Ideas for Developing Students' Reasoning: A Hungarian Perspective

Anita Szombathelyi and Tibor Szarvas

As the end of the 20th century approaches, we start to realize again the significance of proof in mathematics education. The NCTM's (1989) *Curriculum and Evaluation Standards for School Mathematics* cautions against the tendency to completely abandon proofs and focus only on the end results and formulas. In this article, we re-emphasize the importance of proofs in teaching by sharing some of our experiences as students and teachers in Hungary, in addition to our experiences as graduate teaching assistants at an American university. We offer examples and ideas that might help educators develop students' mathematical reasoning skills.

The temptation has always existed to de-emphasize the role of proofs in mathematics teaching, because they are considered too time-consuming for a class period and because of the growing perception that mathematics is primarily needed for applications. However, we believe that danger lurks in this perception. Removing all proofs from calculus, for example, as the Hungarian-born Pólya (1957, 219) wrote, may "reduce the calculus to the level of the cookbook. The cookbook gives a detailed description of ingredients and procedures but no proofs for its prescriptions or reasons for its recipes."

Indeed, students in Kindergarten through Grade 12 are expected to be able to observe patterns and relationships, to use them to make conjectures and to construct arguments by drawing logical conclusions. They are encouraged to use deductive and inductive reasoning, formulate counterexamples and indirect proofs, and use mathematical induction—just as in the typical Hungarian K–12 curriculum. We should start promoting the use of reasoning, introduce critical thinking at an early age and reiterate its importance at each grade level. Teaching students to look for more than one way to establish the validity of the same statement helps them make connections among different mathematical concepts and procedures.

We also believe that it is important to prepare mathematics teachers to emphasize proofs in their lessons. In Hungary, prospective high school teachers are required to present to their professors several theorems covered in their courses, along with complete proofs. Besides developing confidence and ability, this exercise helps teachers more readily incorporate mathematical reasoning in their own instruction.

We next share a series of examples and ideas that can help develop logical thinking and reasoning ability—examples and ideas that we have used successfully in Hungary and in the United States. We then move from such intuitive aspects of proofs as making conjectures to more formal types. We use some classical examples, along with real-life problems that are easily accessible to students.

Making Conjectures

Beginning at an early age, students are frequently asked to observe patterns and relationships. They should be encouraged to make conjectures about the particular case of a problem and to test these conjectures. During that process, they might devise a proof for the general case. They can start with simple problems and, using the same techniques and reasoning, gradually proceed to more abstract levels.

Example 1

Calculate $2^3 \cdot 2^2$, $2^5 \cdot 2$, $2^4 \cdot 2^2$.

$$a^n \cdot a^m = (\overbrace{a \cdot a \cdot \ldots \cdot a}^{n})\,(\overbrace{a \cdot a \cdot \ldots a}^{m}) = a \cdot a \cdot \ldots \cdot a \cdot a \cdot a \cdot \ldots \cdot a = a^{n+m}$$

Example 2

Examine the differences of consecutive square numbers.

Students will quickly notice the pattern: $2^2 - 1^2 = 3$, $3^2 - 2^2 = 5$, $4^2 - 3^2 = 7$, ... In general, for n and $n + 1$, $(n + 1)^2 - n^2 = 2n + 1$. After students arrive at a conjecture, they can easily prove why this general statement is always true.

Using basic examples, we can increase the opportunities for our students to give complete proofs. Making a conjecture is an achievement, but students should recognize that it is not yet a proof. They should be encouraged to furnish a more formal argument.

It should be clear to them that proving one particular case is not equivalent to proving the statement in general. For many students, this concept is difficult to comprehend.

True-or-False Questions

Children are introduced to true-or-false questions when they are very young. We think that they must always give reasons for their answers. Otherwise, true-or-false problems are nothing but an easier version of multiple-choice questions, and then we really do not assess students' mathematical thinking. A rigorous, formal proof is desirable for a true statement, whereas a counterexample should be provided for a false statement. Students use counterexamples as solutions to different problems at a very early age, but they do not have a word for it—think of such simple statements as "Every flower is pink." This method may avoid the dilemma pointed out by Galbraith (1995, 416), "the fact that a single exception disproves a generalization is not accepted by students"; they "do not understand what constitutes a counterexample, namely, an instance that satisfies the conditions but not the conclusion of the statement."

Example 3

Prove or disprove: Straight lines that have no point in common are parallel.

Example 4

True or false? Give reasons. If $f'(c) = 0$ for a function f, then f has a maximum or minimum at $x = c$.

Such examples as these have two parts. First, students must decide whether they think that the statement is true, and then they must either prove it or find a counterexample, that is, an example that satisfies the given conditions but does not satisfy the conclusion of the statement. Students who try to learn by simply memorizing everything can find these problems to be very challenging. Recalling formulas and theorems without really understanding them will not help students find counterexamples. These types of problems promote mathematical thinking and reasoning, not guessing or working from memory.

Examining Flawed Arguments

In these problems, students are asked to find errors in the argument. It is useful to introduce students to proofs that have hidden errors or logical inconsistencies and ask them to go through the argument step-by-step, searching for the flaws and correcting them if possible. This exercise is an excellent opportunity to discuss common mistakes, different methods of proofs and so on.

Example 5

Can you find an error? Start with the equation

$$x + y = -z$$

Multiply both sides by 4 and then by 5, and exchange the sides of the second equation:

$$4x + 4y = -4z$$
$$-5z = 5x + 5y$$

Adding the two equations yields

$$4x + 4y - 5z = 5x + 5y - 4z.$$

Adding 9z to both sides, we obtain

$$4x + 4y + 4z = 5x + 5y + 5z.$$

Then

$$4(x + y + z) = 5(x + y + z),$$

Which implies that

$$4 = 5.$$

Proving by Contradiction

Proving by contradiction also builds reasoning skills. Moreover, these types of proofs build on students' interests in making logical arguments that are not necessarily straightforward to others. We start with given conditions, assume the conclusion to be false and arrive at a contradiction.

Example 6

Ten teams are playing in a championship in which each team meets every other team exactly once. So far, 11 games have been played. Prove that one team has played at least three games.

Proof: Let us assume the contrary. Each team therefore has played at most two games; that is, at most $(10 \cdot 2)/2 = 10$ games have been played, in contradiction with the wording of the problem. Therefore, a team must have already played at least three games.

Example 7

Prove that the sum of a rational number and an irrational number is irrational. Q is the set of rationals; I, the set of irrationals; and Z, the set of integers.

Proof: Let $x \in Q$ and $y \in I$. Then $x = p/q$. For some $p, q \in Z(q \neq 0)$. Assume that $x + y \in Q$. Then $x + y = p/q + y = r/s$ for some $r; s \in Z\ (s \neq 0)$. So $y = r/s - p/q = (rq - ps)/sq \in Q$, which contradicts the hypothesis.

We think that using mathematical symbols to rewrite expressions and whole sentences is another area that needs more emphasis. Students, especially those who plan to attend college, need to acquire a basic familiarity with symbols and notation.

Working Backward

This method of proof should be used very carefully, always paying attention to working with equivalent statements.

Example 8

Prove that for two positive numbers a and b, the arithmetic mean $(a + b)/2$ is always greater than or equal to their geometric mean $\sqrt{ab}$. Furthermore, equality occurs if and only if $a = b$.

Proof 1:

$$\frac{a+b}{2} \geq \sqrt{ab},$$

$$a + b \geq 2\sqrt{ab},$$

$$a - 2\sqrt{ab} + b \geq 0,$$

$$(\sqrt{a} - \sqrt{b})^2 \geq 0,$$

which is always true. Furthermore,

$$(\sqrt{a} - \sqrt{b})^2 = 0,$$

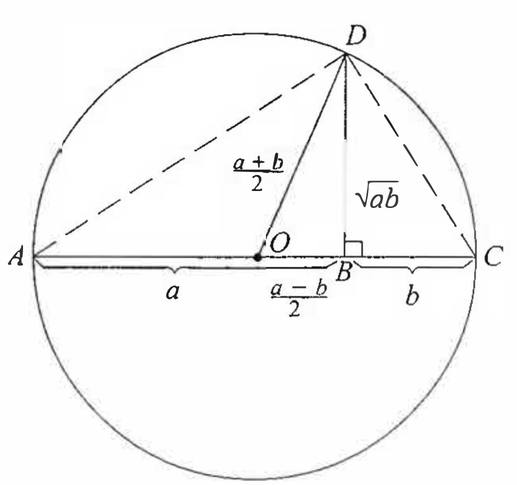
$$\sqrt{a} = \sqrt{b},$$

$$a = b.$$

Another way to solve this problem is by using a geometric construction, as in proof 2.

Proof 2: See Figure 1. We used Thales' theorem and the Pythagorean theorem, along with the fact that the length of the hypotenuse of any right triangle is always greater than the length of any of its legs.

Figure 1
Geometric Proof of Example 8



Proving by Mathematical Induction

This method is often deemed uninteresting because it repeats the same technique for different problems. We do not believe that proofs by induction can be skipped because they are "too simple." We had to learn and use induction in high school in Hungary, and when we became teachers, we expected our students to do the same. We think that expecting students to learn to use mathematical induction is justified because students analyze and work with such different ingredients of mathematical arguments as hypotheses and conclusions. Also, when proving a statement by induction, students learn to use symbols and how to express themselves in the language of mathematics.

Example 9

Prove that

$$s_n = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

This proof is an example of a classical application of mathematical induction. The important point is to conjecture the sum of the first n positive integral numbers. Too often, students accept the formula as a fact without a formal proof.

Proof: For $n = 1$, the statement is trivial:

$$1 = \frac{1(1+1)}{2}$$

To carry out the inductive step, let k be an arbitrary positive integer, and suppose that

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}.$$

To show that the assertion holds true for $n = k + 1$, we use the induction hypothesis along with some algebra:

$$1 + 2 + \cdots + k + k + 1 = (1 + 2 + \cdots + k) + k + 1$$

$$= \frac{k(k+1)}{2} + k + 1$$

$$= \frac{k(k+1)}{2} + \frac{2(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

The principle of mathematical induction guarantees that the statement is true for all positive integers n, and the proof is complete.

In addition to mathematical induction, several other methods of proving this theorem exist. In school, we were always encouraged to find more than one way to solve a problem. Students should learn that several ways can be used to arrive at a conclusion. Students can come up with different processes, along with their advantages and disadvantages.

A way to prove example 9 not using mathematical induction is the following:

Proof: $s_n = 1 + 2 + 3 + \cdots + (n - 1) + n,$

$$\frac{\begin{aligned} s_n &= n + (n-1) + (n-2) + \cdots + 2 + 1 \end{aligned}}{2s_n = (n+1) + (n+1) + (n+1) + \cdots + (n+1)}$$

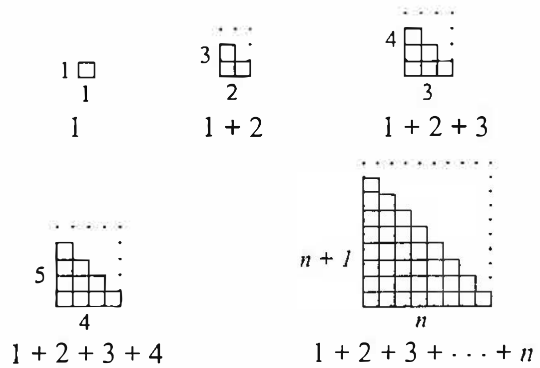
$$2s_n = n(n+1)$$

$$s_n = \frac{n(n+1)}{2}.$$

Proof: See Figure 2.

Figure 2

Proof 3 of Example 9



Deductive Method

The deductive method is widely used to establish the validity of theorems, propositions and corollaries. Previously proved results, definitions and the given hypotheses are applied in a straightforward manner to reach the desired conclusion.

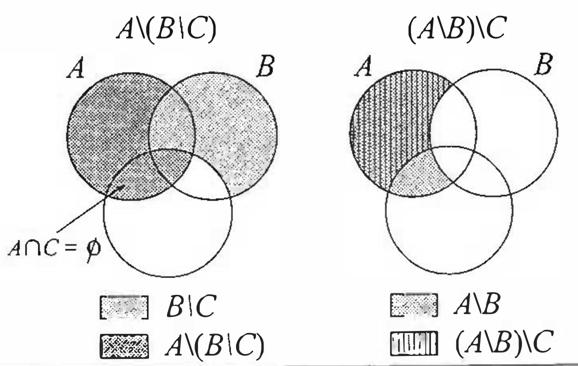
Example 10

Prove that if $A \cap C = \phi$, then $A\backslash(B\backslash C) = (A\backslash B)\backslash C$. (Note that in this article $B\backslash C$ should be taken to mean all points B not in C.)

Proof: See Figure 3.

Figure 3

Proof of Example 10



Example 11

Theorem. A circle can be inscribed in a quadrilateral if and only if the sums of the lengths of the opposite sides of the quadrilateral are equal.

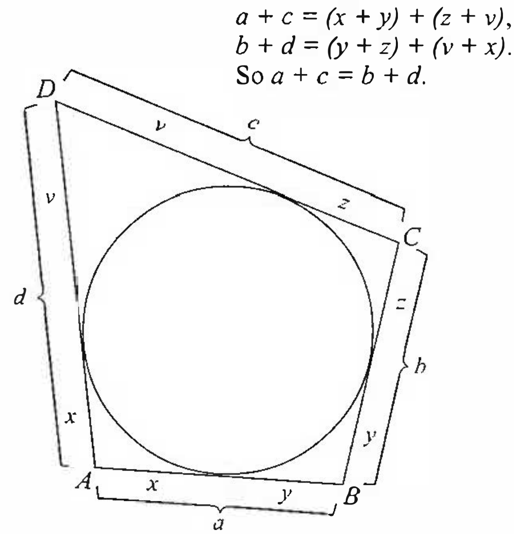
Proof: Part 1: "Necessity"

Hypothesis. The given quadrilateral *ABCD* has a circle inscribed in it.

Conclusion. $a + c = b + d$ (see Figure 4). Here we used the theorem that the segments that are tangent to a circle from an outside point have equal lengths.

Figure 4

Part 1 of Example 11



Part 2: "Sufficiency"

Hypothesis. $a + b = b + d$ in *ABCD* (see Figure 5).

Conclusion: ABCD is a quadrilateral in which a circle can be inscribed.

We prove the second part by contradiction. Assume that $a + c = b + d$ in *ABCD* but that no circle can be inscribed in it. Construct a circle inside the quadrilateral that touches three of the four sides (see Figure 5). Draw a tangent to the circle from point *C*. Let *E* be the point that is the intersection of the tangent line and side *AB*; and let $x = AE$, $y = EC$. Then $EB + EC = BC$. But $EB + EC > BC$ by the triangle inequality, showing that the indirect assumption was false.

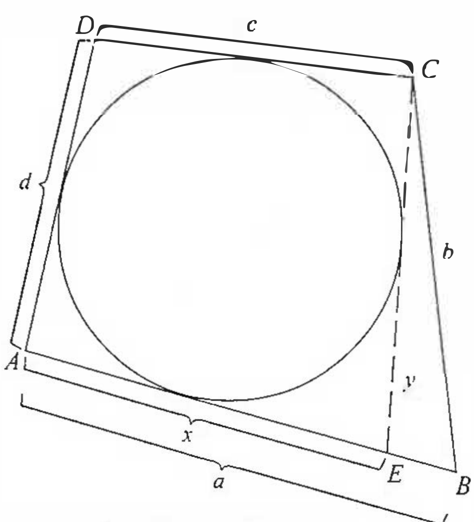
The theorem in example 11 requires a combination of different proof techniques. It is an excellent opportunity to introduce students to discovery learning, conjecture making, testing and proving. They can construct a quadrilateral with an inscribed circle either with a ruler and compass or by using geometric software. Excellent geometric software packages include The Geometer's Sketchpad (Jackiw 1995) and Cabri II (Bellemain and Laborde 1995). Technology can be combined with cooperative learning to get high results in the classroom.

As the next step, students measure the lengths of opposite sides and try to find some relationship between them. The more data they have, the easier to conjecture—they can record one another's findings. The teacher assists as students set up at least the "necessity" part of the theorem and try to prove it. This straightforward deductive proof requires no tricks or fancy techniques. The other direction—if $a + c = b + d$ in a quadrilateral, then a circle can be inscribed in it—needs some thought but can easily be proved using the indirect method.

Geometric constructions are important in gaining insight. They help students develop mathematical thinking and understanding and offer excellent opportunities for exploration, recognizing patterns, making and testing conjectures, and fun, of course.

Figure 5

Part 2 of Example 11



$a + c = b + d$ by hypothesis,
$x + c = y + d$ by part 1,
$a - x = b - y$.
So $EB = BC - EC$ or $EB + EC = C$.
But $EB + EC > BC$
by the triangle inequality.

More generally, geometric proofs have an important place in our mathematics instruction. "The study of the elements of plane geometry yields still the best opportunity to acquire the idea of rigorous proof. . . . If general education intends to bestow on the student the ideas of intuitive evidence andlogical reasoning, it must reserve a place for geometric proof" (Pólya 1957, 215, 217).

These examples illustrate the types of exercises that can help students develop good reasoning skills and the confidence and ability to prove statements. Of course, every single statement need not be prepared along with its proof in the mathematics classroom, but with all the additional content that has been introduced into the high school curriculum to meet the needs of the 21st century, proofs can permeate the curriculum in much the same way that problem solving now does. We, as teachers, should always decide what is important but must emphasize the reasoning aspect of instruction. We must find ways to encourage our students to justify their responses and structure their arguments using mathematical notations. They will then have the tools necessary for formal proofs. This process should begin early, for it will empower students to make mathematical connections later on. They will be freed from the frustrations of a multitude of seemingly unrelated rules and techniques. Last, but most important, they will see the real beauty of mathematics.

References

Bellemain, F., and J-M. Laborde. Cabri Geometry II. Dallas, Tex.: Texas Instruments, 1995. Software.

Galbraith, P. "Mathematics as Reasoning." *Mathematics Teacher* 88 (May 1995): 412–17.

Jackiw, N. The Geometer's Sketchpad 3. Berkeley, Calif.: Key Curriculum, 1995. Software.

National Council of Teachers of Mathematics (NCTM). *Curriculum and Evaluation Standards for School Mathematics.* Reston, Va.: NCTM, 1989.

Pólya, G. *How to Solve It: A New Aspect of Mathematical Method.* 2d ed. Princeton, N.J.: Princeton University Press, 1957.

Reprinted with permission from The Mathematics Teacher, *Volume 91, Number 8 (November 1998), pages 677–81, an NCTM publication. Minor changes have been made to spelling and punctuation to fit ATA style.*