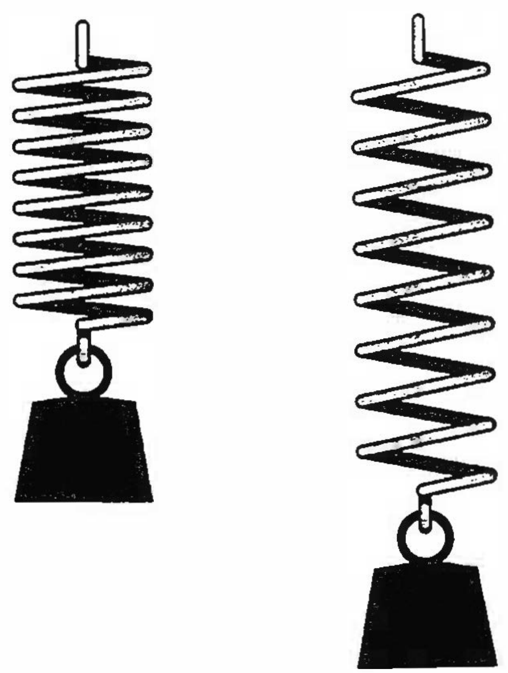
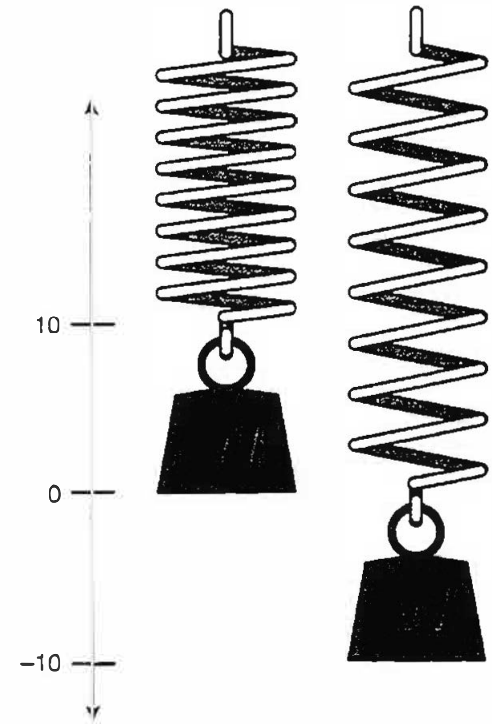


Figure 2
Quantifying the Motion



the question, "Measure from where?" is sure to arise. Restarting the oscillation, the teacher might ask where an appropriate "zero point" would be. For convenience, negotiating the equilibrium point as the zero point is fairly easy, and measuring the "deflection," or amount of stretch, should seem reasonable. One possibility is illustrated in Figure 2. The situation has been quantified, and the function sought can be described numerically. For example, we are looking for a mathematical function that has value −10, then 0 then 10, 0, −10, 0, 10, ... It is hoped that the notion of a sine or cosine function will follow. In the author's experience, it always has!

The teacher will also need to negotiate with students an appropriate sinusoid for this problem. In so doing, a second critical quantity, time, should enter the discussion. In deciding which sinusoid to use in the model, students will need to focus on the known ordered pair at the start of the oscillations: When the time is 0, the displacement of the weight is −10. What should emerge form the discussion is a tentative model: $y = A \cos Bx$.

Once a tentative model has been elicited from the students, the remaining task is to associate the constants A and B with the measurable physical quantities present in the problem. Students have had no trouble connecting the deflection of the weight with the amplitude of the graph of the cosine function and, hence, with A. Thus, if the original deflection of the weight was, say, 10 cm, the tentative model can be adjusted to $y = -10 \cos Bx$.

What has often caused students more difficulty was connecting the constant B with something. This something is, of course, related to the period of the graph of the cosine function, but how is the period of a cosine graph related to the present situation?

Here students are being asked to make a connection between the period of a cosine graph and the period of an oscillation. Having made this connection, students will usually see that the period of oscillation is really a period of *time* and, hence, that the independent variable in this situation is time. However, the really tricky part remains: can the periods of oscillation be measured with some degree of accuracy, and how is that period related to the constant B?

Students usually devise some effective means to measure the period of oscillation. Most often, they have suggested measuring the time required for a certain number of oscillations, say, 5, and then dividing by 5. A more sophisticated group might suggest taking several measurements and averaging them. This pursuit might lead to a discussion of "outliers" and their possible causes, as well as their resolution!

Having a measure of the period of oscillation, students then need to connect that number with the constant B in their model. The teacher might ask them to recall the relationship of B to the period of a cosine graph:

$$\text{period} = \frac{2\pi}{B},$$

from which it follows that

$$B\,(\text{period}) = 2\pi$$

and

$$B = \frac{2\pi}{\text{period}}.$$

For example, if the period of oscillation was 1.6 seconds, we would have

$$B = \frac{2\pi}{1.6},$$

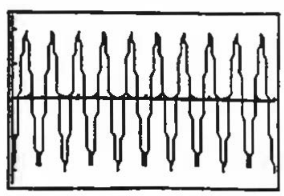
or

$$B = \frac{\pi}{0.8}.$$

Thus, our tentative model can be further adjusted to yield

$$y = -10\cos\left[\left(\frac{\pi}{0.8}\right)t\right].$$

Figure 3
The Graph of $y = -10\cos[(\pi/0.8)t]$ for $0 \le t \le 16$



At this point, students benefit from examining the graph of this function with the aid of a graphing utility. It seems preferable that students do so using their own graphing calculator, but such activities have also been successfully conducted using a graphing utility projected on the overhead projector. Whichever means is used, students must determine whether the graph of this function indeed models the physical situation. Some discussions of an appropriate "viewing rectangle" should precede the graphing activity.

Here a word of caution is in order. When using a graphing utility to graph periodic functions, one must think carefully about the size of the viewing rectangle with respect to the period of the function. In the present situation, for example, a period of oscillation of 1.6 seconds has been assumed. What would happen if an attempt was made to graph this model of $0 \le x \le 150$? Since the weight might continue oscillating for several minutes, it might, in fact, seem quite reasonable to use such a domain for x, as it represents only a 2.5-minute span.

However, the graph that many utilities would produce in such a viewing rectangle is very misleading. See Hansen (1994) for a discussion of graphical misrepresentations that occur when the domain divided by the period of a function is a multiple of the number of pixels in the width of the screen of the graphing utility. I have found that 12–15 cycles of a periodic function are the maximum that can be conveniently displayed with reasonable accuracy using a graphing utility such as the TI-81. To require more than that is to push the technology beyond its limits.

Figure 3 shows a graph of the first model. At this point, students are usually quite pleased with themselves for having produced this model. They are quite unprepared for the next question, which the reader may already have guessed, "How could we improve this model?"

Building a Better Model

Once the existing model has been suggested as problematic, on reflection, students will see that they have modeled a "perpetual motion machine." This notion should begin the search for a better model, one that accounts for the damping of the motion. Once again, students will need to make a connection, this time between the coefficients A and B and the physical reality that the motion is "slowing down."

Asking students the question, "What physical quantity have we been treating as a constant, although it is not really a constant?" helps them to associate A, or the amplitude, with the damping effect. Students can then be encouraged to try out various variable expressions in place of the constant -10 in their model. For example, Demana and Waits (1989) suggest the equivalent of $-10 + t$ in the problem set cited earlier. Figure 4 shows a second model, using $A = -10 + t$.

While furnishing a model of the damping effect, this function has the undesirable property that it seems to show the motion starting up again after stopping! This shortcoming leads to a part of the situation that is difficult to model: an amplitude is sought that will approach zero, then *equal* zero for some value of t and *all larger* values of t.

Figure 4
The Graph of $y = (-10 + t)\cos[(\pi/0.8)t]$ for $0 \le t \le 16$

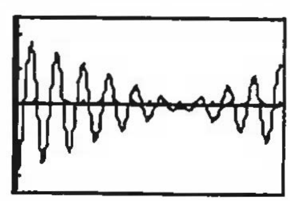
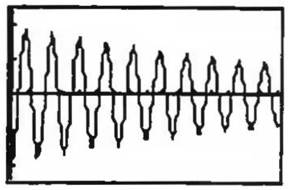
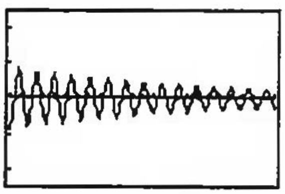


Figure 5

(a) The graph of $y = -10\ e^{-0.05t} \cos[(\pi/0.8)t]$ "going strong" in a (0, 16) by (–12, 12) viewing rectangle.



(b) The graph of $y = -10\ e^{-0.05t} \cos[(\pi/0.8)t]$ "still going" in a (16, 40) by (–12, 12) viewing rectangle



(c) The graph of $y = -10\ e^{-0.05t} \cos[(\pi/0.8)t]$ "coming to rest" in a (40, 80) by (–12, 12) viewing rectangle

Searching for the Best Model

Although such a function might be piecewise defined, the model that physicists have suggested uses a function that is asymptotic with $y = 0$. Students who have had some experience with the graphs of exponential functions should be able to make a connection here. If they are familiar with the graphs of $y = e^x$ and $y = e^{-x}$, then the customary model of damped motion can be constructed. If not, then making a connection with $y = 2^x$ may do.

In any event, the connection that needs to be made is that multiplying a function that is asymptotic with zero by a constant such as –10 produces a function that remains asymptotic with zero. Of course, students must also recognize that a function is sought that asymptotic with the *positive* x-axis. Thus, our model could be adjusted to

$$y = -10e^{kt} \cos\left[\left(\frac{\pi}{0.8}\right)t\right].$$

With the aid of the graphing utility, students can explore the effect of various values of k on the model. Students might be encouraged to find the value of k that best models their situation. This task could be accomplished by measuring the time it takes for the weight to come to rest and searching for the value of k whose graph best depicts that aspect of the situation. Figures 5a, b and c depict such a model graphed in different viewing rectangles to show the "coming to rest" process. Note that in this example, one graph is clearly not sufficient.

Even this exponential model, which is the one usually used in physics, is not a perfect descriptor of the physical situation. After all, we would probably agree that the weight does indeed eventually come to rest, but y does not equal zero for any value of x in the domain of these models. What makes this the best model, in fact, what make any model a better model, is that it mimics more of the physical aspects of the situation than do other models.

Bibliography

Davis, P. J., and R. Hersh. *The Mathematical Experience.* Boston: Houghton Mifflin, 1981

Demana, F., and B. Waits. *Precalculus: A Graphing Approach.* Reading, Mass.: Addison-Wesley, 1989.

Hansen, W. "Using Graphical Misrepresentations to Stimulate Student Interest." *The Mathematics Teacher* 87 (March 1994): 202–5.

National Council of Teachers of Mathematics. *Curriculum and Evaluation Standards for School Mathematics.* Reston, Va.: Author, 1989.

Reprinted with permission from the NCTM publication The Mathematics Teacher *Volume 88, Number 1 (January 1995) pages 18–22). Minor changes have been made to spelling and punctuation to fit ATA style.*

Making Connections Using Embedded Software

Claire Groden and Laurie Pattison-Gordon

The NCTM's *Curriculum and Evaluation Standards for School Mathematics* (1989) calls for increased attention to "connecting mathematics to other subjects and to the world outside the classroom." Often, these connections are made with interdisciplinary projects and through the study of mathematics in a real-world situation. We can also make connections by using software created for practical, real-world applications in the mathematics classroom.

Design Your Own Home

Can students draw a bird's-eye view of their house or apartment building and the land it takes up? Can they draw an accurate map of the interior? We found this exercise to be interesting for students and adults. Students are particularly intrigued by drawing "dream houses" and can easily do so with a program such as Abracadata's Design Your Own Home software. Issues of scale, area, ratio, angle and perimeter are naturally confronted in this task. In designing a dream house, students move from the computer to the drawing board or three-dimensional models. They make decisions about the appropriate scale for the particular medium, confront transformations as they flip and rotate elements in their design, and measure area and perimeter and experiment with different models for a fixed area.

For an interesting problem, have all the students in a class create the blueprint of a 1,500–square-foot house. As different designs emerge, many mathematically rich questions arise:

- How does this space compare with that in your own home?
- Which design will be the most expensive to build? The easiest to heat? Why?

To begin another problem that can branch in many different directions, give the class a map of a town, real or contrived. Divide the students into groups and by lottery assign each group a plot of land. Each group's task is to build dwellings on its assigned plot. Group members may choose to live in one house, in a multidecker house, in separate houses and so on. Rules can be established for the use of space, such as that no building may be within 100 feet of a street. This task can be simplified by giving everyone the same plot on which to build and comparing building decisions. Or it can be expanded by discussing real issues that architects must address—sunlight, building-code restrictions and so on. Once the space is designed, the teams can present their designs.

Having the groups examine their use of space by designing the interior may be very revealing. This design can be done with a drawing program such as Aldus SuperPaint or with Design Your Own Home: Interiors. When asked to create house interiors, many adults do not demonstrate a good sense of space; they often can fit 10 bathtubs into their bathrooms, sometimes none at all!

Bridgebuilder

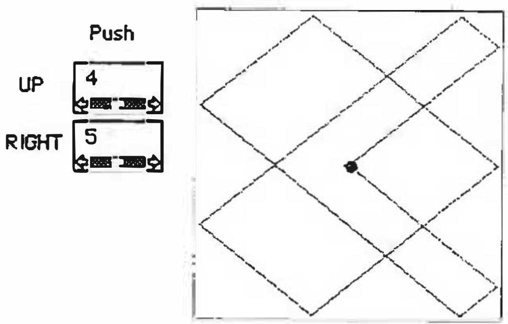
If students are exploring the mathematics of bridges, they can experiment with building truss bridges on Bridgebuilder software, in which they use triangles and symmetry to create a stable bridge. They can tweak and refine their design for efficiency using data about bridge members that are furnished by the software. As students manipulate the data, the idea of isolating variables is likely to emerge as an efficient and important strategy.

Physics Explorer

Mathematics has been called the language of science, and science can be an effective lens through which mathematical relationships can be observed and recorded. The school's science department may be able to supply software environments rich with mathematical data. For example, students can explore relationships between numerical input and bouncing balls as they experiment with a billiard table built on Physics Explorer software. They can learn to narrow and focus their investigation, look for counterparts, find different patterns, and recognize dead ends. In the example shown in Figure 1, students enter a number for upward thrust and a number for right thrust. Clicking "GO" launches the ball on the path determined by these numbers. Given the freedom to explore, some students note the geometric shapes composed, others focus on the numbers and paths, and some focus on both. When asked to predict the outcome before launching a ball, students create many conjectures, such as that shown in Figure 1.

Figure 1

Conjecture: the number of bounces needed to move from the centre and back to the centre is the sum of the UP and RIGHT numbers

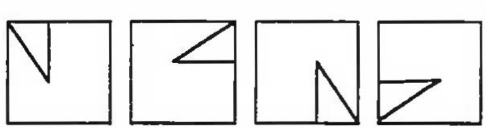


Aldus SuperPaint

Although Aldus SuperPaint is a software program worthy of discussion in detail, only one of the applications found to be of value in mathematics classrooms is presented here. Mathematics textbooks often include activities in which students explore the mathematics of quilt designs. Indeed, many quilters use computers to create their designs. How do they do it? What are the essential elements? Aldus SuperPaint and any number of paint and draw programs give the opportunity to explore color variations (combinatorics), composition (Why do some squares make more patterns than others when put together in a quilt?), geometric shapes (What shapes are formed if these squares are put together?), and continuity (How can I connect the line segments in this square to line segments in other squares?).

Figure 2

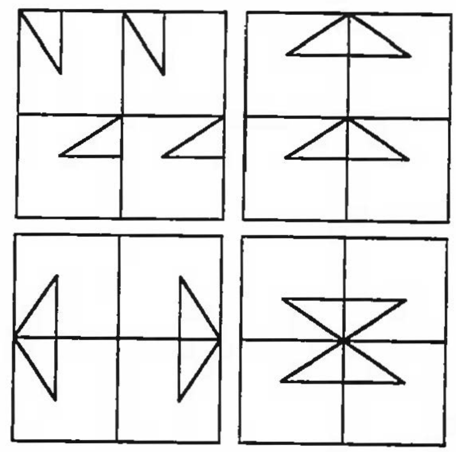
Escher's Two-Line–Segment Drawings



Perhaps students would like to emulate Escher in his musings by creating a two-line–segment pattern (Figure 2) in a square and exploring the squares created by putting four of the two-line–segment squares together (Figure 3) (Schattschneider 1990). Which two-line–segment patterns create the largest number of different squares? If they find a pattern they like, they can scale down their four-square pattern, copy it, and make the whole quilt by putting together as many squares as they like. In the process, they may compose or reveal other geometric shapes.

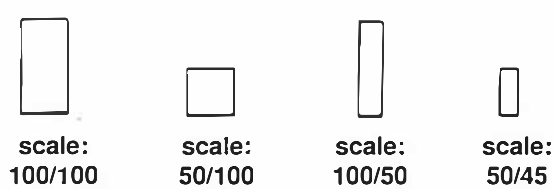
Figure 3

Four-Square Patterns Composed from the Two-Line–Segment Drawings



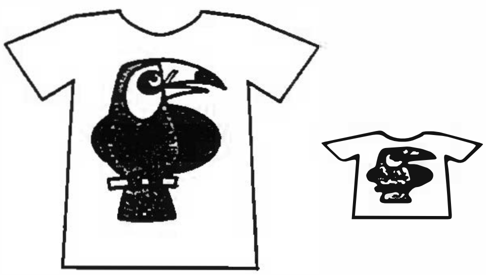
Paint-and-draw programs offer a new medium for exploring existing geometry activities. Ibe (1987), in the article "Mathematics and Art for One Shape," asks readers to rotate and translate a square, adding to the preimage, to create different figures. The Replicate function in Aldus SuperPaint allows students to translate this problem to the screen. Replicate also allows users to rotate and dilate simultaneously to create spirals. Scaling can be done so that each axis is scaled by a different amount. New problems emerge, for example, can students scale a rectangle and turn it into a square? Distorting different figures by scaling the axes unevenly gives a good visual sense of what happens when the scales on a graph are changed (Figure 4).

Figure 4



Rescaling can also be played in reverse by giving a starting and ending state. Can students resize the shirt on the right in Figure 5 to its original size on the left? Are more than one series of rescalings possible?

Figure 5



Sprout

Start with the familiar pen or garden problem. Students are given a length of fence. What it the biggest garden or pen they can make? To extend the investigation, use Sprout software and have students choose and plant vegetables and collect data about the number of people that can be fed vegetables. A case might be made against cultivating the greatest area for the greatest production, depending on how the rows are planted and what crops are grown.

ET-MITT (Empowering Teachers—A Mathematical Inquiry Through Technology) is a National Science Foundation–funded project in which graduate courses for middle school teachers are being developed at Bolt, Beranek and Newman in collaboration with the University of Massachusetts, Lesley College and EDCO Collaborative. The focus of the project is the use of technology to foster an inquiry approach to teaching and learning mathematics, as demonstrated in this article.

Incorporating these software programs into a mathematics classroom makes a quiet statement in and of itself. It states that art, architecture and science belong in the realm of mathematics and that mathematics can be found in the worlds of art, architecture and science—and in the art class and the science class, too.

References

Ibe, M. D. "Mathematics and Art from One Shape." In *Geometry for Grades K–6: Readings from the Arithmetic Teacher,* edited by J. M. Hill, 61–62. Reston, Va.: NCTM, 1987.

NCTM. *Curriculum and Evaluation Standards for School Mathematics.* Reston, Va.: Author, 1989.

Scattschneider, D. *Visions of Symmetry: Notebooks, Periodic Drawings and Related Works of M. C. Escher.* New York: Freeman, 1990.

List of Software

Aldus SuperPaint. Silicon Beach Software, PO Box 261430, Sandiego, CA 92196-1430.

Architecture: Design Your Own Home. Abracadabra, PO Box 2440, Eugene, OR 97402; phone (800) 451-4871.

Bridgebuilder—Preengineering Software. 1266 Kimbro Drive, Baton Rouge, LA 70808; phone (504) 769-3738.

Physics Explorer: One Body. Sunburst Communications, 101 Castleton Street, PO Box 100, Pleasantville, NY 10570-0100; phone (800) 321-7511.

Sprout, Abracadata, PO Box 2440, Eugene, OR 97402; phone (800) 451-4871.

Reprinted with permission from the NCTM publication The Mathematics Teacher *Volume 88, Number 6 (September 1995) pages 500–02). Minor changes have been made to spelling and punctuation to fit ATA style.*

The Use of Technology in the Learning and Teaching of Mathematics, Position Paper

National Council of Teachers of Mathematics

Technology is changing the ways in which mathematics is used and is driving the creation of new fields of mathematical study. Consequently, the content of mathematics programs and the methods by which mathematics is taught and learning assessed are changing. The ability of teachers to use the tools of technology to develop, enhance and expand students' understanding of mathematics is crucial. These tools include computers, appropriate calculators (scientific, graphing, programmable and so on), videodisks, CD-ROM, telecommunications networks by which to access and share real-time data, and other emerging educational technologies. Exploration of the perspectives these tools provide on a wide variety of topics is required by teachers.

It is the position of the National Council of Teachers of Mathematics that the use of the tools of technology is integral to the learning and teaching of mathematics. Continual improvement is needed in mathematics curricula, instructional and assessment methods, access to hardware and software, and teacher education.

- Although the nature of mathematics and societal needs are forces that drive the curriculum, the opportunities that technology presents must be reflected in the content of school mathematics. Curricular revisions allow for the de-emphasis of topics that are no longer important, the addition of topics that have acquired new importance and the retention of topics that remain important. In the implementation of revised curricula, time and emphasis are to be allocated to the topics according to their importance in an age of increased access to technology. Instructional materials that capitalize on the power of technology must be given a high priority in their development and implementation. The thoughtful and creative use of technology can greatly improve both the quality of the curriculum and the quality of students' learning.
- Teachers should plan for students' use of technology in both learning and doing mathematics. A development of ideas is to be made with the transition from concrete experiences to abstract mathematical ideas, focusing on the exploration and discovery of new mathematical concepts and problem-solving processes. Students are to learn how to use technology as a tool for processing information, visualizing and solving problems, exploring and testing conjectures, accessing data and verifying their solutions. Students' ability to recognize when and how to use technology effectively is dependent on their continued study of appropriate mathematics content. In a mathematics setting, technology must be an instructional tool that is integrated into daily teaching practices, including the assessment of what students know and are able to do. In a mathematics class, technology ought not be the object of instruction.
- Every student is to have access to a calculator appropriate to his or her level. Every classroom where mathematics is taught should have at least one computer for demonstrations, data acquisition and other student use at all times. Every school mathematics program should provide additional computers and other types of technology for individual, small-group and whole-class use. The involvement of teachers by school systems to develop a comprehensive plan for the ongoing acquisition, maintenance and upgrading of computers and other emerging technology for use at all grade levels is imperative. As new technology develops, school systems must be ready to adapt to the changes and constantly upgrade the hardware, software and curriculum to ensure that the mathematics program remains relevant and current.
- All professional development programs for teachers of mathematics are to include opportunities for prospective and practicing teachers to learn mathematics in technology-rich environments and to study the use of current and emerging technologies. The preparation of teachers of mathematics requires the ability to design technology-integrated classroom and laboratory lessons that promote interaction among the students, technology and the teacher. The selection, evaluation and use of technology for a variety of activities such as

simulation, the generation and analysis of data, problem-solving, graphical analysis and geometric constructions depends on the teacher. Therefore, the availability of ongoing in-service programs is necessary to help teachers take full advantage of the unique power of technology as a tool for mathematics classrooms.

The National Council of Teachers of Mathematics recommends the appropriate use of technology to enhance mathematics programs at all levels. Keeping pace with the advances in technology is a necessity for the entire mathematics community, particularly teachers who are responsible for designing day-to-day instructional experiences for students.

Reprinted with permission from the NCTM, February 1994. Minor changes have been made to spelling and punctuation to fit ATA style.

Jones Collected One Dollar

"While we're waiting for the others," said Jones to Brown, "let's do something with these dice."

"Okay," said Brown. "What do you suggest?"

"We'll each throw two dice," said Jones, "and multiply together the numbers which turn up. For example, if you throw a 6 and a 3, your product is 18. Then the one with the lower product pays the other $1 per point on the difference. One's maximum gain would thus be $35."

"Suits me," said Brown. "If we get the same product, we both throw again?"

"That's right."

Jones threw first. His throw produced a 4 and a 3.

"Ha," he said. "That's not too bad. There are 19 chances in 36 that you get a lower product, and only 13 chances that you get a higher one . . . Like to pay me a dollar to call the whole thing off?"

"I think I'd better," said Brown. "You're not pulling my leg about the odds?"

"Indeed I'm not," said Jones.

So Brown paid $1. Was he well advised to do so?

A Collection of Connections for Junior High Western Canadian Protocol Mathematics

Sol E. Sigurdson, Thomas E. Kieren, Terri-Lynn McLeod and Brenda Healing

We have put together "A Collection of Connections" comprising 12 uses of junior high school mathematics. These activities support the communication and connection strands of the Western Canadian Protocol mathematics curriculum. In using them, teachers may adapt them extensively. They can serve as a basis of one to three mathematics periods. We have also found that teachers need to plan if they intend to incorporate them into their teaching units. They can be used as end-of-unit activities or as focal activities in the development of a unit. We would encourage teachers to use the activities as a means of bringing mathematical skills to life. These contexts provide an opportunity to enhance a student's view of mathematics. As one Grade 9 student said at the conclusion of one of these activities, "That just proves that mathematics is everywhere."

The following are samples from the number and algebra strand.

Number (Square Roots and Powers)
The Musical Scale
The Musical Scale Student Activities
Algebra
The Clock Maker and the Pendulum
The Clock Maker and the Pendulum Student Activities

The Musical Scale

Intent of the Lesson

The mathematical basis of the musical scale is shown to have two aspects, reflected in the two parts of the lesson. In Activity 1, simple fractions determine the frequency of the notes in the *do, re, mi* scale. In Activity 2, powers and roots are used to compare frequencies and to justify the use of the black keys. The mathematics of the lesson include reciprocals, multiplying fractions and squares.

General Question

The basis of all our music is the 8-note scale:

do, re, mi, fa, sol, la, ti, do

These 8 notes get represented several times at higher and lower frequencies. We are all familiar with the piano keyboard. We notice that, in addition to the 8 white notes, we have five black notes. This is the 12-note scale because the 13th note is really the first note of the next 12 tones. The question we are asking in this lesson is how we got the 8-note scale and how the 12-tone scale works. We are going to use our knowledge of fractions to help us in this understanding.

Another interesting mathematical question about music is, if we can play only 12 different notes, how can we make so much different music? It is a mathematics problem to see how many different arrangements of 12 notes are possible. The number must be very large because we have millions of tunes. In addition to the 12 notes, we have the length of time each note sounds (1 beat, 2 beats, ½ beat, ¼ beat and so on). We also have the length of the interval between the notes. The 12 different notes, the lengths of the notes as they sound and the length of the space between the notes can be arranged in a very large number of ways to make a lot of different music.

However, this is not the question that we are dealing with today. Today we are going to investigate how we came to have 12 different notes. As you might imagine, it, like a lot of mathematics, started with Pythagoras.

Teaching Suggestion

This lesson can be taught as a whole or in two separate parts: Activity 1 and Activity 2.

If a piano keyboard and a guitar are available, they can be used effectively in this lesson. Many classes will have a guitar player and a piano player who can assist in this lesson. A knowledge of music is helpful.

Preliminary Activity

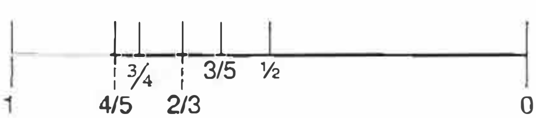
1. The pitch of a string, that is, whether it sounds high or low, depends on the frequency of its vibrations. For example, long strings vibrate slowly and therefore produce low notes. Short strings vibrate fast and produce high notes. (This idea can be illustrated nicely on a guitar.)
2. The frequency is inversely related to the length of a string.
   For example, if L (length of the string) gives a particular F (frequency of vibration) then ½ of L gives 2F (two times the frequency)
   and 2L gives ½ F
   and 1/3L gives 3 times F.
   Other examples might include 1/10 of L will give a note of 10F.
   Rule: The new frequency is found by multiplying the old frequency by the reciprocal of the change in length. If the length is doubled, the frequency is halved. If the length is ¼, the frequency is four times.
3. On particularly difficult concept is that a string which is divided in half produces that same note as the original string but twice as high. (This can be illustrated with all strings on a guitar.)

Although these concepts are not essential for the lesson, the students should have as good a grasp as possible of them. With these understandings of the physics of the vibrating string, our musical scale investigation can begin.

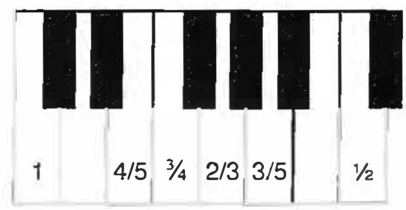
Answering the General Question

Activity 1: The Scale of Eight Notes

Pythagoras took a string and noticed how, when he divided it in half, he got the same note at twice the frequency. He wanted to divide this musical interval *do* (low) to *do* (high) into a series of notes. His first discovery was that when he divided this length of string into simple fraction ratios he got nice sounding notes. So that besides the ½ ratio, the other ratios were 2/3, ¾, 4/5 and 3/5. These are the simplest fractions we have. On a guitar string they look like this:



The teacher could have students measure the length of the guitar string from bridge to nut and find these lengths by calculating the fraction and measuring. The notes that we get from these simple fractions, *mi, fa, sol* and *la*, are shown below on the piano keyboard.



Now the teacher or a helper can find these fractional lengths on the guitar and play them on a keyboard and agree with the class that they do sound nice when compared tot eh original note.

Let us assume that the original note has a frequency of 256Hz, the frequency of middle C. Remembering our rule, we can figure out the frequency of the new notes. If the length is ½ of the original length the frequency will be two times 256 = 512. How about 2/3? We multiply 256 by 3/2. In this way, the frequencies of the other fractions of the length can be found.

Teaching Suggestion

We can figure these frequencies out by doing the fraction in two stages. First, doubling the length means ½ the frequency and taking 1/3 of that means tripling the frequency so the new frequency is 3/2 of 256 = 384.

Using the Table

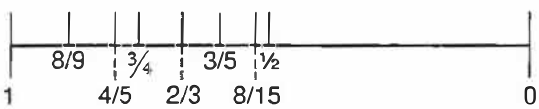
Making a table such as the one below can help keep track of these ideas. (The two blank spaces are for two additional notes which we will discuss later.) The first four notes that we have found are *mi, fa, sol* and *la*. We noticed that the frequency factor goes from 1 to 2. (In the table, the improper fractions are in parentheses, which is a useful form for comparison.)

Note	Length	Frequency Factor	Frequency
do	one unit	1	256.0
mi	4/5	(5/4) 1 ¼	320.0
fa	¾	(4/3) 1 1/3	341.3
sol	2/3	(3/2) 1 ½	384.0
la	2/3	(5/3) 1 2/3	426.7
do	½	2	512.0

These five ratios, *mi, fa, sol, la* and *do*, are simple ratios of the Pythagorean scale. To make the scale sound smoother, notes were added in the gaps between the *do* and the *mi* and between the *la* and the *do*. Two notes were added so that the first new note had a frequency 1/8 more than 1 and at the other end of the scale the new note had a frequency 1/8 less than 2. These two notes complete our eight-note scale.

Note	Length	Frequency Factor	Frequency
do	one unit	1	256.0
re	8/9	(9/8) 1 1/8	288.0
mi	4/5	(5/4) 1 ¼	320.0
fa	¾	(4/3) 1 1/3	341.3
sol	2/3	(3/2) 1 ½	384.0
la	3/5	(5/3) 1 2/3	426.7
ti	8/15	(15/8) 1 7/8	480.0
do	½	2	512.0

Charting the fractional lengths of the string on a number line, we notice that fractions of the length of the strings are not evenly spaced between ½ and 1. However these 8 fractional lengths make up the pleasant sounding musical scale: *do, re, mi, fa, sol, la, ti, do.*



Now we have an eight-note scale. By adding another eight notes and another we can have a series of notes and can play a wide range of *do*, *re*, *mi* and so on, repeated.

Activity 2: The Twelve-Tone Scale

We would like to think that in this scale the change is constant from one note because then we would have an even scale. However, the pattern of fractional lengths above suggests it is not. Let us examine how the frequency changes from note to note. How much higher is the frequency from one note to the next? That is, what do we need to multiply the frequency of *do* by to obtain the frequency of *re, re* to *mi; mi* to *fa* and so on? This number is referred to as the *multiplier*. It is the number we multiply the previous note by to obtain the next note; that is, the ratio of the note to the previous note.

Note	Length	Frequency Factor	Frequency	Multiplier
do	one unit	1	256	------
re	8/9	9/8	288	1.125
mi	4/5	5/4	320	1.111
fa	¾	4/3	341.3	1.066
sol	2/3	3/2	384	1.125
la	3/5	5/3	426.7	1.111
ti	8/15	15/8	480	1.125
do	½	2	512	1.066

As we calculate the multipliers (with a calculator) of these notes we notice that there are two (approximate) numbers. The multiplier is either 1.125 or 1.066. In fact, we note that the square root of 1.125 is approximately 1.066. Alternatively the square of 1.066 is 1.125. So the multiplier is either m or m^2 where m is 1.066.

As we go from *do* to *do*, the multipliers are: *do* × m^2 = *re*, *re* × m^2 = *mi*, *mi* × m = *fa*, *fa* × m^2 = *sol*, *sol* × m^2 = *la*, *la* × m^2 = *ti* and *ti* × m = *do*. The pattern of multipliers is m^2, m^2, m, m^2, m^2, m. This means that we *do not* have an evenly increasing scale. Between *do* and *re* the frequency increases by two jumps of m, but between *mi* and *fa* it only increased by one jump of m. A natural thing to do would be to add a note between *mi* and *fa* that only increased by one jump of m. A natural thing to do would be to add a note between *do* and *re* so that the frequency takes a jump of m to the new note and another jump of m from the new note to *re*. In this way, if we added five notes (one wherever we had an m^2 jump), the result is a 12-note scale, each note spaced a frequency m times higher than the previous.

We get the common keyboard of C, C#, D, D#, E, F, F#, G, G#, A, A#, B, C. (# means sharp.)

The advantage to the 12-note scale is that because all notes are spaced evenly apart we can start a *do-re-mi-fa-sol-la-ti-do* scale on any note. Once we start on any note, we go m^2, m^2, m, m^2, m^2, m^2, m. An m^2 jump means we go up two notes, while an m jump means we go up one note.

When we start with C we can go the usual *do*, *re*, *mi*, *fa*, *sol*, *la*, *ti*, *do* by staying on the white notes. What are we doing is starting with a frequency and following the pattern:

do, m^2, m^2, m, m^2, m^2, m^2 and arriving at *do*.

If we wanted to run the same scale beginning at D, which notes would we have to pick? Which note is m^2 above D? It is E#? Now which note is m^2 above E#? It is F#. So we see we can create a *do*, *re*, *mi*, *fa*, *sol*, *la*, *ti*, *do* pattern. In fact, we can see that we can do it by starting on any note.

What is the exact value of m? In going from *do*, to *do* there are 12 multiplications by m. We know in going from *do* to *do* that the frequency doubles. Therefore $m^{12} = 2$. What number multiplied together 12 times equals 2? It is the 12th root of 2, which is 1.059. By spacing the notes out mathematically even, we get a scale that can be repeated and that can be started at any note. However, the new set of notes does not follow the same frequency as Pythagoras' original nice sounding notes. In his spacing m = 1.066 and m^2 = 1.111 or 1.125, while in the even spacing the m = 1.059 and m^2 = 1.121. These are only approximations. Instruments like guitars and pianos are tuned to the mathematical pattern. These instruments cannot play the natural scale exactly. The violin, where players find their own notes, can play a scale that is perfectly in tune.

Materials

A guitar can be used in Activity 1 to emphasize the fractional relationships of the frequencies and a piano keyboard is useful in Activity 2 where the focus is one the black keys. Calculators are needed.

Modifications

The names of notes like C# and D# can be avoided if necessary. In music, a jump of m^2 in any frequency is called a "full tone," while a jump of m is a "half tone."

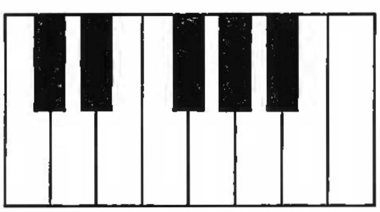
The Musical Scale Student Activities

General Question

The basis of all our music is the 8-note scale:

do, re, mi, fa, sol, la, ti, do.

These 8 notes get repeated several times at higher and lower frequencies. We are all familiar with the piano keyboard. We notice that, in addition to 8 white notes, we have 5 black notes.



This is the 12-note scale because the 13th note is really the first note of the next set of 12 notes. The real question is how we got the 8-note scale and how the 12-note scale works. We are going to use our knowledge of fractions and powers to find out.

Another interesting mathematical question about music is: if we can play only 12 different notes, how can we make so much different music? It is a mathematics problem to see how many different arrangements of 12 notes are possible. The number of arrangements must be very large because we have millions of tunes. In addition to the different arrangements of the 12 notes, we have the length of time each note sounds (1 beat, 2 beats, ½ beat, ¼ beat and so on). We also have the length of the interval. We also have the length of the interval between two notes that are played in sequence. The 12 different notes, the length of time the notes sound and the length of the time between the notes can be arranges in a very large number of ways to make a lot of different music.

However, this is not the question that we are dealing with today. Today we are going to investigate how we came to have 12 different notes. As you might imagine, it, like a lot of mathematics, started with Pythagoras.

Activities

1. a) A string 60 cm long gives a frequency of 256 vibrations per second. What will be the frequency of the vibrations when the string
   i) is shortened to of this length?
   ii) is shortened to ½ of this length?
   iii) is increased to 1½ times its length?

   b) What is the rule for finding the new frequency of a string when the length is changed to some fraction of its original length?
   Use F_n for the new frequency, F_o for the original frequency and f for the fraction.

   c) A string has a frequency of 152 vibrations per second. Use this rule (Question 1b) to find the frequency of the string which is 8/9 of its original length.

2. Use your calculator to fill in the table below for the frequencies of the notes.

Note	Length	Frequency Factor	Frequency
do	one unit	1	256
re	8/9	9/8	
mi	4/5	5/4	
fa	¾	4/3	
sol	2/3	3/2	
la	3/5	5/3	
ti	8/15	15/8	
do	½	2	

3. a) When we divide a string in half, we get a note of two times the frequency. This is the same note as the original at twice the frequency. Draw a diagram of a string and explain why it works this way.

   b) Find the frequencies of the C-note one octave, two octaves, three octaves and four octaves above middle C. The frequency of middle C is 256.

   c) Find the frequency of the C-notes one octave, two octaves, three octaves and four octaves below middle C.

4. a) The piano keyboard has 12 notes and then it repeats. The C note is middle C. In question 3 the frequency of C notes one, two, three and four octaves above and below middle C were calculated. The frequency for other notes can be found in the same way. Look on the chart

and find the frequency of D (next to middle C). What is the frequency of note D, an octave above?

b) We have two patterns of notes:
   i) The note to the right of any note has a frequency how many times higher?
   ii) The note an octave above any note has a frequency how many times higher?

5. Use your calculator to fill in the table below. The *multiplier* refers to the number we multiply the previous note by to obtain the next note. It is the ratio of the note to the previous note. Use your information from question 2 to fill in the column for frequency.

Note	Length	Frequency Factor	Frequency	Multiplier
do	one unit	1		----------
re	8/9	9/8		
mi	4/5	5/4		
fa	¾	4/3		
sol	2/3	3/2		
la	3/5	5/3		
ti	8/15	15/8		
do	½	2/1		

6. a) We have seen in the scale that we have developed that the relationships between the frequencies are not all the same. For example, some m^2 intervals are 1.125 and some are 1.111. Also if $m = 1.066$, then m^2 will be 1.136. Experiment with your calculator and find what m would be in order that all intervals are equal. In other words, what is m if

   $1 \times m \times m \times m$ (12 times) = 2?

   b) Pianos are made so the intervals are all the same. We can call this the mathematically perfect scale. Explain why a scale played on a piano is never exactly in tune.

   c) Violin players make their own notes, so they can always make scales that are musically exactly in tune. Does a violin player need to know the mathematics of music to play correctly? Would knowing it help the player to understand why the piano never sounds completely in tune?

The Clock Maker and the Pendulum

Intent of the Lesson

The time for one full swing of a pendulum is directly related to the square root of the length of the pendulum. The graph of the period of the swing against the length of the pendulum will produce a curved line graph. However, the graph of the square of the period against the length of the pendulum produces a linear graph. The mathematics involved is variables, graphing, linear equations and squares and square roots of numbers.

General Question

The swinging pendulum of a grandfather clock used to be a familiar sight. In designing clocks, clock makers find it convenient to have the pendulum swing in times of one second, two seconds and three seconds. When constructing a pendulum of various string lengths and a weight, it is obvious that not all pendulums swing at the same rate. How long does it take for the pendulum to make one full swing? A full swing is most easily understood as the movement from the extreme left position, across to the extreme right position and back to the extreme left position. As is the case with many crafts, experience has taught clock makers what length to make pendulums. They probably do not know why it works that way. The purpose of this activity is to examine why it works. Understanding the pendulum is largely a mathematical problem.

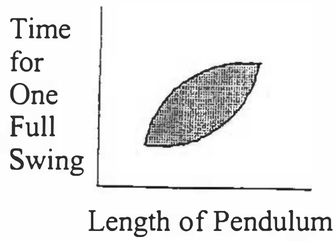
How fast a pendulum swings does not depend on how hard it is pushed. It also does not depend on the heaviness of the weight at the end of the pendulum. How fast a pendulum swings depends only on the length of the pendulum.

Discussion Questions

- Describe what a full swing of a pendulum is. (Pendulum starts at the left, swings all the way to the right and returns to the left.)
- Describe it using other starting points.
- In this lesson, which description is the most appropriate?
- Which takes longer to make one full swing, the short or the long pendulum? (Long)
- How can the approximate length of the pendulum be found that would make the time for one full swing equal to one second?
- What is the general relationship between the length of the pendulum and the time for one full swing? (Using the length as the x-axis and the time for

one full swing as the y-axis, the graph moves from the lower left to the upper right.)

- What kind of relationship would be preferred? (A straight line.)



[The shaded area shows the general region of where the points of the graph will lie. The shorter the pendulum, the shorter the time for one full swing.]

Through a teacher demonstration, have the class count the number of seconds it takes a pendulum to make 10 swings. The time for one swing is easily determined.

The general question, then, is to find the length of the pendulum which takes one second (or two seconds or three seconds) to make one full swing. Also, the *relationship* between the time for one full swing and the length of the pendulum should be examined.

The important concept being developed here is the mathematical relationship between the length of the pendulum and the time for one full swing. If the relationship is a straight line, the equation can easily be worked out between the two variables. Unfortunately, the graph of the relationship between the period and the length of the pendulum is not a straight line. This problem will be dealt with later.

Preliminary Activity

The Time for One Full Swing

To find the length for one full swing, students should find the length of time for 20 or 30 swings and then calculate the length for one full swing. Note that the timer should start when the person counting says zero and stop when the counter says 20 or 30 (whatever is agreed upon). A stop watch will be useful but not essential for this activity. The diagram below illustrates how counting is to be done. The mathematical problem of why to begin with *zero* should be discussed fully. It can be illustrated as follows. Suppose the time for a frog to make 20 jumps is to be counted. The diagram below shows the path of the jumping frog:

In counting 20 jumps, we could say "zero" at point A, "one" at point B and so on until point C is reached and numbered 20. If timing started at "zero" and stopped when 20 was reached, this would be the time for 20 jumps. The same type of thinking can be applied to full swings of a pendulum.

Have counting start when the swing is in a leftmost position. Start the pendulum swinging and start counting, beginning with zero. Again, the teacher should demonstrate this technique.

Discussion Questions

- If 20 swings take 20 seconds, what is the time for one swing? (1 second)
- If 20 swings take 33 seconds, what is the time for one swing? (1.65 seconds)
- How would the number of swings per second be found? (1/time for one swing)
- If the time for one swing was two seconds, what is the number of swings per second? (½ per second)
- If the number of swings per second was three, what is the time for one full swing? (1/3 second)
- What mathematical term describes how these two ways of talking about the swinging pendulum are related? (Inverse)

The time for one full swing is the period, while the number of swings per second is the frequency. The discussion will mainly be limited to the period. The procedure for finding the period is the time 20 or 30 full swings of the pendulum and make the calculation.

A Linear Relationship

Recall how to find a relationship between two variables:

- If I rent a car for $15 per day, what is the relationship between the cost and the number of days it is rented? What are the two variables? The graph is a straight line.
- If I make $1.25 for each box of candy I sell, what is the relationship between the number of boxes I sell and my total earnings. What are the two variables? The graph is a straight line.

In fact, if a graph turns out to be a straight line, it is very easy to find the relationship between the two variables.

Answering the General Question

The goal is to determine the relationship (equation) between the length of the pendulum and the period. Remember, the period is the "time for one swing."

Length of the Pendulum

The length of the pendulum is found by measuring the distance from the centre of the weight to the fixed end of the string: this can be determined to the nearest millimetre. The *length of the pendulum*, not the length of the string, is what needs to be found.

Time for One Full Swing

Once the pendulum is swinging freely (a few swings after it has been dropped), begin the swing at the extreme left and conclude the swing when it returns. How long will it take for 20 or 30 full cycles? One full swing can then be calculated.

Trials

The class can be divided into 10 groups. Each group is given a ruler, a piece of string and a weight and assigned a particular length of pendulum with which to work. Lengths from 0.2–2 m are recommended. At least 20 full swings should be timed. Each group should do three separate counts (of 20 swings) with the same length and find an average "time for one swing." Then 10 trials from 10 groups should be recorded, perhaps on the blackboard, as:

Trial Number Name of group)	length of pendulum	Time for one swing (period)	(Time for one swing)^2
1*	0.2		
2	0.4		
3	0.6		
4	0.8		
5			
6			
7			

* Group 1 is the teacher.

Collect the group data, make a table of values and graph them. Student in the class can draw his or her own graphs from this data which they can compare with other group members.

Teaching Suggestions

The teacher should note that the time for one swing with the pendulum of zero length is zero. In other words the graph goes through the point (0,0). Including the origin is useful in seeing that the graph is a curved line.

In noting that these points fit onto a curved line, it may be obvious that one or more of the points do not fit the general pattern. This can be used as a point of discussion with the class. A graph is a useful way of detecting patterns and noting deviations. In any case, an accurate graph will have to be drawn if a smooth curve is to be detected. The graph of the period squared should be drawn on the same graph as the initial curve.

Students should appreciate the importance of getting data that is on a straight line going through the origin and how this simplifies the mathematical problem of finding a relationship. The extra column of the table may be used for $\frac{\text{(time for one swing)}^2}{\text{length of the pendulum}}$.
This should be approximately equal to the value K.

The Graph

The graph of the period against the length is a curved line. If the time for one full swing is squared, that is, the period is squared and the graph plotted, the points do fall in a straight line passing through the origin. This means that there is a linear relationship between the length and the square of the period. Some mathematical experimenting can be done (with a calculator) to find the multiplier "K" in the equation.

$L = KP^2$

Back to the Clock Maker

Returning to the original question of the clock maker, there are two ways assistance can be offered. The clock maker could be given the formula or he can be given a carefully constructed graph. The graph of the relationship between the length of the pendulum and the period (the curved line) would allow him to predict precisely the length for any desired period.

Provided the number "K" is retained, the formula itself is easily remembered. The formula is convenient to use because the clock maker knows the period he wants and through substitution can determine the corresponding length of the pendulum. For a period of one second the length is simply K. For a period of two seconds the length of pendulum is 4K.

Materials

Ruler, string and weights to which a string can be attached are needed, as well as calculators to find the coefficient in the linear relationship. As noted preciously, a stopwatch or other timing instrument would be useful.

Modifications

There is a temptation to also discuss frequency, even though this is unnecessary. Even the use of the word "period" is not essential.

Another way of writing the relationships is $P = G(L)^{½}$.

Written in this form, G is related to the constant of gravity, a very important number. Therefore, mathematical graphing has led to a scientific truth.

As a warm-up activity, each group of students could be given both a 50 g and a 500 g weight. By making two pendulums of the same length, the students should be able to determine that the weight does not influence the time for one full swing of the pendulum. This also gives the students the opportunity to practice making accurate measurements before they do the actual activity.

Answering the general question can be done in two ways: empirically (doing many trials to find the answer) or mathematically (using the graph of a few trials to find the answer). The focus of the activity could be on either or both of these methods.

The Clock Maker and the Pendulum Student Activities

General Question

The time for one full swing of a pendulum is determined by the length of the pendulum. Can you determine which length will give a time for one full swing of exactly one second? Of exactly two seconds?

Activities

1. a) The time for 30 full swings of a pendulum was 40.7 seconds. What is the time for one swing?
   b) How many swings will it make in one second?
   c) What is the mathematical relationship between the answers to question 1a) and 1b)? Why does this seem reasonable?
2. a) In counting the swings of a pendulum, the rule is to begin the stopwatch at the count of zero and stop it at the count of 30. Why not begin with 1? Explain this.
   b) Do the second thousand years after Christ was born begin in the year 2000 or in the year 2001? How does this relate to question 2a)?
   c) The same mathematics problem occurs in counting our pulse after exercise. Explain.
   d) What is the counting issue that is common to these three problems?
3. a) Mathematical relations between two variables that give a straight line as a graph are the best known relations in science. The most famous of these is the relation between the distance traveled and the time spent traveling of an object moving at a constant speed. The relationship is d(istance) = s(peed) × t(ime). What are some units for speed and time that could be used?
   b) A bullet traveled 400 metres in 2.2 seconds. What is the distance-speed-time equation for the bullet?
   c) An Arctic tern traveled 10,000 kilometres in 40 days. What is the distance-speed-time equation for the Arctic tern? If the "speed" of the tern is not constant, why does our equation still work?
4. a) In the Leaning Tower of Pisa, Galileo noticed that the time for an object to fall was related to the height at which it was dropped. Is the graph of the height against time to fall a straight line?

Time to Fall (Seconds)	Height (Metres)	Square of Height	Square Root of Height
1.8	16		
2.25	25		
2.7	36		
3.15	49		
3.6	64		

   b) Make a graph of the square of the height against the time to fall using the data in the chart provided. Is this a straight line?
   c) Make a graph of the square root of the height against the time to fall. Is this a straight line?
   d) Wherever the graphs in questions 3a, 3b and 3c were a straight line, find the value of K in the equation connecting time of fall (t) to height (h).
5. a) Here are some accurate measurements of a pendulum made by scientists on the moon. Find the square of the time for one full swing and figure out what K would be in this case? Why is K smaller on the moon?

Length	Time for One Full Swing	Square of Time for a Swing
0	0	
0.10	.32	
0.20	.45	
0.30	.55	
0.40	.63	
0.50	.7	
0.60	.75	
0.70	.84	
0.80	.9	
0.90	.95	
1	1	
1.2	1.1	
1.4	1.18	
1.6	1.25	
1.8	1.35	
2	1.4	

b) Select any eight points and graph them accurately. Make a large graph and make sure you start at (0,0). Notice the shape. Write an explanation for what the graph tells you about the relationship between length of the pendulum and time for one swing.

c) Square the time of the period and plot this. This should give you a straight line. What is the equation of this line? It is the form $y = Kx$.

d) You may have noticed that K can be figured out from just one swing. Why, then, is it necessary to make a graph to figure this out?

Note: Those readers interested in the entire volume of "A Collection of Connections" may contact Sol E. Sigurdson, Faculty of Education, Department of Secondary Education, University of Alberta, Edmonton T6G 2G5; phone 492-0753.

The authors acknowledge the financial support for this project provided by the Alberta Advisory Committee for Educational Studies (AACES) and the Central Research Fund of the University of Alberta. Reprinted with permission from the authors. Minor changes have been made to spelling and punctuation to fit ATA style.

How Many Additional Workers?

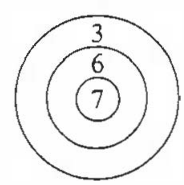
Twenty workers did ¼ of a job in 8 days. Then, because of an emergency, it became necessary to complete the job in the next 5 days. How many additional workers were added to the crew of 20 to accomplish this?

Calendar Math

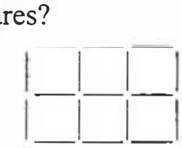
Arthur Jorgensen

Here are math exercises for the month of September 1998.

1. In the Johnson house there are 10 pieces of furniture that are either 4-legged chairs or 3-legged stools. Altogether there are 37 legs. How many chairs and stools are there? This problem can be solved in several ways.
2. If a digital (12-hour) clock is used, how many times in each day
   a. does the display show 3 consecutive numerals in ascending order? (for example, 1, 2, 3)
   b. show 3 consecutive numerals in descending order? (for example, 3, 2, 1)
   c. show 3 numerals which are identical? (for example, 1, 1, 1)
3. If Mary says that 10 + 4 = 2 and 4 6 = 10, what is she referring to?
4. When does ¾ + 5/6 = 8/10?
5. If squares are worth 2 points, circles are worth 3 points, triangles are worth 4 points and rectangles are worth 5 points, draw a figure worth 27 points. (For example, some students can be asked to draw figures and the others are asked to determine their value.)
6. A hardware store sells bulbs for $1.50 which cost $1.00. What fraction of the cost was the gain?
7. If an electrician can install a switch in 3/10 of an hour, how many can he or she install in 6 hours?
8. How many bottles each containing ¾ of a litre can be filled from a jug containing 7½ litres?
9. How many cars will be required to haul 33 passengers to the show if each car can carry only 5 passengers?
10. Tom is hired to work at the service station. He will be paid $1 for the first hour, $2 for the second hour, $4 for the third hour and so on. If he works for 7 hours, how much will he have earned?
11. A pet shop sold a dog and a cat for $84. If the dog was sold for $12 more than the cat, how much was each worth?
12. It takes one minute to cut through a log. At this rate, how long would it take to cut a log into 7 pieces?
13. Mr. Smith can pile the wood in 2 days. If his son does it, it will take 4 days. If they work together, how long will it take them to pile the wood?
14. As one looks at this sequence of numerals, what are possibilities for the next 3 elements of the sequence? 2, 4, 6, ___, ____, ____.
15. If consonants are worth 10 points and vowels are worth 0 points, what is your name worth? This problem can be modified in several ways.
16. Tom buys a hamburger for $2.10 and a soft drink for $0.95. How much change does he get from a $5 bill?
17. In the Zendell family, Lucy is 3 years older than John, but 2 years younger than Susan. The sum of their ages today is 29 years. How old is each of the children?
18. How many ways can you make change for a quarter, if you may use any combination of pennies, nickels and dimes?
19. A farmer wants to plant 5 trees. The distance from one tree to the next tree is 10 metres. How far is it from the first tree to the last tree?
20. What is the highest score below 50 that is impossible to score on the given dart board?



21. Seventeen toothpicks are arranged to make 6 squares. Can you remove 5 of the toothpicks and leave 3 squares?

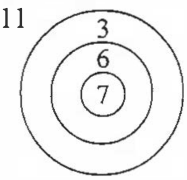


22. Sandy sold 15 comic books for $7.50. On each book she wrote a price of $0.50. Did Sandy correctly price the books?
23. How many 4-digit numerals can you make using the digits 1, 9, 9, 3?
24. Make 2 triangles with 5 matches.
25. Melissa has seven coins which are composed of quarters and dimes. If their total value is $1.30, how many of each coin does she have?

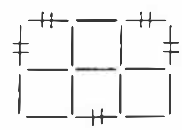
26. A man sold a bike for $90, bought it back for $80 and resold it for $100. What did he make or lose on the total deal?
27. In solving a problem, Tim divided instead of multiplying a number by 8. The answer he got was 6. What was the correct answer?
28. Use the clues to identify the suspect: I am greater than 0.28. I am less than 0.8. My denominator is divisible by 2. My numerator is a prime number. Suspects: 1/8, 9/10, 3/5, 3/8, 7/6, 4/7, 5/6.
29. Fill in the license plate with 3 prime numbers whose sum is 12. D ___ ___ E ___ W.
30. Find a 2-digit number where the product of the digits is 4 times their sum. You may want to try this problem to find numbers where the product is 2, 3, 5, ... times the sum. Look for a pattern.

Answers

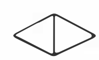
1. 7 chairs, 3 stools
2. a. 8, b. 10, c. 10
3. The clock time
4. When you are adding ratios
5. Each student's answer can be different
6. ½
7. 20 switches
8. 10 bottles
9. 7 cars
10. $127
11. Cat is worth $36, dog is worth $48
12. 6 minutes
13. 1 days
14. a. 8, 10, 12; b. 10, 16, 26
15. Answers will differ
16. $1.95
17. John is 7, Lucy is 10, Susan is 12
18. 12 ways
19. 40 metres
20. 11



21.



22. Yes. 15 × $0.50 = $7.50
23. 10
24.



25. 4 quarters, 3 dimes
26. He needs $30
27. 384
28. 3/8
29. 2, 3, 7
30. 12 and 36 are two possible answers

Revolving Wheels

The front wheel of a bicycle is four times as large as the rear wheel, and the rear wheel makes one complete revolution each time the pedals make 1/3 of a revolution. How many revolutions do the front wheels make when the pedals make 8 revolutions?

Math Projects for Science Fairs

Katherine Heinrich

The Canadian Mathematical Society (CMS) (http://www.camel.math.ca) is concerned with the support and promotion of mathematics in Canada—through the teaching of mathematics, the popularization of mathematics and the creation of new mathematics (mathematics research). For many years, the CMS has sponsored the Canadian Mathematical Olympiad and in 1995 was responsible for the 36th International Mathematics Olympiad held in North York, Ontario (http://camel.math.ca/CMS/Olympiads/).

In 1996, the CMS established the CMS Awards, to be presented annually at the Canada-Wide Science Fair. At the 1997 and 1998 fairs the first prize award is $300 and a calculator. The second prize is one calculator at each of the junior, intermediate and senior levels. The awards criteria are outstanding projects in the mathematical sciences or making extensive use of mathematics in a project.

To date, there have not been many mathematics projects in science fairs; we believe that one reason for this might be that it is not clear what a mathematics project might involve. To help shed some light on this problem, we have prepared a list of possible projects and references on topics that could make exciting and interesting projects. The references are in parentheses following the ideas. But first some warning: the list is quite incomplete (as are some of the references) and not all the ideas have been fully thought out. This is intentional. After all, it is to be *your* project. Some are ideas more interesting than others, some will require more mathematics background than others and some have more scope for exploration than others. But all are related to areas of mathematics that a CMS member has found an exciting and rewarding place to explore and study.

We hope you enjoy them and that you discover wonderful and amazing things (as we did).

Numbers

1. Investigate "big" numbers. What is a big number? The following examples might guide your investigation. A bank is robbed of 1 million loonies. How long would it take to move them? How much would they weigh? How much space would they take up? How big a swimming pool do you need to contain all the blood in the world? Is 10^{100} very big? What is the biggest number anyone has ever written down (check the *Guinness Book of Records* over the last few years)? How did this number come about?
2. How do computer barcodes (the ones you see on everything you buy) work? This is an example of coding theory at work. Find others. Investigate coding theory—there are many books with titles like "an introduction to coding theory" (this is not about secret codes). (Gallian 1991, 1993, 1995)
3. Infinity comes in different "sizes." What does this mean? How can it be explained? (Kamke 1950; Huntington 1955; or refer to any book on Set Theory.)
4. It is easy to check if a number is divisible by 10 by looking to see if its last digit is a 0. How many other "tests of divisibility" can you find? Divisibility by 5 or 7 or 9? Why do they work? (Gardner 1969)
5. Most computers these days can handle sound one way or another. They store the sound as a sequence of numbers. Lots of numbers. 40,000 per second, say. What happens when you play around with those numbers? For example, add 10 to each number. Multiply each number by 10. Divide by 10. Take absolute values. Take one sound, and add it to another sound (that is, add up corresponding pairs of numbers in the sequences). Multiply them. Divide them. Take one sound, and add it to shifted copies of itself. Shuffle the numbers in the sequence. Turn them around backwards. Throw out every third number. Take the sine of the numbers. Square them. For each mathematical operation you can play the resulting sound on the computers speakers, and hear what change has occurred. A little bit of programming, and you can get some very bizarre effects. Then, try to make sense of this from some sort of theory of signal processing. You will first have to discover how sound is stored.
6. Find out all you can about the Fibonacci Numbers, 0,1,1,2,3,5,8, ... In particular, where do they arise in nature? For example, look at the spirals on a pinecone—following the pattern of

the cone, one spiral will go left, the other right. The cone will be covered by "parallelograms," the number of seeds on each side of the parallelogram will (always?) be two neighboring Fibonacci Numbers. For example, 5 and 8. It is similar for pineapples, petals and leaves on plants.

7. What is the Golden Mean? Study its appearance in art, architecture, biology and geometry and its connection with continued fractions, Fibonacci Numbers. What else can you find out?
8. Find out all you can about the Catalan Numbers, 1,1,2,5,14,42, ...
9. Investigate triangular numbers. If that's not enough, do squares, pentagonal numbers, hexagonal numbers and so on. Venture into the third and even fourth dimensions. (Conway and Guy 1996)
10. Build models to illustrate asymptotic results such as Stirling's formula or the prime number theorem.
11. There is a well-known device for illustrating the binomial distribution. Marbles are dropped through the top and encounter a number of pins before dropping into cells where they are distributed according to the binomial distribution. By changing the position of the pins one should be able to get other kinds of distributions (bimodal, skewed, approximately rectangular and so on). Explore.
12. Investigate the history of π and the many ways in which it can be approximated. Calculate new digits of π—see http://www.cecm.sfu.ca/~pborwein/ to discover what this means.
13. Use Monte Carlo methods to find areas or to estimate π. (Rather than using random numbers, throw a bunch of small objects in the required area and count the numbers of objects inside the area as a fraction of the total in the rectangular frame.)
14. Explore Egyptian fractions. In particular consider the conjectures of Erdös and Sierpinski: Every fraction of the form $4/n$ or $5/n$, $n \geq 3$ can be written in the form $1/a + 1/b + 1/c$, where $a < b < c$, and a, b and c are positive integers. See what you can discover. (Stewart 1964; Sondheimer and Rogerson 1981)
15. Look at how different bases are used in our culture and how they have been used in other cultures. Collect examples: time, date and so on. Look at how other cultures have written their number systems. Demonstrate how to add using the Mayan base 20, and maybe compare to trying to add with Roman numerals (is it even possible?) Explore the history and use of the abacus. (Bakst 1965; Ifrah 1985)
16. There are several methods of counting and calculating using your fingers and hands. Some of these methods are still in common usage. Explore the mathematics behind one of them. (Ifrah 1985)

Scheduling

1. At certain times charities call households offering to pick up used items for sale in their stores. They often do a particular geographical area at a time. Their problem, once they know where the pick-ups are, is to decide on the most efficient routes to make the collection. Find out how they do this and investigate improving their procedure. A similar question can be asked about snow plows clearing city streets, or garbage collection. (See Euclidean tours, Chinese postman problem—information can be found in most books on graph theory but one of particular interest is Behzad and Chartrand 1971.)
2. How should one locate ambulance stations, so as to best serve the needs of the community? How do major hospitals schedule the use of operating theatres? Are they doing it the best way possible so that the maximum number of operations are done each day? The reference given above may help.
3. How does the NBA work out the basketball schedule? How would you do such a schedule bearing in mind distances between locations of games, home team advantage and so on? Could you devise a good schedule for one of your local competitions? (Dinitz, Lamken and Wallis 1995)
4. How would a factory schedule the production of bicycles? Which parts are put together first? How many people are required to work at each stage of the production? (See Steen 1978, chapter by R. L. Graham.)
5. Look for new strategies for solving the traveling salesperson problem.

Games

1. What is game theory all about and where is it applied?
2. Study games and winning strategies—maybe explore a game for which the winning strategy is not known. Analyze subtraction games (nim-like games in which the two players alternately take a number of beans from a heap, the numbers being restricted to a given subtraction set). (See Berlekamp, Conway and Guy 1982—this book contains hundreds of other games for which the

complete analysis is unknown; for example, Toads and Frogs. See also, Guy 1991—pay special attention the last section; Garder 1959–61 [vol. 1].)

3. Ten frogs sit on a log—five green frogs on one side and five brown frogs on the other with an empty seat separating them. They decide to switch places. The only moves permitted are to jump over one frog of a different color into an empty space or to jump into an adjacent space. What is the minimum number of moves? What if there were 100 frogs on each side? Coming up with the answers reveals interesting patterns depending on whether you focus on color of frog, type of move or empty space. Proving it works is also interesting—it can lead to recursion. There is also a simple proof that is not immediately obvious when you start. Look for and explore the other questions like this—one of the most famous is the Tower of Hanoi.
4. Try the "Monty Hall" effect. Behind one of three doors there is a prize. You pick door #1. He shows you that the prize wasn't behind door #2 and gives you the choice of switching to door #3 or staying with #1. What should you do? Why should you switch? Make an exhibit and run trials to "show" this is so. Find the mathematical reason for the switch.
5. A graph is a mathematical structure made up of dots (called vertices) and lines joining pairs of dots (called edges). There are many games that can be played on graphs, and much mathematics involved in finding winning strategies. (See the MegaMath Web site at http://www.c3.lanl.gov/mega-math for ideas.)
6. Investigate card tricks and magic tricks based in mathematics. Some of the best in the world were designed by the mathematician/statistician Persi Diaconis. (Albers 1995; Gardner 1959–61)
7. All forms of gambling are based on probability. Investigate how much casinos anticipate winning from you when you play blackjack, roulette and so on. Study a variety of lotteries and compare them. Should one ever buy a lottery ticket? Why does three-of-a-kind beat two pairs in poker? Discover why the different types of hands are ranked as they are. (Gardner 1969; Colbourn 1995)

Geometry

1. Pool problems: if you have a rectangular table without friction and send a pool ball at an angle θ, will it return to the same spot? Investigate using a diagram in Sketchpad (or Cabri). If it does not return to the same spot, will it pass over all points on the table? Does the answer depend on the dimensions of the table? Make a sketch in which you can change the dimensions of the table and the direction of the ball, and explore the path through 10 or 20 bounces. What happens on a circular pool table? Make a dynamic geometry sketch.
2. Flatland and sphereland. If you lived in flatland (the plane) could you build a bicycle which exists in the plane and works? Could you do the same on the sphere? Explore other "machines" in a flat space. (Dewdney 1984; Hinton 1907; see also good descriptions of the problem in Gardner 1959–61, 1990.)
3. There are many aspects of spherical geometry that could be investigated. Explore congruences of triangles on a sphere. Other useful tools that are also available are a plastic sphere, with hemispherical "overhead transparencies," great circle ruler, compass and so on. One can also make very effective models with plastic spheres from a craft shop and cut-off plastic containers for rulers.

   Explore quadrilaterals and their symmetries on a sphere. Is there a family which shares most of the properties of a parallelogram? What symmetry do they have? Which two properties (for example, opposite angles equal) are sufficient to prove all the other properties?
4. What equalities of lengths and angles are sufficient to prove two sets of four points (quadrilaterals or quadrangles . . .) are congruent? (Leads directly to unsolved research problems in computer-aided design. For further references, e-mail whitely@mathstat.yorku.ca.)
5. Build models showing that parallelograms with the same base and height have the same areas. (Is there a three-dimensional analog?) This can lead to a purely visual proof of the Pythagorean theorem, using a physical model based on dissections. The formula for the area of a circle can also be presented in this way, by building an exhibit on the Pythagorean theorem but with the basis that "the area of the semicircle on the hypotenuse is equal to the sum of the areas of the semicircles on the other two sides." (Jacobs 1970)
6. Study the regular solids (Platonic and Archimidean), their properties, geometries and occurrences in nature (for example, virus shapes, fullerene molecules, crystals). Build models. (Gardner 1990, 1959–61 [vol. 2]; Jacobs 1970)
7. Consider tiling the plane using shapes of the same size. What's possible and what isn't? In

particular, it can be shown that any four-sided shape can tile the plane. What about five sides? Make sketches in a geometry program—Sketchpad, Cabri, or using Kali (available free from the Geometry Centre, or Reptiles: demo version available at the Math Forum at Swarthmore)—-these can be found at the Web sites. (See the Martin Gardner books; Grunbaum and Shephard 1987; Steinhaus 1969.)

8. Draw and list any interesting properties of various curves; evolutes, involutes, roulettes, pedal curves, conchoids, cissoids, strophoids, caustics, spirals, ovals and so on. (Coxeter et al. 1938—it has lots of other ideas too; Lockwood 1963)
9. Make a family of polyhedra, for example, the Archimidean solids, or Deltahedra (whose faces are all equilateral triangles), or equilateral zonohedra, or, for the very ambitious, the 59 Isocahedra. (Ball 1962—full of ideas; Coxeter et al. 1938; Wenninger 1971; Schattschneider and Walker 1987; Senechal and Fleck 1988)

   What polyhedral shapes make fair "dice"? What are the physical properties? What are the geometric properties? What is the root of the word "polyhedra" (and why does this fit with the use as dice)? Can you list all possible shapes? What numbers of faces can appear? What other (nonpolyhedral) shapes are actually used in games?

   What polyhedral shapes appear in crystals? List them all. Why do these appear? Why don't other shapes appear? What is the connection between the big outside shape and the inside "connections of molecules"? (Senechal 1990)
10. What is Morley's triangle? Draw a picture of the 18 Morley triangles associated with a given triangle ABC. Find 18 more triangles for each of the triangles BHC, CHA, AHB, where H is the orthocentre of ABC. Discover the relation with the nine-point circle and deltoid (envelope of the Simson or Wallace line).
11. Investigate compass and straight-edge constructions showing what's possible and discussing what's not. For example, given a line segment of length one, can you use the straight edge and compass to "construct" all the radicals? Investigate constructions using origami (paper folding). Can you construct all figures that are constructed with ruler and compass? Can you construct more figures? References can be found in articles in *Math Monthly*, *Math Magazine*.
12. The cycloid curve is the curve traced by a point on the edge of a rolling wheel. Study its tautochrone and brachistochrone properties and its history. Build models. Suppose all cars had square wheels. How would you design the road so that you always had a smooth ride? What about other wheel shapes? (Wagon 1991)
13. Find as many triangles as you can with integer sides and a simple linear relation between the angles. What about the special case when the triangle is right-angled?
14. What is a hexaflexagon? Make as many different ones as you can. What is going on? (Garner 1988, 1959–61 [vol. 1])
15. A kaleidoscope is basically two mirrors at an angle of $\pi/3$ or $\pi/4$ to each other. When an object is placed between the mirrors, it is reflected 6 or 8 times 9 (depending on the angle). Construct one. Investigate its history and the mathematics of symmetry. Make models of kaleidoscopes in a dynamic geometry program (Cabri or Geometers Sketchpad). Demonstrate why only certain angles work. (Ball 1962; Hodgson 1987)
16. You make a tangram puzzle by dividing a two- or three-dimensional object into many geometrical pieces, so that the original object can be reconstructed in more than one way. Burr puzzles are interlocking assemblies of notched sticks. For example, there are Burr puzzles that look like spheres or barrels when they are completed. (See Coffin 1990 for information on how to construct your own.)
17. Build rigid and nonrigid geometric structures. Explore them. Where are rigid structures used? Find unusual applications. This could include an illustration of the fact that the midpoints of the sides of a quadrilateral form a parallelogram (even when the quadrilateral is not planar). Are there similar things in three dimensions? Are there plane frameworks (rigid bars and flexible joints) that are rigid but contain no triangles? Are all triangulated spheres rigid (either made of sticks and joints or of hinged plastic pieces "polydron")? What is the formula for the number of bars in a triangulated sphere, in terms of the number of vertices? How does this formula relate to other rigid frameworks in three-dimensional space?

    Consider a plane "grid" composed of squares (say four squares by four squares) made of bars and joints. Which diagonals of squares will make this rigid? What is the minimum number? Can you give a recipe for deciding which diagonals will work? (There is a COMAP module related to this problem.) If the grid is composed of a trapezoid and its image after a half turn, alternating, does the same recipe work? (This is a research problem which has *not* been thoroughly worked out! E-mail whiteley@mathstat.yorku.ca.)

18. The Art Gallery problem: What is the least number of guards required to watch over all paintings in an art gallery? The guards are positioned at specific locations and collectively must have a direct line of sight to every point on the walls. (Tucker 1994; Wagon 1991)
19. The Parabolic Reflector Microphone is used at sporting events when you want to be able to hear one person in a noisy area. Investigate this, explaining the mathematics behind what is happening.

Combinatorics

1. An international food group consists of 20 couples who meet four times a year for a meal. On each occasion, four couples meet at each of five houses. The members of the group get along well together; nonetheless, there is always a bit of discontent during the year when some couples meet more than once! Is it possible to plan four evenings such that no two couples meet more than once? There are many problems like this. They are called combinatorial designs. Investigate others.
2. What is the fewest number of colors needed to color any map, if the rule is that no two countries with a common border can have the same color. Who discovered this? Why is the proof interesting? What if Mars is also divided into areas so that these areas are owned by different countries on earth? They too are colored by the same rule but the areas there must be colored by the color of the country they belong to. How many colors are now needed? (Hutchinson 1993; Ball 1962; Steen 1978)
3. Discover all 17 "different" kinds of wallpaper. (Think about how patterns on wallpaper repeat.) How is this related to the work of Escher? Discover the history of this problem. (Shephard 1976; Coxeter 1971; Conway and Coxeter 1996)
4. Investigate self-avoiding random walks and where they naturally occur. (Slade 1996)
5. Investigate the creation of secret codes (ciphers). Find out where they are used (today!) and how they are used. Look at their history. Build your own using prime numbers. (Fellows and Koblitz 1993; Ball 1962)
6. It is easy to cover a chessboard with dominoes so that no two dominoes overlap and no square on the chessboard is uncovered. What if one square is removed from the chessboard? (Impossible—why?) What if two adjacent corners are removed? What if two opposite corners are removed? (Possible or impossible?) What if any two squares are removed? What about using shapes other than dominoes (for example, 31×1 squares joined together)? What about chessboards of different dimensions? See the following problem as well. (Golumb 1965)
7. Polyominoes are shapes made by connecting certain numbers of equal-sized squares together. How many different ones can be made from two squares? from three, from four, from five? Investigate the shapes that polynominoes can make. Play the "choose-up" Pentomino game. (Golumb 1965; Gardner 1959–60 [vol.1], 1977)
8. Find pictures which show that $1 + 2 + ... + n = n(n+1)/2$; that $1^2 + 2^2 + ... + n^2 = n(n+1)(2n+1)/6$; and that $1^3 + 2^3 + ... + n^3 = (1 + 2 + ... + n)^2$. How many other ways can you find to prove these identities? Is any one of them "best"? (See Sondheimer and Rogerson 1981, or "Proofs Without Words," a regular feature of *Mathematics Magazine*.)

Others

1. Build a true-scale model of the solar system—but be careful because it cannot be contained within the confines of an exhibit. Illustrate how you would locate it in your town. Maybe even do so!
2. What is/are Napier's bones and what can you do with it/them?
3. Discover how to construct the Koch or "snowflake" curve. Use your computer to draw fractals based on simple equations such as Julia sets and Mandelbrot sets. (Peterson 1988; see Lauverier 1991 for example programs)

   What is fractal dimension? Investigate it by examining examples showing what happens to lines, areas, solids or the Koch curve, when you double the scale.
4. Gardner (1982) defines a paradox to be "any result that is so contrary to common sense and intuition that it invokes an immediate emotion of surprise." There are different types of paradoxes. Find examples of all of them and understand how they differ.
5. Knots. What happens when you put a knot in a strip of paper and flatten it carefully? When is what appears to be a knot really a knot? Look at methods for drawing knots. (Steinhaus 1969;

Farmer and Stanford 1996). Also check out these Web sites: KnotPlot—http://www.cs.ubc.ca/nest/imager/contributions/scharein/KnotPlot.html, Mathmania—http://www.csc.uvic.ca/~mmania/ and MegaMath—http://www.c3.lanl.gov/mega-math/.)

6. Another source of knots is the stonework and ornamentation of the Celts. Investigate Celtic knotwork and discover how these elaborate designs can be studied mathematically. (Cromwell 1993; Meehan 1991)
7. Learn about origamic architecture by making pop-up greeting cards. (Chatani 1986)
8. Is there an algorithm for getting out of two-dimensional mazes? What about three-dimensional? Look at the history of mazes (some are extraordinary). How would you go about finding someone who is lost in a maze (two- or three-dimensional) and wandering randomly? How many people would you need to find them?
9. Explore Penrose tiles and discover why they are of interest. (See Peterson 1988, and most books on tiling the plane.)
10. Investigate the Steiner problem—one application of which is concerned with the location of telephone exchanges to minimize costs.
11. Use PID (proportional-integral-differential) controllers and oscilloscopes to demonstrate the integration and differentiation of different functions.
12. Construct a double pendulum and use it to investigate chaos.
13. Investigate the mathematics of weaving. (Grunbaum and Shephard 1980; Clapham 1980)
14. What are Pick's Theorem and Euler's Theorem? Investigate them individually, or try to discover how they are related. (DeTemple and Robertson 1974)
15. Popsicle Stick Weaving: With long flat sticks, which patterns of "weaving over and under" in the plane are stable (as opposed to flying apart)? Find a pattern with four sticks. Is it unique? Does the stability change when you twist one of the sticks in the plane)? Find several patterns with six sticks whose stability depends on the particular "geometry" of where they cross (that is, the pattern becomes unstable if you twist one of the sticks in the plane). Can you give a rule for recognizing the "good geometric positions"? What kinds of "forces" and "equilibria" are being balanced here? What general rules can you give for "good" weavings? (Source of some information: whiteley@mathstat.yorku.ca.)

References

Albers, D. "Professor of (Magic) Mathematics." *Math Horizons* (February 1995): 11–15.

Bakst, A. *Mathematical Puzzles and Pastimes*. Princeton, N.J.: Van Nostrand, 1965.

Ball, W. W. R. *Mathematical Recreations and Essays*. Rev. by H. S. M. Coxeter. New York: Dover, 1962.

Behzad, M., and G. Chartrand. *Introduction to the Theory of Graphs*. Boston: Allyn & Bacon, 1971.

Berlekamp, E. R., J. H. Conway and R. K. Guy. *Winning Ways, for Your Mathematical Plays*. New York: Academic Press, 1982.

Chatani, M. *Paper Magic: Pop-Up Paper Craft: Origamic Architecture*. Tokyo: Ondorisha Publishers, Ltd., 1986.

Clapham, C. J. C. *Bulletin of the London Mathematical Society* 12 (1980): 161–64.

Coffin, S. T. *The Puzzling World of Polyhedral Dissections*. New York: Oxford University Press, 1990.

Colbourn, C. J. "Winning the Lottery." *The CRC Handbook of Combinatorial Designs*, edited by C. J. Colbourn and J. Dinitz, 578–84. Boca Raton, Fla.: CRC Press, 1995:

Conway, J. H., and H. S. M. Coxeter. "Triangulated Polygons and Frieze Patterns." *Mathematics Gazette* 57 (1973): 87–94 (questions), 175–183 (answers).

Conway, J. H., andR. K. Guy. *The Book of Numbers*. New York: Copernicus, 1996. (See Chapter 2.)

Coxeter, H. M. S. et al. *The Fifty-Nine Icosahedra*. Toronto: University of Toronto Press, 1938.

Coxeter, H. S. M. "Frieze Patterns." *Acta Arithmetica* XVIII (1971): 297–310.

Cromwell, P. R. "Celtic Knotwork: Mathematical Art." *The Mathematical Intelligencer* 15, no. 1 (1993): 37–47.

Cundy, H. M., and A. P. Rollett. *Mathematical Models*. Oxford: Clarendon Press, 1966.

DeTemple, D., and J. M. Robertson. "The Equivalence of Euler's and Pick's Theorems." *Mathematics Teacher* (March 1974).

Dewdney, A. K. *The Planiverse: Computer Contact with a Two-Dimensional World*. New York: Poseidon, 1984.

Dinitz, J., E. Lamken and W. D. Wallis. "Scheduling a Tournament." *The CRC Handbook of Combinatorial Designs*, edited by C. J. Colbourn and J. Dinitz, 578–84. Boca Raton, Fla.: CRC Press, 1995.

Farmer, D. W., and T. B. Stanford. "Knots and Surfaces: A Guide to Discovering Mathematics." *Mathematical World* 6. 1996.

Fellows, M., and N. Koblitz. "Kid Kryupto." Proceedings CRYPTO '92. *Lecture Notes in Computer Science* 740 (1993): 371–89.

Gallian, J. "Assigning Driver's License Numbers." *Mathematics Magazine* 64 (1991): 13–22.

———. "How Computers Can Read and Correct ID Numbers." *Math Horizons* (Winter 1993): 14–15.

———. "Math on Money." *Math Horizons* (November 1995): 10–11.

Gardner, M. *The Scientific American Book of Mathematical Puzzles and Diversions*. 2 vols. New York: Simon & Schuster, 1959–61.

———. The *Unexpected Hanging and Other Mathematical Diversions*. New York: Simon & Schuster, 1969.

——. *Mathematical Magic Show: More Puzzles, Games, Diversions, Illusions & Other Mathematical Sleight-of-Mind from Scientific American*. New York: Knopf, 1977.

——. *Hexaflexagons and Other Mathematical Diversions*. Chicago: University of Chicago Press, 1988.

——. *Aha! Gotcha: Paradoxes to Puzzle and Delight*. San Franciso, Calif.: Freeman, 1982

——. *The New Ambidextrous Universe; Symmetry and Asymmetry from Mirror Reflections to Super Strings*. New York: Freeman, 1990.

Golumb, S. W. *Polyominoes*. New York: Scribner's, 1965.

Grunbaum, B., and G. C. Shephard. *Tilings and Patterns*. New York: Freeman, 1987.

——. "Satins and Twills: An Introduction to the Geometry of Fabrics." *Math Magazine* 53 (1980): 139–61.

Guy, R. K., ed. *Combinatorial Games. Proceedings of Symposia in Applied Math*. Providence, R.I.: American Mathematical Society, 1991.

Hinton, C. H. *An Episode of Flatland*. London: Swan Sonnenschein, 1907.

Hodgson, B. R. "La géométrie du kaléidoscope." *Bulletin de l'association mathématique du Québec* 27, no. 2 (1987): 12–24.

Hutchinson, J. *Mathematics Magazine* 66 (1993): 211–26.

Huntington, E. V. *The Continuum and Other Types of Serial Order*. New York: Dover, 1955.

Ifrah, G. *From One to Zero: A Universal History of Numbers*. New York: Viking, 1985.

Jacobs, H. R. *Mathematics, a Human Endeavor; A Textbook for Those Who Think They Don't Like the Subject*. 3d ed. San Francisco, Calif.: Freeman, 1970: 38.

Kamke, E. *Theory of Sets*. New York: Dover, 1950.

Lauverier, H. *Fractals: Endlessly Repeating Geometrical Figures*. Princeton, N.J.: Princeton University Press, 1991.

Lockwood, E. J. *Book of Curves*. Cambridge: Cambridge University Press, 1963.

Meehan, A. *Celtic Design—A Beginner's Manual*. New York: Thames & Hudson, 1991.

Peterson, I. *The Mathematical Tourist: Snapshots of Modern Mathematics*. New York: Freeman, 1988.

Schattschneider, D., and W. Walker. *M. C. Escher Kaleidocycles*. Corte Madera, Calif.: Pomegranate Artbooks, 1987.

Senechal, M. *Crystalline Symmetries: An Informal Mathematical Introduction*. Philadelphia: Hilger, 1990.

Senechal, M., and G. Fleck, eds. *Shaping Space: A Polyhedral Approach*. Boston: Birkhauser, 1988.

Shephard, G. C. "Additive Frieze Patterns and Multiplication Tables." *Mathematical Gazette* 60 (1976): 179–84.

Slade, G. "Random Walks." *American Scientist* (March–April 1996).

Sondheimer, E., and A. Rogerson. *Numbers and Infinity: A Historical Account of Mathematical Concepts*. Cambridge: Cambridge University Press, 1981.

Steen, L. A., ed. *Mathematics Today: Twelve Informal Essays*. New York: Springer-Verlag, 1978. (See chapters by K. Appel, W. Haken and R. L. Graham.)

Steinhaus, H. *Mathematical Snapshots*. 3d ed. New York: Oxford University Press, 1969.

Stewart, B. M. *Theory of Numbers*. New York: Macmillan, 1964.

Tucker, A. "The Art Gallery Problems." *Math Horizons* (Spring 1994): 24–26.

Wagon, S. *Mathematica in Action*. New York: Freeman, 1991.

Wenninger, M. J. *Polyhedron Models for the Classroom*. Cambridge: Cambridge University Press, 1971.

Further Reading

Asher, M. *Ethnomathematics: A Multicultural View of Mathematical Ideas*. Pacific Grove, Calif.: Brooks/Cole, 1991.

Casey, N., and M. Fellows. *This Is Mega-Mathematics: Stories and Activities for Mathematical Thinking, Problem-Solving and Communication*. Los Almos, N.M.: Los Alamos National Laboratory, 1993.

Farmer, D. W. *Groups and Symmetry: A Guide to Discovering Mathematics*. Providence, R.I.: American Mathematical Society, 1996.

Martin Gardner—all his books!

Hoffman, P. *Archimedes Revenge: The Joys and Perils of Mathematics*. New York: Norton, Ballantine Books, 1988.

Klarner, D. ed., *The Mathematical Gardner*. Boston: Prindle, Weber & Schmidt, 1981.

Loeb, A. L. *Concepts and Images, Visual Mathematics*. Boston: Birkhauser, 1993.

Mason, J., with L. Burton and K. Stacey. *Thinking Mathematically*. Reading, Mass.: Addison-Wesley, 1985.

Paulos, J. A. *Beyond Numeracy: Ruminations of a Numbers Man*. New York: Knopf, 1991.

Sloyer, C. *Fantastiks of Mathematiks: Applications of Secondary Mathematics*. Providence, R.I.: Janson, 1986.

Steen, L. A., ed. *On the Shoulders of Giants: New Approaches to Numeracy*. Washington, D.C.: National Academy Press, 1990. (See chapter by T. F. Banchoff.)

Stewart, I. *Another Fine Math You've Got Me Into*. New York: Freeman, 1992.

Stewart, I. *Game Set and Math: Enigmas and Conundrums*. Oxford: Basil Blackwell, 1991.

Web Sites

KnotPlot:	"The KnotPlot Site," Robert Scharein, University of British Columbia, http://www.cs.ubc.ca/nest/imager/contributions/scharein/KnotPlot.html
MacTutor:	"The MacTutor History of Mathematics Archive," School of Mathematical and Computational Sciences, University of St. Andrews, St. Andrews, Scotland, http://www-groups.dcs.st-and.ac.uk/~history/
Mathmania:	"Mathmania," The Erdös for Kids Problem Sponsoring Program, http://www.csc.uvic.ca/~mmania/
MegaMath:	"This is Mega Mathematics!" Los Alamos National Laboratory, http://www.c3.lanl.gov/mega-math/
π	"Dr. Peter Borwein's Home Page," http://www.cecm.sfu.ca/~pborwein/

Many more Web sites can be found starting from the CMS Web site at http://www.camel.math.ca/. Finally, consider a subscription to a math magazine. To subscribe to *Math Horizons*, write to *Math Horizons*, MAA Service Center, PO Box 91112, Washington DC 20090-1112, USA.

To subscribe to *Crux Mathematicorum*, write to *Crux Mathematicorum*, Canadian Mathematical Society, 577 King Edward, Suite 109, PO Box Station A, Ottawa K1N 6N5; or subscribe online by visiting the Web site http://camel.math.ca/CMS/CRUX/ and view a sample issue.

Spelling Test

Each week Jill's class takes a 30-item spelling test. If she scored 20 on the first test, what is the lowest score she can make on the second test in order for the average of her first three tests to be 26?

Watermelon Weight

A giant watermelon weighed 100 lb. and was 99 percent water. While standing in the sun, some of the water evaporated, so it was only 98 percent water. How much did the watermelon then weigh?

Probability: A Study of Chance

Shirley LeMoine

Lesson Plan for Grades 6–8

Overview

The theory of probability is an important branch of mathematics with many practical applications in the physical, medical, biological and social sciences. An understanding of this theory is essential to the understanding of weather reports, medical findings, political doings and the provincial/state lotteries. Students have many misconceptions about probability situations.

Purpose

The purpose of this activity is to begin the process of helping students to learn the basic principles of probability.

Objectives

As a result of this activity the students will conduct an experiment to determine if a game is "fair," collect data (table), conduct an analysis of the game (tree diagram), state and apply the rule (definition) for probability.

Resources/Materials

Overhead grid, overhead, pencils, paper

Activities and Procedures

1. Introduce the activity with a demonstration of the game "Rock, Scissors, Paper."
2. Divide the class into pairs (player A and player B) and have them play the game 18 times.
3. Use an overhead graph grid to graph the wins of player A in red (how many A players won one game, two games and so on). Do the same for all B players in a different color.
4. Help students determine the range, mode and mean for each set of data. Compare the results.
5. Do a tree diagram to determine the possible outcomes.
6. Answer the following questions to determine if the game is fair.
   a. How many outcomes does the game have? (9)
   b. Label each possible outcome on a tree diagram as to wins for A, B or tie.
   c. Count the wins for A. (3)
   d. Find the probability that A will win in any round (3/9 = 1/3). Explain that probability means favorable outcomes/possible outcomes.
   e. Count the probability that B will win in any round. (3/9)
   f. Is the game fair? Do both players have an equal probability of winning in any round? (Yes)
7. Compare the mathematical model with what happened when the students played the game.

Tying It All Together

1. Use this as an introduction to a unit on probability.
2. Follow-up with discussion about how probability is used in different places in the world.
3. Play game again with three students, using the following rules:
   a. A wins if all three hands are the same.
   b. B wins if all three hands are different.
   c. C wins if two hands are the same.

There will be 27 outcomes this time, that is, $3^3 = 27$.

Candy Math

Cheryl Matern

As a high school teacher and parent, I am always trying to design interesting mathematics activities that I can do with my children at home. When my children were working on fractions in school and having some difficulty with the concepts of fractions, I did the following activities with them. The activities will only be successful if you become the "question-asker." In other words, it is important to let your children be the thinkers and answer-givers. In the activities described below, suggested questions are identified by "Q" while desired answers are in parentheses. You will need several pieces of licorice, scissors to cut the licorice, and paper and pencil to write fraction names and fraction symbols.

Activity 1: Naming Fractions

Cut a piece of licorice in half and then cut the halves in half.

Q: What do we call these pieces? (Fourths.)

Pick up one of the pieces.

Q: So what is this piece called? (One fourth.)

Pick up two pieces.

Q: And these are . . . ? (Two fourths.)

Q: Yes, it is two fourths. But is there another name for this? (One half.)

Continue this questioning with three fourths and four fourths.

Q: How many halves in four fourths? (Two.)

Q: How many wholes in four fourths? (One.)

Take a second piece of licorice and one fourth from the old piece.

Q: How much do we have now? (Five fourths or one whole and one fourth.)

Do the same with six fourths, which can also be expressed as one whole and two fourths or one whole and one half. Continue this activity as far as you wish. It is important to stress the different ways we can name fractions as well as focusing on fractions greater than one whole. As you may know, many young children do not think that five fourths is a fraction. This is probably due to the fact that elementary curricula tend to focus on fractions less than one in their introduction to fractions. If your children are at an age when they are working on fraction symbols, write down all of the fractions on paper as you discuss them. You may also write equations for the equivalent fractions.

Activity 2: Adding and Subtracting

Put two fourths on the table.

Q: If I add one more fourth, how much will we have? (Three fourths.)

Q: Is that more or less than one whole piece? (Less.)

Q: Is it more or less than one half? (More.)

Show them a whole and a half for them to compare it to. Now let's eat one fourth.

Q: How much do we have now? (Two fourths.)

Q: What if we eat another fourth? How much will we have left? (One fourth.)

Activity 3: Adding Unlike Denominators

It is very difficult for young children to understand why we need common denominators to add and subtract fractions. I have found that the following informal activity is effective if repeated often with many different fractions.

Take two pieces of licorice. Cut one in half and one in fourths. Take one half and one fourth.

Q: How much do you have? (Three fourths.)

If you need to cut the half into two fourths, do so. Write the addition equation on paper. It is not necessary at this point to explain the common denominator procedure, but rather simply to write the equation. Now take two pieces of licorice and cut one in half and one in thirds. Take one half and one third.

Q: How much do we have now?

They probably will not be able to tell you the answer here, because it is like trying to combine an apple and an orange. What else can you say except that you have an apple and an orange? Well, children will respond the same way by saying, "We have one half and one third." Then you can do your magical demonstration by making the pieces the same size. Cut the halves into thirds (creating six pieces in all) and cut the thirds into halves (creating six pieces in all). Now the one half is three sixths and the one third is two sixths, so one half and one third is five sixths.

Activity 4: Multiplying with Fractions

Take a piece of licorice and cut it in half. Hold up one piece.

Q: What do you have? (One half.)

Write the fraction symbols on paper as children respond. Then cut one of the halves in half.

Q: What do you have now? (One fourth.)

So half of a half is one fourth. Cut one of the fourths in half.

Q: What do you have now? (One eighth.)

To help children know what fraction they have, ask them to pretend you've cut all four fourths in half. Continue cutting in half for as long as you wish, through sixteenths, thirty-seconds, sixty-fourths and so on. (By then the pieces will be pretty tiny!)

Q: Let's look at all of the fractions we just wrote. Do you notice any pattern in the denominators of all of these fractions? (Each denominator is double the one before it.)

Q: So if we cut one sixty-fourth in half, what would that piece be called? (One one-hundred, twenty-eighth.)

Q: Are the fractions getting larger or smaller in value? (The denominators are getting bigger, but the fraction pieces are getting smaller.)

Q: What do you think would be the smallest fraction piece we could get if we could keep cutting pieces in half? (There is none. It would keep getting smaller and smaller, infinitely.)

I have done these activities with my children many times. On those cold, dark winter evenings, my children frequently beg for licorice math after dinner. Even if their teachers are doing hands-on activities with fractions in the classroom, your children will certainly benefit from the one-on-one interaction they receive from you at home. These are just a few ideas, and I hope you will be able to come up with ideas and extensions of your own to help make fractions more meaningful to your children.

Reprinted with permission from Washington Mathematics *(Fall 1996), a publication of the Washington State Mathematics Council. Minor changes have been made to spelling and punctuation to fit ATA style.*

How Many Pages?

If you start reading a book at the top of page *x* and, without skipping any pages, you read to the bottom of page *y*, how many pages will you have read?

A Chocolate Candy Color Distribution: An Enumerative Statistical Experiment

Bonnie H. Litwiller and David R. Duncan

In teaching statistical processes, it is important that there be applications to real-world settings and activities. When this is done, students are more likely to see the meaning of the steps being developed.

One such activity involves using the Chi-Square statistical test and its applications to counting M&M's of different colors. All students are aware that M&M chocolate candies come in different colors. For instance, a package of the plain M&M candy (nonholiday) contains a mixture of six colors: brown, blue, green, orange, red and yellow.

According to the information provided by Mars Incorporated, the manufacturer of M&M's, the following should be the color distribution for the plain chocolate M&M's:

Brown 30%
Yellow 20%
Red 20%
Orange 10%
Green 10%
Blue 10%

We shall test this distribution hypothesis, called the null hypothesis, with a randomly selected set of plain M&M's.

Experiment 1

We combined the contents of seven 1.69 oz packages. Results of counting the different colors in our sample are as follows:

Color	Number
Brown	104
Yellow	73
Red	92
Orange	38
Green	32
Blue	43
TOTAL	382

The expected distribution:

Color	Expected Number
Brown	30% of 382 = 114.6
Yellow	20% of 382 = 76.4
Red	20% of 382 = 76.4
Orange	10% of 382 = 38.2
Green	10% of 382 = 38.2
Blue	10% of 382 = 38.2

To test the null hypothesis, we shall use the Chi-Square statistic. Let us construct Table 1 with column entries as follows:

O = The observed frequencies, the numbers of each color of M&M's actually present in our package.

E = The expected frequencies (if the null hypothesis were true).

$(O - E)^2/E$ = A measure of the discrepancy between O and E.

Table I

Color	O	E	$(O - E)^2/E$
Brown	104	114.6	0.98
Yellow	73	76.4	0.15
Red	92	76.4	3.19
Orange	38	38.2	0.00
Green	32	38.2	1.01
Blue	43	38.2	0.60
TOTAL	382	382	5.93

In the last column (a measure of discrepancy), a small number indicates that O and E are relatively close together, as is the case for yellow. A larger number indicates that O and E are relatively far apart, as is the case for red.

The sum of this discrepancy column, 5.93, is called the computed Chi-Square Statistic (CCSS). A determination must be made as to whether the CCSS is large enough to cause us to reject the null hypothesis. To make this decision a "referee" is needed. This referee is found in the Table Chi-Square Statistic (TCSS).

To read a Chi-Square table, the degrees of freedom must first be determined; that is, the number of categories (colors) −1. In our case, the degrees of freedom is 6 − 1 = 5. This means that if the total number of M&M's were known, and the number in each of five categories were known, the number in the sixth category could be calculated.

The significance level is the probability of rejecting a null hypothesis which is in fact true. This could occur because the sample is not representative of the population. From a Chi-Square table, we find:

Significance Level	TCSS
10%	9.236
5%	11.070
1%	15.085

The decision mechanism for the null hypothesis is:

- If CCSS > TCSS, then CCSS is large in the "judgment of the referee." If this is true, *reject* the null hypothesis.
- If CCSS < TCSS, then CCSS is small in the "judgment of the referee." If this is true, *accept* the null hypothesis.

For Experiment 1, our CCSS of 5.93 is less than any of the TCSS values; for each level of significance, we do *not* reject the null hypothesis. In other words, we retain the assumption that the packaging process includes the percent of M&M's of each color as claimed by the manufacturer.

There are many different types of M&M's besides the plain chocolate, nonholiday variety used in Experiment 1. Distribution information from Mars Incorporated predicts the following percents for other types of M&M's.

Nonholiday

Peanut Butter or Almond	Peanut
Brown 20%	Brown 20%
Yellow 20%	Yellow 20%
Red 20%	Red 20%
Green 20%	Orange 20%
Blue 20%	Green 10%
	Blue 10%

Holiday

Easter (Plain Chocolate, Peanut or Almond)

Yellow	25%
Blue	25%
Green	25%
Pink	25%

Experiment 2: Peanut Butter (Nonholiday)

We used seven 1.63 oz packages for our sample set.

Color	Predicted %	O	E	$(O-E)^2/E$
Brown	20	48	37.2	3.14
Yellow	20	30	37.2	1.39
Red	20	58	37.2	11.63
Green	20	32	37.2	0.73
Blue	20	18	37.2	9.91
TOTAL		186	186	26.80

For 4 degrees of freedom (5 − 1), a Chi-Square table yielded the following values:

Significant Level	TCCS
10%	7.779
5%	9.488
1%	13.277

Since 26.80 is greater than any of the above TCCS statistics, the null hypothesis is rejected for all significance levels. In other words, we *reject* the assumption that the packaging process places equal numbers of M&M's of each color in our set.

Experiment 3: Almonds (Nonholiday)

We used seven 1.31 oz packages for our sample set.

Color	Predicted %	O	E	$(O-E)^2/E$
Brown	20	31	20.4	5.51
Yellow	20	21	20.4	0.02
Red	20	16	20.4	0.95
Green	20	19	20.4	0.10
Blue	20	15	20.4	1.43
TOTAL		102	102	8.01

Using the same degrees of freedom and Chi-Square table as in Experiment 2 we

- reject the null hypothesis at the 10 percent significance level, since 8.01 > 7.779, but

- do not reject the null hypothesis at the 5 percent and 1 percent significance level, since 8.01 < 9.488 and 8.01 < 13.277.

There is enough evidence to cause doubts that the distribution percents are correctly described, but not enough evidence to conclusively prove it. In a legal setting, this is similar to having enough evidence to indict but not convict.

Experiment 4: Peanut (Nonholiday)

We used seven 1.74 oz packages for our sample set.

Color	Predicted %	O	E	$(O-E)^2/E$
Brown	20	25	31.4	1.30
Yellow	20	41	31.4	2.94
Red	20	18	31.4	5.72
Orange	20	20	31.4	4.14
Green	10	26	15.7	6.76
Blue	10	27	15.7	8.13
TOTAL		157	157	28.99

For 5 degrees of freedom, the Chi-Square table entries are the same as Experiment 1. We reject all significance levels since 28.99 is greater than any of the TCSS. The predicted percentages are not confirmed in our sample set.

For Experiments 5, 6 and 7, the Easter pastel colors are used. For each Easter type, the predicted color distributions are 25 percent for each of the colors yellow, blue, green and pink.

For Experiments 5–7, the TCSS values are:

Significance Level	TCSS
10%	6.251
5%	7.815
1%	11.344

Experiment 5: Easter (Plain Chocolate)

We used one 16 oz package.

Color	Predicted %	O	E	$(O-E)^2/E$
Yellow	25	122	128.5	0.33
Blue	25	117	128.5	1.03
Green	25	142	128.5	1.42
Pink	25	133	128.5	0.16
TOTAL		514	514	2.94

We do not reject the null hypothesis at any significance level.

Experiment 6: Easter (Peanuts)

We used one 16 oz package.

Color	Predicted %	O	E	$(O-E)^2/E$
Yellow	25	31	49	6.61
Blue	25	50	49	0.02
Green	25	37	49	2.94
Pink	25	78	49	17.16
TOTAL		196	196	26.73

A resounding rejection of the null hypothesis at all the significance levels is in order!

Experiment 7: Easter (Almonds)

We used one 12 oz package.

Color	Predicted %	O	E	$(O-E)^2/E$
Yellow	25	29	27.5	0.08
Blue	25	23	27.5	0.74
Green	25	16	27.5	4.81
Pink	25	42	27.5	7.65
TOTAL		110	110	13.28

Rejection of the null hypothesis at all levels is again in order.

A variety of conclusions resulted in the different experiments. Sometimes the results were consistent with the distribution predictions, leading to nonrejection of the null hypothesis; sometimes the results were inconsistent with the distribution predictions, leading to rejection of the null hypothesis. On other occasions the results were mixed—"inconsistent enough" to yield null hypothesis rejections at some significance levels but not at others.

Challenges for Readers and Their Students

1. Redo experiments with larger numbers of M&M's. How do your results compare with ours?
2. Investigate Christmas (red and green) and Valentine (red, pink and white) M&M distributions.
3. Find other candies for which predicted color distributions are known and replicate our process.
4. Find other real-world enumerative data for which the Chi-Square Method can be used.

Alternative Instruction and Alternative, Performance-Based Assessment: An Annotated Bibliography

Jane Ehrenfeld

The following annotated bibliography was compiled by Jane Ehrenfeld under the direction of Professor K. Ann Renninger at Swarthmore College in Swarthmore, Pennsylvania. It is hosted by the Math Forum, a virtual centre for mathematics on the Internet, as part of the Forum's series on Learning and Mathematics—Research in Math Education: summaries and discussions of seminal articles on learning and mathematics. These summaries and similar bibliographies can be found on the Web at http://forum.swarthmore.edu/learning.math.html. Reprinted with permission. Minor changes have been made to spelling and punctuation to fit ATA style.

In a recent discussion in the Learning and Mathematics Discussion Series on the Math Forum, I noticed a flurry of interest in alternative instruction. There is a fair amount of information on the Forum concerning alternative instruction, but an annotated list of resources was lacking, so I chose to gather a bibliography on alternative instruction and performance-based assessment.

Portfolio assessment, a significant issue in alternative instruction, is not included; however, I felt that this bibliography would be more accessible to teachers if it presented ways in which assessment could be altered without drastic restructuring of class and curriculum. Some of the articles and books referenced herein have links to portfolio-based instruction and assessment, and can therefore provide resources for teachers interested in pursuing portfolio systems in their classrooms. All of the articles are math-related, and many give concrete lesson plans and suggestions for practice.

The list is in two parts—one for alternative instruction and one for performance-based assessment. Critical sources are the National Council of Teachers of Mathematics' *Curriculum and Evaluation Standards for School Mathematics* (in section 1, number 9), which forms the basis for much of the work in alternative instruction and assessment being currently undertaken, and Ruth Mitchell's *Testing for Learning* (in section 2, number 6), which provides an excellent argument for changing assessment practices in American education.

Note: All of the references cited in this bibliography except Mitchell (in section 2, number 6) and Perrone (in section 2, number 9) are available from ERIC Document Reproduction Service, 7420 Fullerton Road Suite 110, Springfield, VA 22153-2852, 1-800-443-3742.

Reference

Mitchell, R. *Testing for Learning: How New Approaches to Evaluation Can Improve American Schools*. Toronto: Maxwell Macmillan Canada, 1992.

Alternative Instruction

1. Aronson, G., et al. *Using a Calculator. Math in Action*. Workbook. Hayward, Calif.: Janus Book, 1985.

A workbook for slower-paced learners to help them use calculators so that they can focus on conceptual aspects of mathematics rather than on computational skills. Teachers might consider extending the use of calculators in their classrooms, especially when working with these slower-paced learners, to focus on a deeper understanding of the material to supplement math-facts practice.

2. **Cooper, R. *Alternative Math Techniques Instructional Guide.* Harrisburg, Penn.: Pennsylvania State Department of Education. Bureau of Adult Basic and Literacy Education, 1994.**

A guide for children with learning problems that emphasizes alternative methods. Includes a discussion of the principles behind teaching students with special needs, specific lessons and a guideline to the problems that the children might encounter with each lesson.

3. **Dossey, J. A. "Transforming Mathematics Education." *Educational Leadership* 47, no. 3 (November 1989): 22–24.**

Overview of the *National Council of Teachers of Mathematics Standards*, with references. The Standards (which are also referenced below), promote conceptually based instruction that builds a true understanding of mathematics in the students. Suggestions for practice are included. For example, it is recommended that writing about math learning (that is, math journals) be encouraged as essential for linking language and mathematics.

4. **Hiebert, J., and D. Wearne. "Links Between Teaching and Learning Place Value With Understanding in First Grade." *Journal for Research in Mathematics Education* 23, no. 2 (1992): 98–122.**

Compares and contrasts alternative instruction and text-based instruction with regard to a series of lessons on place value. Conclusions are drawn as to the benefits and drawbacks of alternative, conceptually based instruction. A summary of this article can be found in the archives of the Learning and Mathematics Discussions on the Math Forum.

5. **Hiebert, J., and D. Wearne. "Instructional Tasks, Classroom Discourse, and Students' Learning in Second-Grade Arithmetic." *American Educational Research Journal* 30, no. 2 (Summer 1993): 393–425.**

Compares traditional and alternative Grade 2 math classrooms and provides results of the comparison that show positive gains with alternative instruction. The lessons observed focused on place value, and addition and subtraction with multiple-digit numbers, and suggested teaching these concepts so that the underlying mechanisms will be apparent to the children.

6. **Leonard, B. "Get Moving in Math!" *Instructor* 94, no. 1 (August 1984) 74–76, 78.**

Describes Math Lab, an alternative, conceptually based mathematics curriculum. Includes lessons for each conceptual level of instruction: concrete, semi-concrete, semi-abstract and abstract.

7. **Madsen-Nason, A., and P. E. Lanier. *Pamela Kaye's General Math Class: From a Computational to a Conceptual Orientation.* Research Series No. 172. East Lansing, Mich.: Institute for Research on Teaching, College of Education, Michigan State University, 1986.**

A three-year case study of a teacher's curriculum change and an analysis of the outcomes, including many positive results for the students. The implication is that focusing on fostering student understanding, rather than on skills alone, will produce students with more mathematical competence, higher levels of effort and improved attitudes regarding math. Some of the areas of focus are communication in the classroom, social issues that facilitate math learning and changes in curriculum to improve conceptual instruction.

8. **Marshall, G. "A Changing World Requires Changes in Math Instruction." *Executive Educator* 12, no. 7 (July 1990): 23–24.**

Outlines the Standards of the National Council of Teachers of Mathematics with recommendations regarding alternative instruction and assessment. As opposed to the other Standards-based article listed above, this article translates the Standards into practical applications, above and beyond the suggestions already included within the Standards. Some of the changes involve using software to increase challenges in math, cooperative learning formats and testing that go beyond the multiple-choice model.

9. **National Council of Teachers of Mathematics. *Curriculum and Evaluation Standards for School Mathematics.* Reston, Va.: Author, 1989.**

The Council's guide to reforming and improving math education. The emphasis is on conceptual, contextualized instruction and includes strong recommendations for moving away from traditional, noncomprehension-based drill-and-practice math instruction. An example of this change in focus might be teaching multiplication not as facts but as a system, in which the children construct the multiplication table on their own and in the process learn why it is that this table exists. The Standards contain guidelines, discussions about learning and practice and curricular recommendations for each grade level.

10. **Peterson, P. L., et al. *Profiles of Practice: Elementary School Teachers' Views of Their Mathematics Teaching.* East Lansing, Mich.: Institute for Research on Teaching, College of Education, Michigan State University, 1991.**

Examines a sample of schools from three states and evaluates elementary math teachers' goals and activities. Provides groupings for different teaching

styles (that is, teachers who use manipulatives frequently, teachers who rely only partially on the drill-and-practice technique and so on), with numbers of teachers per group and an analysis of the results.

11. **Remillard, J. *Is There an Alternative? An Analysis of Commonly Used and Distinctive Elementary Mathematics Curricula.* Elementary Subjects Center, Series No. 31. East Lansing, Mich.: Institute for Research on Teaching, College of Education, Michigan State University, 1991.**

A description and analysis of one math textbook and three alternative curricula for elementary math education. Findings show that the textbook emphasized computational skills and math facts, while the alternative curricula emphasized comprehension and application of math knowledge.

12. **Westbury, I, et al., ed. *In Search of a More Effective Mathematics Education: Examining Data from the IEA Second International Mathematics Study.* Norwood, N.J.: Ablex, 1994.**

A series of studies on new directions in mathematics education, including, for example, an article on successful teaching of problem-solving to Grade 8 students, and one entitled, "What Makes For Effective Math Instruction? Japanese and American Classrooms Compared," which brings an international perspective to the issue of alternative instruction.

13. **Winograd, K. "Writing, Solving, and Sharing Original Math Story Problems: Case Studies of Fifth Grade Children's Cognitive Behavior." Paper presented at the annual meeting of the American Educational Research Association, Chicago, April 1991.**

This paper examines different aspects of the process of children writing, solving and sharing math problems and the possibility of using this form of instruction as an alternative to purely text-based instruction. Findings indicate that having children create and share their own math word problems in small groups is a positive method of alternative instruction.

Alternative, Performance-Based Assessment

1. **Baker, E. L. "Issues in Policy, Assessment, and Equity." In *Focus on Evaluation and Measurement.* Vols. 1 and 2. Proceedings of the National Research Symposium on Limited English Proficient Student Issues, 1992.**

An article that looks at the relationship between policy and alternative assessment, which is presented as a way to achieve more equity in education (especially for limited English proficient—LEP—students). Major concerns include

- that LEP students are not being assessed because of a lack of tools suited to their needs,
- that LEP students are failing at assessments because of their lack of English abilities, and
- that LEP students are not getting the chance to learn.

Includes definitions and descriptions of alternative assessment methods that address these concerns. (For more on limited English proficiency issues, see the annotated bibliography by Kristen Lockwood.)

2. **Baron, J. B. "SEA Usage of Alternative Assessment: The Connecticut Experience." In *Focus on Evaluation and Measurement.* Vols. 1 and 2. Proceedings of the National Research Symposium on Limited English Proficient Student Issues, 1992.**

This article looks at state-level interest in alternative assessment and examines alternative assessment programs in Connecticut. Includes a discussion of the guidelines for effective performance-based assessment, and the problems of using this type of assessment with limited English proficient students.

3. **ERIC Clearinghouse on Information and Technology. *Alternative Assessment and Technology. ERIC Digest.* ERIC Clearinghouse on Information and Technology, Syracuse NY, 1993. 4-194 Center for Science & Technology, Syracuse University, Syracuse NY, 13244.**

Discusses performance-based and portfolio assessment, with an emphasis on the contribution of technology to these approaches. Provides an overview of the alternative assessment methods of the Center for Technology in Education (CTE) and examines the projects that CTE has run in various high schools. For example, students are asked to present projects they have done orally, answering questions, giving a presentation and defending their ideas. These presentations are then judged according to criteria such as clarity, coherence, responsiveness to questions and monitoring of listeners' understanding, which takes the place of traditional assessment.

4. **Hange, J. E., and H. G. Rolfe. "Creating and Implementing Alternative Assessments: Moving Toward a Moving Target." Paper presented at the annual meeting of the American Educational Research Association, New Orleans, April 1994.**

Findings from the first year of a study in which 22 Virginia teachers implemented alternative assessment techniques in their math classrooms. The results after a second year of study included

videotapes of how to run a workshop on implementing alternative assessment in classrooms.

5. **Jorgensen, M.** ***Assessing Habits of the Mind. Performance-Based Assessment in Science and Mathematics.*** **ERIC Clearinghouse for Science, Mathematics and Environmental Education, 1994.**

Describes and discusses methods, reasons and questions concerning performance-based assessment in textbook form for teachers to use while instituting performance-based assessment methods in their classrooms. (Available from ERIC/CSMEE, The Ohio State University, 1929 Kenny Road, Columbus, OH 43210.)

6. **Mitchell, R.** ***Testing for Learning: How New Approaches to Evaluation Can Improve American Schools.*** **New York: The Free Press, 1992.**

Discusses the idea that methods of assessment strongly influence methods of instruction and provides recommendations for change. Mitchell claims that teachers have been trained to teach to multiple-choice tests, and that this approach precludes an emphasis on understanding and connectedness of knowledge. If tests were different, she argues—meaning that tests should focus more on conceptual than on factual knowledge—then the way teachers teach could provide more of an emphasis on conceptual, contextualized knowledge.

7. **Northwest Regional Educational Lab., Portland, OR.** ***Test Center. Math Assessment Alternatives.*** **Department of Education, Washington, D.C., 1992.**

An annotated bibliography of alternative assessment related articles. Fifty-six references are included, covering both elementary and secondary education.

8. **Office of Educational Research and Improvement (ED), Washington, D.C. "Science and Math Assessment in K–6 Rural and Small Schools."** ***Rural, Small Schools Network Information Exchange*** **14 (Spring 1993).**

Contains reprints of 31 journal articles and other papers concerning assessment (with an emphasis on alternative assessment) in rural and small schools. This collection also includes articles on portfolio assessment for teachers interested in that means of assessment.

9. **Perrone, V., ed.** ***Expanding Student Assessment.*** **Alexandria, Va.: Association for Supervision and Curriculum Development, 1991.**

A collection of essays on alternative assessment and new directions for change. The essays also discuss the problems with current testing methods, and focus on the assessment issue from a classroom-based perspective. The major argument presented throughout the book is that evaluation strategies should more closely approximate what teachers and students are doing in the classrooms, and what is happening in the classrooms must focus on understanding and connection with knowledge, rather than on memorization of isolated facts. Therefore, methods such as multiple-choice tests are seen as deficient, and other means of assessment, such as graded presentations, might be more effective, given that they connect to what is happening in the classroom and encourage conceptual understanding of the material covered.

10. **Ryan, P.** ***Teacher Perspectives of the Impact and Validity of the Mt. Diablo Third-Grade-Curriculum-Based Alternative Assessment of Mathematics (CBAAM).*** **Far West Lab. for Educational Research and Development, San Francisco, 1994.**

Examines teacher perceptions and activities following implementation of an alternative assessment program in California, and presents results and analysis. Teachers resequenced their curricula, introduced new content and emphasized educational processes more than they had prior to the program's implementation. Evidence was provided for the significant short-term impact of the program and the suggested long-term impact.

11. **Shepard, L., et al.** ***Second Report on Case Study of the Effects of Alternative Assessment in Instruction. Student Learning and Accountability Practices. Project 3.1. Studies in Improving Classroom and Local Assessments.*** **Washington, D. C.: Office of Educational Research and Improvement (ED), 1994.**

Three papers on the results of a comparison study of traditional and alternative assessment. The alternative assessment was performance-based, rather than involving traditional assessments which generally test for isolated facts rather than applied knowledge and student understanding of concepts. One study looks at alternative assessment in 13 Grade 3 classrooms, one looks at interviews with students in these classrooms, and one examines the work and roles of the teachers who participated in the study.

12. **Smith, L., et al.** ***Assessment of Student Learning in Mathematics.*** **Columbia, S.C.: South Carolina Center for Excellence in the Assessment of Student Learning, College of Education, University of South Carolina, 1993.**

Assessment in the mathematics classroom in light of modern goals for math education. Emphasizes alternative assessment and reviews different performance-based assessment methods including

1) open-ended questions,
2) mathematical investigations and projects,
3) writing activities in mathematics,
4) observations and interviews,
5) enhanced multiple-choice questions and
6) portfolio assessments.

The implication is that these are all positive means of assessment that are in line with conceptual understanding and alternative instruction. Grading and scoring techniques are also discussed.

13. Williams, M. *Renewal that Fits: Preparing Educators for Reforming Schools*, 1995.

Discusses a program at Morehead State University in Kentucky that, among other things, provides training for teachers in performance-based assessment methods. The program was constructed to help public schools adapt to state-mandated educational reforms. Suggests that there might be a useful paradigm for teacher training in performance-based assessment methods. 1995. Available from EDRS.

Area of Triangle

Show, without the use of trigonometry, that the area of a triangle with sides *a, b, c* and angle at vertex *A* equal 60 is given by:

$$S = \frac{\sqrt{3}}{4}[a^2 - (b - c)^2].$$

If the angle at vertex *A* is 120, then the area *S* of the triangle is given by:

$$S = \frac{\sqrt{3}}{12}[a^2 - (b - c)^2].$$

MCATA Executive 1997–98

President
Florence Glanfield
8215 169 Street NW
Edmonton T5R 2W4
Res. 489-0084
Fax 483-7515
glanfiel@gpu.srv.ualberta.ca

Past President and 1998 Conference Cochair
George Ditto
610 18 Avenue NW
Calgary T2M 0T8
Res. 289-1709
Bus. 777-8927
Fax 777-7229
gditto@cbe.ab.ca

Vice Presidents
Cynthia Ballheim
612 Lake Bonavista Drive SE
Calgary T2J 0M5
Res. 278-2991
Bus. 228-5810
Fax 229-9280
ballheimcj@aol.com

Betty Morris
10505 60 Street W
Edmonton T5M 2M1
Res. 466-0539
Bus. 441-6104
Fax 459-0187
morrisb@ecs.edmonton.ab.ca

Secretary
Donna Chanasyk
13307 110 Avenue NW
Edmonton T5M 2M1
Res. 455-3562
Bus. 459-4405
Fax 459-0187
donnac@connect.ab.ca

Treasurer
Doug Weisbeck
66 Dorchester Drive
St. Albert T8N 5T6
Res. 459-8464
Bus. 434-9406
Fax 434-4467
weisguy@v-wave.com

NCTM Representative
TBA

Newsletter Editor
Art Jorgensen
4411 5 Avenue
Edson T7E 1B7
Res. 723-5370
Fax 723-4471
ajorgens@telusplanet.net

Journal Editor
Klaus Puhlmann
PO Box 6482
Edson T7E 1T9
Res. 795-2568
Bus. 723-4471
Fax 723-2414
klaupuhl@gyrd.ab.ca

Webmaster
Dick Pawloff
4747 53 Street
Red Deer T4N 2E6
Res. 347-3658
Bus. 343-3733
Fax 347-8190
dpawloff@tac.net

Department of Education Representative
Daryl Chichak
1826 51 Street NW
Edmonton T6L 1K1
Res. 450-1813
Bus. 427-0010
Fax 469-0414
dchichak@edc.gov.ab.ca

1998 Conference Cochair
Bob Michie
149 Wimbledon Crescent SW
Calgary T3C 3J2
Res. 246-8597
Bus. 289-9241
Fax 777-7309
bmichie@cbe.ab.ca

Mathematics Representative
Michael Stone
Room 470 Math Sciences Bldg.
University of Calgary
2500 University Drive NW
Calgary T2N 1N4
Res. 286-3910
Bus. 220-3964
Fax 282-5150
mgstone@acs.ucalgary.ca

Faculty of Education Representative
Dale Burnett
Faculty of Education
University of Lethbridge
Lethbridge T1K 3M4
Res. 381-1281
Bus. 329-2416
or 329-2457
Fax 329-2252
dale.burnett@uleth.ca

Directors
Daryl Chichak
1826 51 Street NW
Edmonton T6L 1K1
Res. 450-1813
Bus. 427-0010
Fax 469-0414
dchichak@edc.gov.ab.ca

Rick Johnson
8 Grandin Woods Estates
St. Albert T8N 2Y4
Res. 458-5670
Bus. 476-7953
Fax 459-0512
rjohnson@freenet.edmonton.ab.ca

Graham Keogh
37568 Rge Rd 275
Red Deer County T4S 2B2
Res. 347-5113
Bus. 342-4800
Fax 343-2249
gkeogh@rdcrd.ab.ca

Richard Kopan
72 Sunrise Crescent SE
Calgary T2X 2Z9
Res. 254-9106
Bus. 777-7520
Fax 777-7529
kopanr@cadvision.com

Elaine Manzer
9502 79 Avenue
Peace River T8S 1E6
Res. 624-3988
Bus. 624-4221
Fax 624-4048
manzere@prsdnet.ab.ca

Sandra Unrau
11 Hartford Place NW
Calgary T2K 2A9
Res. 284-2642
Bus. 777-6920
Fax 777-6922
sunrau@cbe.ab.ca

PEC Liaison
Carol D. Henderson
860 Midridge Drive SE Suite 521
Calgary T2X 1K1
Res. 256-3946
Bus. 938-6666
Fax 256-3508

ATA Staff Advisor
David L. Jeary
SARO
540 12 Avenue SW Suite 200
Calgary T2R 0H4
Bus. 265-2672
or 1-800-332-1280
Fax 266-6190
djeary@teachers.ab.ca

ISSN 0319-8367
Barnett House
11010 142 Street NW
Edmonton AB T5N 2R1