Middle School Students' Reasoning About Geometric Situations

William M. Carroll

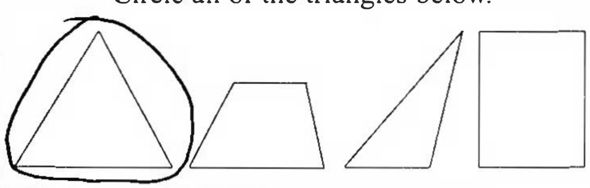
Along with increased emphasis on reasoning, communication and problem solving, the National Council of Teachers of Mathematics (1989, 1991, 1995) has called for a change in assessment techniques. In contrast to short-answer questions, assessments that elicit writing, diagrams and other representations offer better windows into students' understandings and misconceptions about mathematics. This article describes some short geometry tasks that go beyond the simple recognition of figures and properties. Because they have been field-tested with students using various mathematics curricula, we have collected hundreds of student responses to these questions that seem to represent a good range of students' geometric thinking and development. From these responses, along with ideas about the development of geometric thinking (Fuys, Geddes and Tischler 1988), we have developed scoring rubics to go along with many of these questions. The rubrics and questions might be useful to middle school teachers who are developing short open-ended questions that encourage and assess students' thinking.

Questions that require written reasoning or student-generated illustrations are useful for several reasons. First, although short-answer questions are easy to correct, they provide limited information about students' thinking. For example. the cause of the mistake in Figure 1 is unclear. Did the student have a narrow working definition for *triangle*, miss the second triangle or simply misread the question? Without this information, it is difficult for the teacher to plan relevant instruction.

Figure 1

Single-Answer Question That Does Not Illustrate a Student's Reasoning

Circle all of the triangles below.



Second, questions that require writing, drawing or other representation encourage and require more complex thinking. The question in Figure 1 would reveal more about the student's thinking if the direction "Explain why the figures not circled are not triangles" was added. In constructing an explanation, students are more likely to reason about the more relevant properties of the figure and, as they do so, to access and integrate previous knowledge. Responses to a question like this one are fairly easy to interpret, and they can assist the teacher in identifying misconceptions that their class may hold.

Third, these types of activities and assessment more closely resemble the activities that we value in the mathematics classroom. The assessment requires a student to use and integrate information actively. Many open-ended questions can also encourage multiple ways of thinking about the problem.

Three activities that we have used to assess more complex geometric thinking are described here. Some samples of students' work and rubrics that we have developed are also provided. They are well suited for Grades 5–8.

Task 1: Properties of Triangles

The first question asks students to reason about whether a triangle can be constructed with two right angles:

> Sheila said, "I can draw a triangle with two right angles." Do you agree with Sheila? Explain your answer.

Even though the question itself is fairly straightforward, the range of student responses is quite wide. Some students skip the question or answer without using geometric language or apparent reasoning. For example, a response like "I disagree with Sheila because you can't do it," provides no evidence of geometric understanding. At the other end, students use their knowledge of triangles and angles to provide an explanation that amounts to an informal proof: "No. Because the sum of the angles in a triangle is 180 degrees. And two right angles make 180 degrees.

So there would be no third angle for the triangle." A few students noted that two sides would be parallel and so the figure could not be a triangle, or they drew several counterexamples. Of course, a triangle could be constructed with two right angles on a sphere or on other non-Euclidean surfaces. Although no students responded, "Perhaps this would be possible on another type of surface that isn't flat," such a response would be acceptable and classified as a high-level response. In between these two examples, many of the Grades 5–6 students showed fair understanding but were limited in their ability to explain. Another group attempted to answer, but its members were limited by their misconceptions, for example, "There is one right and one left angle in a triangle, no matter how you draw it."

Some of the students clearly lacked an understanding of *right angle*, thinking that it referred to orientation. Most classes knew the basic definition but would benefit from explorations of figures that could be made with right angles. To deepen their understanding and facility, these students might use geoboards or dot paper to attempt to construct different types of triangles, prompted by such questions as "Which of the following triangles can you make? Look for a pattern, and explain what you found." These findings could be recorded and discussed, and reasoning about the relationship between properties could be emphasized.

In conjunction with these tests, we also conducted individual interviews. Interestingly, although many of the students "knew" earlier in the interview that the sum of the interior angles of a triangle was 180 degrees, few used this information spontaneously when this problem was posed. Instead, most attempted to draw such a figure, then stated that it was impossible. They were unable to use their factual knowledge without working concretely or visually. Such questions afford an opportunity to integrate geometric ideas that are otherwise loosely connected, that is, to generalize knowledge to more abstract understandings.

Table 1 describes five levels of response that we found and some illustrative responses. We think that the levels correspond fairly well to the van Hiele model of geometry, which describes a progression in thinking from recognition without reasoning to analyzing properties separately. Note that students can achieve the higher levels by various approaches to the question.

Table 1

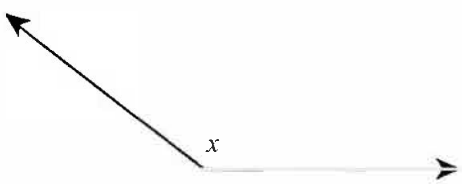
Rubric and Sample Responses to Sheila's Triangle

Level	Description and Sample of Students' Responses
0	No response or off task; geometric language is not used: "I disagree with Sheila because you can't do it."
1	Incorrect response, but some reasoning is attempted: "Yes, because all triangles have a right angle and a left angle." "Yes, you make one at the top and one at the bottom." Partially correct response, but reasoning is weak: "No, because all triangles have right angles."
2	Correct response, but reasoning is not complete: "No, because you can only put 1 right angle in a triangle." "No, it would have to be a square or a rectangle."
3	Correct response and good reasoning. Explanation goes beyond level 2 but relies on concrete or visual understanding rather than on abstract knowledge of properties: "Because if you put 2 right angles together, you already have 3 sides, and the sides are not closed." "No, because if you draw 2 right angles ⊔ and try to connect them, you get a square or a rectangle. Two right angles is already 3 sides."
4	Exemplary response. Student used knowledge of triangles and angles: "Triangles have 3 angles and 180°. If there are 2 right angles, then it would equal 180°. But that is only 2 angles." "How could you possibly have 2 right angles equaling 180° when you have 2/3 of a triangle done?" "You would have 2 parallel sides."

Task 2: Estimating the Measure of Angles

The second task was developed to assess students' knowledge of angular measurement, especially their use of such benchmark angles as 90 degrees, 180 degrees, and 45 degrees, to estimate the size of given angles. For example, a student who is familiar with common angles would recognize that the angle is greater than 90 degrees. We are also interested in whether students feel comfortable giving an estimate rather than an exact answer. One form of this task is illustrated in Figure 2.

Figure 2
Angle Estimation



Estimate the measurement of angle x in degrees.
Angle x is about _____degrees.
Explain or show how you got your estimate.

Because students were asked to explain their reasoning or the method they used, the range of students' reasoning, and of errors in reasoning, was apparent. Students could rely on visualizing the benchmark angles for a fairly complete answer. The largest group of Grades 5–6 students fell in this category: "It's a little more than 90 degrees but less than 180 degrees." Frequently, students drew in a right angle or stated that they pictured a right angle fitted into the obtuse angle.

Although this response was completely satisfactory, some students went further, using more precise estimates. A typical response was, "I made the angle 90 degrees and looked at the remaining part and saw that it was about half of 90 degrees, which is 45 degrees. I added 90 degrees and got 135 degrees."

Many students were successful on this task. Large numbers of middle school students, especially those who have had geometry experiences, have a good picture of benchmark angles and are able to use them successfully. This use of benchmark angles seems like a real-life skill that can promote estimation skills. The rubric that we have developed with four levels of responses, is shown in Table 2.

Task 3: Hidden Geometry Figures

The third task involves more problem solving. Like the first task, Sheila's triangle, this one requires students to consider geometric properties of polygons—

Table 2
Rubric and Sample Responses to Angle Estimation

Level	Description and Sample of Students' Responses
0	No response or off task.
1	Answer is not between 90 degrees and 180 degrees, but some attempt to explain is made: "70°, because the angle is less than a straight line." Answer is between 90 degrees and 180 degrees, but the student does not provide good reasoning: "I looked at it and I knew it was about 120°."
2	Student gives answer between 90 degrees and 180 degrees and includes use of benchmark angles: "I knew it was bigger than a right angle, but not 180°." "120°. It looks a little more than 90° but less than 90° away from 180°."
3	Student uses more precise benchmarks, perhaps in two steps. Estimate is within 15 degrees of exact answer: "The angle is about 90 and a half, so I divided 90 in half and add it to 90°." "130°. I drew a right angle and then counted up by 10 degrees."

parallel sides or types of angles. If two of the three sides showing are parallel, is it possible for the hidden figure to be a triangle? (See Figure 3.) What properties distinguish a square or a trapezoid from other figures? These questions require more than naming figures; they require reasoning about what makes the figures unique, combined with using the problem-solving strategies needed for sorting the figures. For example, three of the figures in Figure 3 can be trapezoids, and all could be hexagons; but only the second can be a triangle, and only the fourth can be a square. Note that the figures are not the standard geometric figures generally shown in books, that is, a regular hexagon or a triangle with the base at the bottom.

These problems can lead to nice discussions of geometric properties. In some classes observed, students worked in small groups, actively drawing and discussing the properties of the given polygons. Some examples of students' work are shown in Figure 4. A scoring rubric for the assessment of the activity, along with response levels, is shown in Table 3. The responses in Figure 4 correspond to those levels, with only the last response including correct drawings and names.

The three rubrics illustrated in this article vary in levels from five (Table 1) to three (Table 3). The complexity of the rubric should mirror both the complexity of the task and the purpose the teacher has in mind. Often a three-point rubric is sufficient for teachers who want to assess the range of student understanding for the purpose of planning instruction.

Another purpose for developing rubrics is to include students in the assessment process. Often students are not clear about what differentiates an excellent response from a poor response. A clearly stated rubric, along with some examples, can clarify

Figure 3

Hidden Figures Task

Gina drew some shapes: a triangle, a square, a trapezoid and a hexagon.
She covered most of each figure, as shown below.
Can you tell which figure is which? Write the name below each figure.
Then try to draw the rest of the figure.

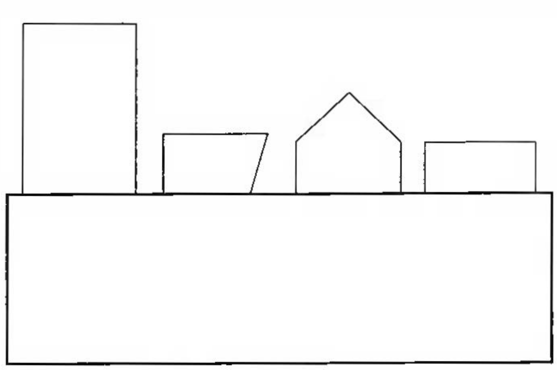
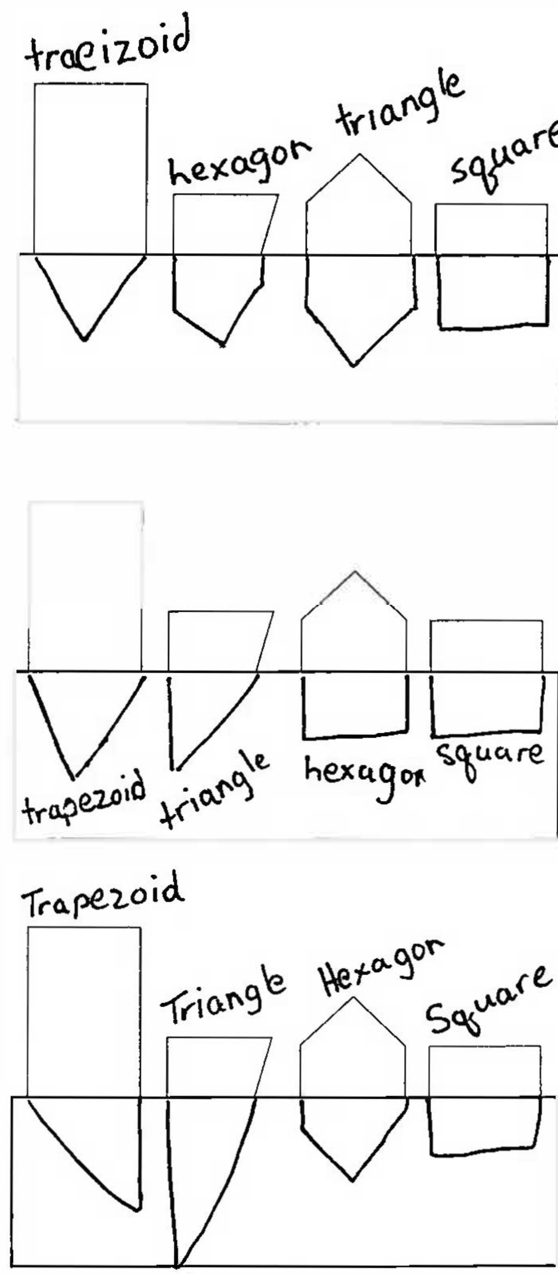


Figure 4

Responses to the Hidden-Figures Question



these standards for the students. Teachers have indicated that this process is ongoing. Early in the year, students are unclear about how to express their reasoning or what constitutes a good explanation, but with experiences and exemplars, they improve their responses.

An additional example of hidden geometry figures is illustrated in Figure 5. This form of the question asks students to include an explanation about their reasoning. The reader is invited to consider the possible responses that students might give and how a rubric might be developed to assess these responses. Then test it with students to see how well their responses fit the rubric and what adjustments are necessary in it.

Figure 5

Another Hidden-Figure Task

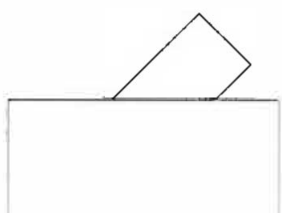
A figure is partly hidden. Which of the following might it be? Circle all the possible answers.

rectangle

triangle

trapezoid

square



Choose one of the figures that you did not circle, and explain why you did not choose it.

Conclusion

Research has shown that many secondary school students in the United States are ill-prepared for formal geometry classes (Senk 1989). Often, junior high and senior high school students lack experiences in reasoning about geometric properties. To prepare students for more formal thinking in the secondary school, geometry activities in the middle school must go beyond simple visual exercises. Rigorous proofs are not necessary at this age, but students should be able to use ideas about geometry to construct informal arguments, which helps them better understand the structure of geometry. These arguments might involve oral or written responses.

Teachers have been quite positive about the types of questions illustrated in this article. They can be used as individual or group activities or assessments or both. Because the questions emphasize reasoning, problem solving and communication, teachers have reported that these types of questions help them implement the NCTM (1989, 1995) standards in their classrooms. Students are also often enthusiastic about engaging in these types of activities, as opposed to simple classification and vocabulary activities.

As the examples in this article illustrate, a good deal of information about students' mathematical knowledge can be gathered from fairly short activities and assessments that involve reasoning. Although longer projects and tasks are also needed, these short activities can make reasoning a regular part of the classroom in a manageable fashion. With some practice, such questions and rubrics are easily developed, especially when teachers work collaboratively. Often, single-response questions can be used as the base from which the question is expanded

Table 3

Rubric for Hidden-Figures Question

Level	Description and Sample of Students' Responses
0-Little progress	Student incorrectly names and draws most of the geometric figures. Response shows little understanding of geometric shapes and their properties. (First response in Figure 4)
1-Shows progress	Student correctly names all geometric figures. However, some drawings show incorrect figure. (Second response in Figure 4)
2-Good understanding	Student correctly names and draws all geometric figures. Drawings illustrate correct properties of the figure. (Third response in Figure 4)

and students are asked to explain their thinking in words or drawing.

Having students draw, explain and elaborate on their answers has several benefits. They must apply knowledge more fully when reasoning is required. In the process of explaining their reasoning, they more fully integrate previous knowledge and learning occurs. As suggested by the assessment standards document (NCTM 1995), assessment and learning are not separate processes. They can be used as activities to develop and discuss reasoning or assessments. These types of questions are also more like the mathematics we expect people to need as a life skill. Real problem situations require planning, reasoning and communication.

Teachers who have attempted to construct reasoning questions and rubrics often report that the process gives them a better insight into their students' thinking. Considering the range of possible responses and misconceptions is helpful in planning instruction and activities.

Perhaps more important, when more open, more complex assessment tasks are used, students' thinking is more clearly revealed and information that is crucial to planning individual and class instruction can be gathered. As our results indicate, middle school students are quite capable of developing good reasoning about geometric situations when they have had substantial experiences in geometry throughout elementary school. However, many students fail to go beyond a simple visualization of geometric figures. Challenging students to apply their knowledge in situations that require application, explanation and illustration is one step toward improving geometric reasoning.

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Length of the Belt

A drive belt runs around three pulleys that are in a triangular arrangement to one another. The distances between the pulley shafts are 1.5 m, 2.0 m and 2.5 m, respectively. All three pulleys have a radius of 50 cm. How long is the drive belt?