The Locker Problem: An Explanation Using the Properties of Primes

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The Problem

A small-town high school, we'll call it Hamlet High, has exactly 100 students and exactly 100 lockers, one for each student. After a lesson on combinations, the Math 30 class at Hamlet High decided to conduct an experiment. With the authority that comes from seniority, they lined up all the students in the hall after school, each student next to his or her locker. Then, at their bidding, student 1 opened every locker, student 2 closed every second locker, student 3 changed the state of every third locker and so on, for what the Math 30 class hoped would be all 100 students. But after student 25, amidst rumblings of mutiny, changed the state of every 25th locker, the students finally revolted and headed for Hamlet's pizza place. The Math 30 class, left with an unfinished experiment, might have abandoned it. But after studying and arguing over the incomplete results, they concluded that they could predict a priori which lockers would be open without having to coerce a single student's cooperation. Moreover, they were audacious enough to declare that the pattern they had found could be extended to 1,000 mutinous students, or even to 10,000 such students.

What pattern did the Math 30 class discover? What is its mathematical basis?

A Solution

The Math 30 class concluded that, at the end of the experiment, the lockers whose numbers were perfect squares, that is, lockers 1, 4, 9, 16, 25, 39, 49, 64, 81 and 100, would have been open and the remainder closed. Alan constructed a computer program with a 100 × 100 array to mimic the experiment, but Theresa, who, along with a great many mathematicians, was of the opinion that a computer program did not constitute a proof, went further. She discovered that perfect squares have an odd number of factors, whereas other counting numbers have an even number of factors. The insight behind her discovery came from examining the factors of the numbers 36 and 40.

36 has factors 1, 2, 3, 4, 6, 9, 12, 18 and 36. Thus locker number 36 would have its state changed by students 1, 2, 3, 4, 6, 9, 12, 18 and 36 and not by any other students. Locker 36, having had its state changed an odd number of times, would end up in a state opposite to its initially closed state—that is, open.

Theresa pursued the analysis further. The prime factorization of 36 is

$$36 = 2^2 \cdot 3^2$$

She constructed the factors of 36 from this prime factorization as follows:

$$\begin{array}{lll} 2^0 \cdot 3^0 = 1 & 2^0 \cdot 3^1 = 3 & 2^0 \cdot 3^2 = 9 \\ 2^1 \cdot 3^0 = 2 & 2^1 \cdot 3^1 = 6 & 2^1 \cdot 3^2 = 18 \\ 2^2 \cdot 3^0 = 4 & 2^2 \cdot 3^1 = 12 & 2^2 \cdot 3^2 = 36 \end{array}$$

These factors of 36 were obtained systematically by using combinations of 2s and 3s: no 2s, one 2, or two 2s in combination with either no 3s, one 3, or two 3s.

Theresa then determined the prime factorization of 40:

$$40 = 2^3 \cdot 5^1$$

From this prime factorization, she constructed the factors of 40 as follows:

$$\begin{array}{ll} 2^0 \cdot 5^0 = 1 & 2^0 \cdot 5^1 = 5 \\ 2^1 \cdot 5^0 = 2 & 2^1 \cdot 5^1 = 10 \\ 2^2 \cdot 5^0 = 4 & 2^2 \cdot 5^1 = 20 \\ 2^3 \cdot 5^0 = 8 & 2^3 \cdot 5^1 = 40 \end{array}$$

These factors are the possible combinations of 2s and 5s: no 2s, one 2, two 2s, or three 2sin combination with no 5s or one 5. Since there are 8 such factors, locker 40 would have its state changed an even number of times and would be in the same state in which it started—that is, closed.

The Mathematician's Solution

Theresa ended her initial investigation here but later generalized the solution as follows.

As with 36, so with any perfect square. Suppose that N is a perfect square. Then its prime factors each occur an even number of times. So let

$$N = p_1^{2k_1}, p_2^{2k_2} \ldots p_t^{2k_t}$$

where each p_i is a prime and each k_i is a counting number (making $2k_i$ an even counting number). Then the number of factors of N is $(2k_1 + 1)(2k_2 + 1) \ldots (2k_t + 1)$. This is a product of odd numbers and therefore odd. Thus locker N will have its state changed an odd number of times. Having started out closed, it will end up open.

As with 40, so with any counting number that is not a perfect square. Suppose that M is such a number. Then at least one of the prime factors of M occurs an odd number of times. Suppose that prime factor is p_j and that it occurs $2k_j - 1$ times. Let

$$M = p_1^{k_1}, p_2^{k_2} \ldots p_j^{2k_j - 1} \ldots p_t^{k_t}$$

where each p_i is prime and each K_i a counting number. Then the number of factors of M is $(k_1 + 1)(k_2 + 1) \ldots (2k_j) \ldots (k_t + 1)$. This product has an even factor, $2k_j$ (and perhaps others as well) and is therefore even. Thus locker M has its state changed an even number of times and ends up in its starting state, namely closed.

Reference

Stewart, B. M. *Theory of Numbers*. 2d ed. New York: Macmillan, 1965.

Any book on number theory should contain some treatment of the properties of primes. The Stewart book, pages 62–68, is one that I happen to have in my office library.

The Window

A window, whose four sides are each 1 m long, is to be divided into eight individual panes using cross pieces. Each pane is to be 50 cm × 50 cm. Can this problem be solved?

Value of Sheep

Two shepherds owning a flock of sheep agree to divide its value: A takes 72 sheep and B takes 92 sheep and pays A $350. What is the value of a sheep?