The Ancient Problem of Trisecting an Angle

Sandra M. Pulver

The problem of trisecting an angle dates back to the ancient Greeks, and as early as the 5th century BCE, Greek and Muslim geometers devoted much time to this puzzle. This problem is one of the Three Famous Problems, which also include doubling the cube and squaring the circle. These three great construction problems of geometry could not be solved using an unmarked straightedge and compass stone, the only implements sanctioned by the ancient Greeks. But it was not until the 19th century that advances in the algebra of the real number system allowed us to make instruments which made possible these constructions that were impossible with the straightedge and compass alone.

This problem is certainly the simplest one of the three famous problems to comprehend, and because the bisection of an angle presented no difficulty to the geometers of antiquity, there was no reason to suspect that its trisection might prove impossible.

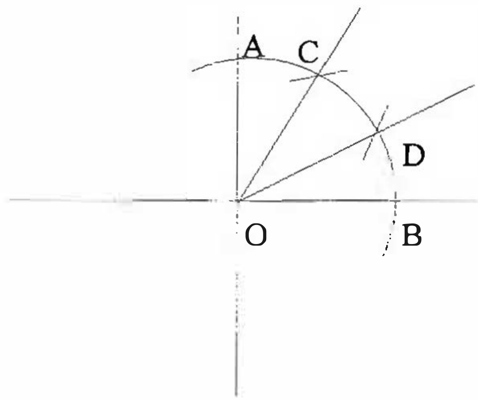
The multisection of a line segment with Euclidean tools is a simple matter, and it may be that the ancient Greeks were led to the trisection problem in an effort to solve the analogous problem of mutisection an angle. Or perhaps, more likely, the problem arose in efforts to construct a regular nine-sided polygon, where the trisection of a 60° angle is required.

The angle trisection problem is not entirely unsolvable using the classical method of compass and straightedge. Even the Greeks knew this, but they were searching for a generalized construction (such as one of the angle bisection) that could be used to trisect any angle.

Actually, there are an infinite number of angles that can be trisected. Among this group are angles whose degree measure equals $360/n$ where n is an integer not evenly divisible by three.

That is, a 90° angle can be trisected because n in $360/n$ is four, which is not evenly divisible by three. Figure 1 is a trisected 90° angle.

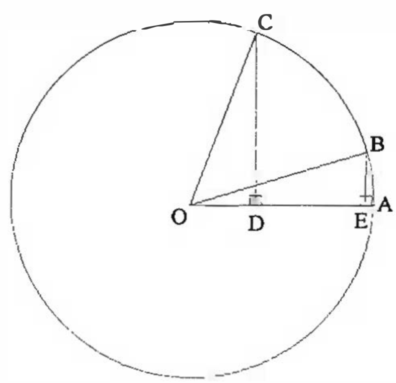
Figure 1
Trisection of a 90° Angle, ∡ AOB



The trisection of the 90° angle can be done quite simply using the following method:

Construct a 90° angle. Then, draw arc AB. Without changing the size of the compass opening, place the compass at point B and draw an arc intersecting arc AB at point C. Line OC is the line trisecting right angle AOB. Line OD also trisects right angle AOB using the same method outlined above, and placing the compass at point A. Line OD also bisects 60° angle COB.

Figure 2



However, there are, of course, an infinite number of angles that cannot be trisected by means of compass and straightedge. These are angles whose degree measures are equal to $360/n$, where n is an integer divisible by three; for example, a 60° angle cannot be trisected because n, in $360/n$, would be six, which is divisible by three. To prove that general angle trisections are impossible with just an unmarked straightedge and compass, we use the special case of a 60° angle.

In Figure 2, suppose $\measuredangle$ COA = 60° and $\measuredangle$ BOA = 20°. For the proof we make use of the following trigonometric identity:

$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

Let $3\theta = 60°$ and let $x = 2\cos\theta = 2\cos 20°$. Then

$$\cos 60° = \frac{x^3}{2} - \frac{3x}{2}$$

$$2\left(\frac{1}{2}\right) = \left(\frac{x^3}{2} - \frac{3x}{2}\right)2$$

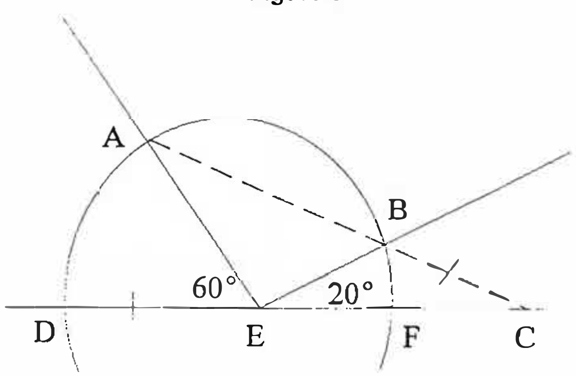
$$1 = x^3 - 3x$$

$$x3 - 3x - 1 = 0$$

This cubic equation is irreducible. Thus, its roots cannot be constructed with a straightedge and compass. From this, we can conclude that the construction of trisecting the general angle cannot be performed by the use of the straightedge and compass alone.

One ingenious method for trisecting angles was presented by Archimedes who used the marking of two points on a straightedge to mark off a line segment (that is, not using the classical rules of compass and straightedge). Figure 3 is a trisected 60° angle.

Figure 3

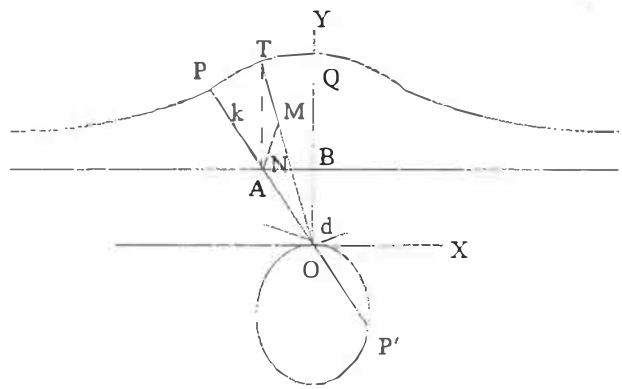


Archimedes' method is the following: Draw angle AED. Draw a semicircle through the angle whose radius is the same length of DE. Extend DE to the right. With the compass open to the length of radius DE, hold the legs of compass against the straightedge and hold the straightedge so it passes through point A. Adjust the straightedge till the points marked by the compass intersect points B and C (where BC is equal to DE). Arc BF is one third arc AD. Since central angles are congruent in degree measure to their intercepted arcs, $\measuredangle$ BEF is one third the degree measure of $\measuredangle$ AED.

Another method of trisecting an angle is by using the Conchoid of Nicomedes (c. 1800 BCE). To define a conchoid, we refer to Figure 4.

Figure 4

Conchoid of Nicomedes



Nicomedes took a fixed point O, which is d distant from a fixed line AB, and drew OX parallel to AB and OY perpendicular to OX. He then took any line OA through O and on OA made AP = AP' = k, a constant. Then the locus of points P and P' is a conchoid. The equation of the curve is

$(x^2 + y^2)(x - d)^2 - k^2x^2 = 0.$

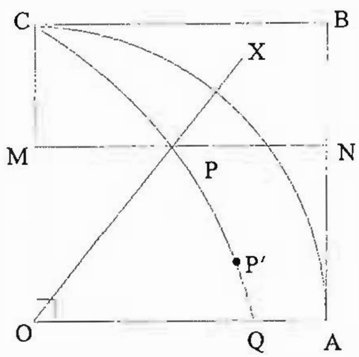
To trisect a given angle, let $\measuredangle$ YOA be the angle to be trisected. From point A, construct AB perpendicular to OY. From point O as pole, with AB as a fixed straight line, 2(AO) as a constant distance, construct a conchoid to meet OA produced at P and to cut OY at Q. At A, construct a perpendicular to AB meeting the curve at T. Draw OT and let it cut AB at N. Let M be the midpoint of NT.

Then MT = MN = MA.
But NT = 2(OA) by construction of the conchoid.
Hence MA = OA.
Hence $\measuredangle$AOM = $\measuredangle$AMO = 2 $\measuredangle$ATM = 2 $\measuredangle$TOQ.
That is, $\measuredangle$AOM = $^2/_3$ $\measuredangle$YOA, and $\measuredangle$TOQ = $^1/_3$ $\measuredangle$YOA.

Hippias of Elis (b. 460 BCE) wrestled with this problem and, realizing the inadequacy of ruler-and-compass method, resorted to other devices. These involved the use of curves other than the circle, and the one employed by Hippias was the quadratrix, so called because it serves equally well for the problem of quadrature (squaring the circle) as for the dividing of an angle into three, or indeed, any number of equal parts.

The quadratrix of Hippias may be defined as follows: Let the radius OX of a circle rotate uniformly about the centre O from OC to OA, at right angles to OC. At the same time, let a line MN parallel to OA move uniformly parallel to itself from CB to OA. The locus of the intersection P of OX and MN is the quadratrix (Figure 5).

Figure 5
Quadratrix of Hippias



In the trisection of an angle, X is any point in the quadrant AC. As the radius OX revolves at a uniform rate from OC to OA, MN always remains parallel to OA. Then if MN is one nth of the way from CB to OA, the locus of point P, the intersection of OX and MN, is one nth of the way from OC around to OA. If, therefore, we make CM = $^1/_3$ (CO), MN will cut CQ at a point P such that OP will trisect the right angle. In the same way, by trisecting OM we can find a point P' on CQ such that OP' will trisect angle AOX, and so for any other angle. This method evidently applies to the multisection as well as to the trisection of an angle.

In this article I have cited only three of the most ancient methods. There are many other ways to trisect an angle by using other techniques such as the "Tomahawk" and the Mira.

Other ingenious mathematicians of recent times have developed original methods to trisect angles. Leo Moser from the University of Alberta trisected angles with the use of an ordinary watch. He said that if the minute hand passed over an arc equal to four times the measure of the angle to be trisected, the hour hand would move through an arc exactly one third the measure of the given angle to be trisected. Alfred Kempe, a London lawyer, developed a linkage method of folding parallelograms so that the two opposite sides cross.

Bibliography

Allman, G. J. *Greek Geometry from Thales to Euclid.* New York: Arno, 1976.

Bold, B. *Famous Problems of Mathematics, A History of Constructions with Straight Edge and Compass.* New York: Van Nostrand, Reinhard, 1969.

Scott, J. F. *A History of Mathematics from Antiquity to the Beginning of the Nineteenth Century.* London: Taylor & Francis, 1975.

Smart, J. R. *Modern Geometries.* Pacific Grove, Calif.: Brooks/Cole, 1994.

Smith, D. E. *History of Mathematics.* Volume 1, 2. New York: Dover, 1923, 25.

Tuller, A. *A Modern Introduction to Geometries.* Princeton, N.J.: Van Nostrand, 1967.

Yates, R. C. *The Trisection Problem.* Baton Rouge, La.: Franklin, 1942.

Five-Digit Prime Numbers

Is it possible to find a five-digit prime number whose sum of the digits is 21?