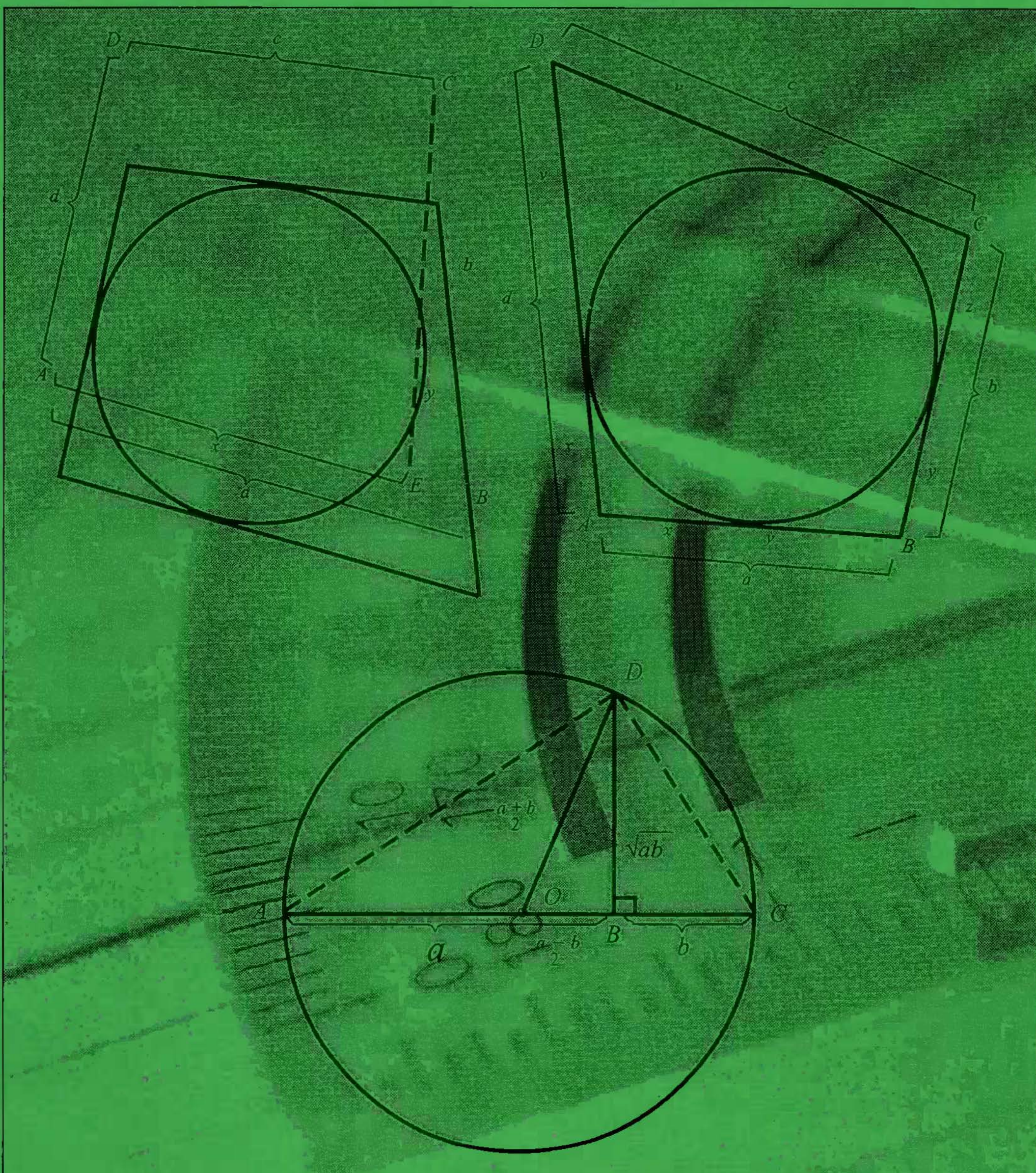
delta-k

JOURNAL OF THE
MATHEMATICS COUNCIL
OF THE ALBERTA
TEACHERS' ASSOCIATION

Volume 36, Number 2 June 1999



GUIDELINES FOR MANUSCRIPTS

delta-K is a professional journal for mathematics teachers in Alberta. It is published to

- promote the professional development of mathematics educators, and
- stimulate thinking, explore new ideas and offer various viewpoints.

Submissions are requested that have a classroom as well as a scholarly focus. They may include

- personal explorations of significant classroom experiences;
- descriptions of innovative classroom and school practices;
- reviews or evaluations of instructional and curricular methods, programs or materials;
- discussions of trends, issues or policies;
- a specific focus on technology in the classroom; and
- a focus on the curriculum, professional and assessment standards of the NCTM.

Manuscript Guidelines

1. All manuscripts should be typewritten, double-spaced and properly referenced.
2. Preference will be given to manuscripts submitted on 3.5-inch disks using WordPerfect 5.1 or 6.0 or a generic ASCII file. Microsoft Word and AmiPro are also acceptable formats.
3. Pictures or illustrations should be clearly labeled and placed where you want them to appear in the article. A caption and photo credit should accompany each photograph.
4. If any student sample work is included, please provide a release letter from the student's parent allowing publication in the journal.
5. Limit your manuscripts to no more than eight pages double-spaced.
6. A 250–350-word abstract should accompany your manuscript for inclusion on the Mathematics Council's Web page.
7. Letters to the editor or reviews of curriculum materials are welcome.
8. *delta-K* is not refereed. Contributions are reviewed by the editor(s) who reserve the right to edit for clarity and space. **The editor shall have the final decision to publish any article.** Send manuscripts to Klaus Puhlmann, Editor, PO Box 6482, Edson, Alberta T7E 1T9; fax 723-2414, e-mail klaupuhl@gyrd.ab.ca.

Submission Deadlines

delta-K is published twice a year. Submissions must be received by August 31 for the fall issue and December 15 for the spring issue.

MCATA Mission Statement

Providing leadership to encourage the continuing enhancement of teaching, learning and understanding mathematics.

delta-K

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GUEST EDITORIAL

What is mathematics? I suspect that if Pontius Pilot were alive today, he would not stay around for an answer. However, I would like to take this opportunity to play with the question a bit.

My first thought is that a multifaceted answer is called for. Mathematics is many things to many people, which gives rise to my second thought: mathematics, at least as most of us know it, is fundamentally a human enterprise. As such, it is also a cultural enterprise. All known human civilizations have had mathematics as part of their heritage. It is as pervasive as language. A nice third thought: mathematics is a language, one that has a higher degree of universality than other languages around the globe.

It is tempting to say that mathematics is a set of problems, but I do not like that. The word *problems* has a mildly negative connotation. I prefer saying that mathematics is a set of questions, which has the additional advantage of implying that someone must ask the questions.

Let's come full circle for a moment. What is mathematics? I wonder what answers a Grade 3 class would give? A Grade 10 class? I wonder what insights such answers might give educators?

What are some of the essential ideas of mathematics? This is a good question for adults. I wish it were a good question for children, but I suspect they would find it too confusing. Then again, I might be wrong. Let me attempt to enumerate a few ideas that I consider essential, like numbers. And this leads to the idea that many kinds of numbers exist. Have we identified them all, or does the future hold new surprises? The idea of proof would appear to be important, but probably not for many children. Infinity is a neat idea, one that children can enjoy playing with. Probability, or chance, is another idea that many children can relate to. What about the idea of variable? Of function? Or fractal? Courses about mathematics should be clear about the ideas they offer.

Mathematics is also about attitude. Both mathematics and language have a developmental aspect. Language, and attitudes about language, tend to improve with age, at least during the school years. What about mathematics? Why are there so few mathematics majors in university? Why do so few people have mathematics as a recreational hobby?

How many dead mathematicians can you name? In addition to the name, what else can you say about the person? How many living mathematicians can you name? How many female mathematicians?

What constitutes a good question in mathematics? What types of questions could a Grade 5 class generate? A Grade 12 class? This opens the door further (I began the process earlier) to the topic of children's mathematics. This is not so much a question of what they can do, as it is a question of their understanding. Children represent one way of categorizing a part of our culture. What about other cultures and other times? How did the Romans do arithmetic? How did the Mayans write their numbers? Could this idea be correlated with the social studies curriculum? Even though the Klingons have a language, I do not recall hearing if they have a different mathematical system.

I recently reread *A History of Reading* (Manguel 1996). At one point, Manguel discusses the tradition of scholasticism from the Middle Ages, saying that it was "a method of preserving rather than eliciting ideas" (p. 73) and that "understanding was not a requisite of knowledge" (p. 76). This is followed by some quotes from Franz Kafka: "One reads in order to ask questions" (p. 89) and "A book must be the axe for the frozen sea within us" (p. 93).

Mathematicians are proficient at substitution. Try substituting *mathematics* for words like *reads* or *book* (with appropriate grammatical adjustments) in the Kafka quotations.

Mathematics is also about beauty. I remember the surprise a Grade 3 class showed when I said this. Images of fractals are fairly well known today, but there is intrinsic beauty in many line diagrams that children have produced with Logo procedures. There is also the beauty of an elegant procedure. "A rose by any other name would smell as sweet." What type of rose is mathematics? What type of rose is mathematics education?

I love mathematics. It leads to so many questions.

Reference

Manguel, A. *A History of Reading*. Toronto: Knopf, 1996.

Dale Burnett

From the President's Pen

Mathematics education is changing because our mathematical needs have changed over time. These changes include, but are not limited to, the role of technology, the needs of society, international competitiveness and what we know about how children learn. We must not look back because the "good ol' days" are gone forever.

People must look toward the future and so should organizations. The MCATA executive met the weekend of January 22–23. We renewed our commitment to provide leadership to encourage the continuing enhancement of the teaching, learning and understanding of mathematics. We have heard from many members their concerns regarding our fall conference. We have achieved our success and our influence by being responsive to our members and, as such, a good part of our January Thinkers meeting was devoted to working with John Thorpe, NCTM executive director, to establish ways to make our affiliation a strong and mutually beneficial one. Please continue to let your MCATA executive know about your concerns and how we can better serve your needs.

Cynthia Ballheim

The Right Angle

Daryl M. J. Chichak

This article has been changed from its original format. To receive the originally formatted document and other information on achievement tests and diploma exams, go to the Alberta Education Web site at http://ednet.edc.gov.ab.ca.

New Information on the Grade 6 Mathematics Achievement Test 1999

The Operations and Number Sense part of the Grade 6 Mathematics Achievement Test (Part A) is new for the 1998–99 school year. There are 30 multiple-choice questions on this test: 7 addition/subtraction questions, 7 multiplication/division questions, 8 connecting experiences questions and 8 number relationship questions. The questions are integrated within the test. The test was developed to be completed in 30 minutes; however, students may take an additional 10 minutes to complete the test. The students are *not* allowed to use manipulatives or calculators when answering the questions. The test was written Thursday, May 27, 1999, at 9 a.m. A sample test and answer key follow in this article.

Also included in this article are a vocabulary list and seven practice questions on Venn diagrams and stem-and-leaf plots.

Questions or comments regarding this article should be directed to Daryl M. J. Chichak, Grade 6 mathematics assessment specialist, at (780) 427-0010, toll-free 310-0000, e-mail dchichak@edc.gov.ab.ca.

Grade 6 Mathematics Vocabulary List 1999

The following is a list of some of the words that may be found in the 1999 Grade 6 Mathematics Achievement Test. Ensure that your students are familiar with the list.

archeological dig
calculate
century
clerk
comparison
contributes
counterclockwise
coupon
difference
digit
displace
dozen
equilateral triangle
equivalent
expensive
gable
heads/tails
hundreds
hundredths
isosceles triangle
juice concentrate
market price
most probable
ordered pairs
pentagonal
portion
prism
probability
product
quantity
quarters
quotient
remainder
right triangle
roller coaster
scalene triangle
sequence
stem-and-leaf plot
sum
sunstroke
tens
tenths
total cost
Venn diagram
vertices
whole number

Grade 6 Mathematics Test, Part A: Operations and Number Sense

Description

There are 30 multiple-choice questions on this test.

This test was developed to be completed in 30 minutes; however, you may take an additional 10 minutes to complete the test.

Instructions

- You are *not* allowed to use manipulatives or calculators when answering the questions.
- Make sure that the number of the question on your answer sheet matches the number of the question you are answering.
- Read each question carefully and choose the correct or best answer.

Example

How many sides does a triangle have?

A. 2
B. 3
C. 4
D. 5

Answer Sheet

(A) ● (C) (D)

- Use only an HB pencil to mark your answer.
- If you change your answer, erase your first mark completely.
- Try to answer all the questions.

Practice Questions

1. What is the sum of 87.5 and 12.5?
   A. 1 093.75
   B. 100.0
   C. 75.0
   D. 7.0

2. Calculate 2.5×100.
   A. 2.50
   B. 25.0
   C. 250
   D. 2 500

3. A fraction that is equivalent to $^{9}/_{10}$ is
   A. $^{90}/_{110}$
   B. $^{81}/_{90}$
   C. $^{56}/_{60}$
   D. $^{16}/_{20}$

4. The difference between 4 300 and 2 088 is
   A. 2 212
   B. 2 322
   C. 2 388
   D. 6 388

5. A percentage that is equivalent to 7/20 is
   A. 70.0%
   B. 35.0%
   C. 28.0%
   D. 14.0%

6. You have 180 cookies. This is equivalent to
   A. 15 dozen
   B. 18 dozen
   C. 20 dozen
   D. 30 dozen

7. Calculate $483.2 \div 8$.
   A. 604
   B. 406
   C. 60.4
   D. 40.6

8. Calculate $1 000.00 - $178.50.
   A. 821.50
   B. 922.50
   C. 932.50
   D. 1 178.50

9. Calculate 63 736 - 2 947.
   A. 34 266
   B. 59 211
   C. 60 789
   D. 66 683

10. A fraction that is equivalent to 75% is
    A. $^{17}/_{20}$
    B. $^{7}/_{10}$
    C. $^{4}/_{5}$
    D. $^{3}/_{4}$

11. You are paid $4.00/h to babysit. You work for 9.5 hours. How much do you earn?
    A. $40.50
    B. $38.00
    C. $36.00
    D. $13.50

12. When a number is divided by 7, the quotient is 8 remainder 6. What is the number?
    A. 42
    B. 50
    C. 56
    D. 62

13. The difference between $640.00 and $346.84 is
    A. $293.16
    B. $304.26
    C. $306.84
    D. $986.84
14. Calculate 38.7 - 29.007.
    A. 9.693
    B. 9.707
    C. 10.307
    D. 11.703
15. The total number of minutes in 3½ hours is
    A. 210 min
    B. 230 min
    C. 330 min
    D. 350 min
16. The number of nickels in $3.50 is
    A. 14
    B. 35
    C. 70
    D. 350
17. Calculate 7 263 ÷ 9.
    A. 78
    B. 87
    C. 807
    D. 870
18. If one item costs $1.20, what is the cost of 5 items?
    A. $3.80
    B. $5.00
    C. $6.00
    D. $6.20
19. Calculate 730 × 25.
    A. 511
    B. 1 825
    C. 5 110
    D. 18 250
20. How many metres are in 275 cm?
    A. 27 500 m
    B. 2 750 m
    C. 27.5 m
    D. 2.75 m
21. Calculate 3.74 + 2.9 + 48.6 + 0.28.
    A. 9.17
    B. 11.78
    C. 43.42
    D. 55.52
22. What is 98.875 rounded to the nearest hundredth?
    A. 100
    B. 99
    C. 98.9
    D. 98.88
23. A grandmother gives $96 to be shared by her 8 grandchildren. How much does each grandchild receive?
    A. $ 12
    B. $ 88
    C. $104
    D. $768
24. The sum of four numbers is 90. The first three numbers are 27, 38 and 15. What is the fourth number?
    A. 10
    B. 30
    C. 80
    D. 170
25. A pair of equivalent fractions are
    A. $^{7}/_{8}$ and $^{64}/_{72}$
    B. $^{1}/_{4}$ and $^{8}/_{32}$
    C. $^{3}/_{7}$ and $^{24}/_{63}$
    D. $^{3}/_{5}$ and $^{21}/_{40}$
26. Calculate 25 000 000 + 790 021.
    A. 24 209 979
    B. 25 790 021
    C. 54 002 100
    D. 104 002 100
27. The sum of two numbers is 13, and their product is 36. What are the two numbers?
    A. 8 and 5
    B. 7 and 6
    C. 4 and 9
    D. 3 and 10
28. You buy one item for $5.24, one for $12.56 and one for $28.95. What is your change from $50.00?
    A. $96.75
    B. $46.75
    C. $ 4.25
    D. $ 3.25
29. What is the product of 56 and 22?
    A. 78
    B. 112
    C. 224
    D. 1 232
30. What is 8.5 rounded to the nearest whole number?
    A. 9
    B. 8
    C. 7
    D. 5

Part B: Practice Questions

Use the following information to answer questions 1 to 3.

In Language Arts, the students played a word game in which the value of a word was determined by the value of the letters in the following list.

A's = $0.11
B's = $0.12
C's = $0.13
• •
• •
• •
Z's = $0.36

The teacher made the following stem-and-leaf plot of the value of the student's words.

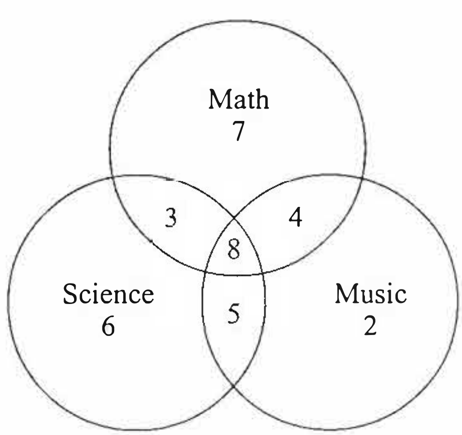
Range of Word Values ($)

$1.7	2 5 8
$1.6	0 1 2 4 8
$1.5	1 3 4 5 5 5 8
$1.4	3 3 4 4 5 5 7 8
$1.3	1 5 6 7 8 8 9 9
$1.2	2 3 6 6 9
$1.1	1 2 3 5
$1.0	0 4 8

1. What is the value of the most expensive word?
   A. $1.00
   B. $1.39
   C. $1.55
   D. $1.78
2. What is the value of the least expensive word?
   A. $1.00
   B. $1.39
   C. $1.55
   D. $1.78
3. What is the difference between the most expensive word and least expensive word?
   A. $0.70
   B. $0.78
   C. $2.78
   D. $2.86

Use the following Venn diagram to answer questions 4 to 7.

The Venn diagram below shows the subject preferences chosen by a group of students.



4. How many students are in the class?
   A. 27
   B. 29
   C. 33
   D. 35
5. How many students chose Science as their only preference?
   A. 3
   B. 5
   C. 6
   D. 8
6. How many students in total chose Math as one of their preferences?
   A. 22
   B. 15
   C. 8
   D. 7
7. How many students chose all three subject areas?
   A. 3
   B. 4
   C. 5
   D. 8

Key to Practice Questions

Part A: Operations and Number Sense

Question	Key	Category
1	B	A/S
2	C	M/D
3	B	NR
4	A	A/S
5	B	NR
6	A	CE
7	C	M/D
8	A	CE
9	C	A/S
10	D	NR
11	B	CE
12	D	M/D
13	A	CE
14	A	A/S
15	A	NR

Question	Key	Category
16	C	CE
17	C	M/D
18	C	CE
19	D	M/D
20	D	NR
21	D	A/S
22	D	NR
23	A	CE
24	A	A/S
25	B	NR
26	B	A/S
27	C	M/D (A/S)
28	D	CE
29	D	M/D
30	A	NR

A/S—addition/subtraction
M/D—multiplication/division
NR—number relationships
CE—connecting experiences

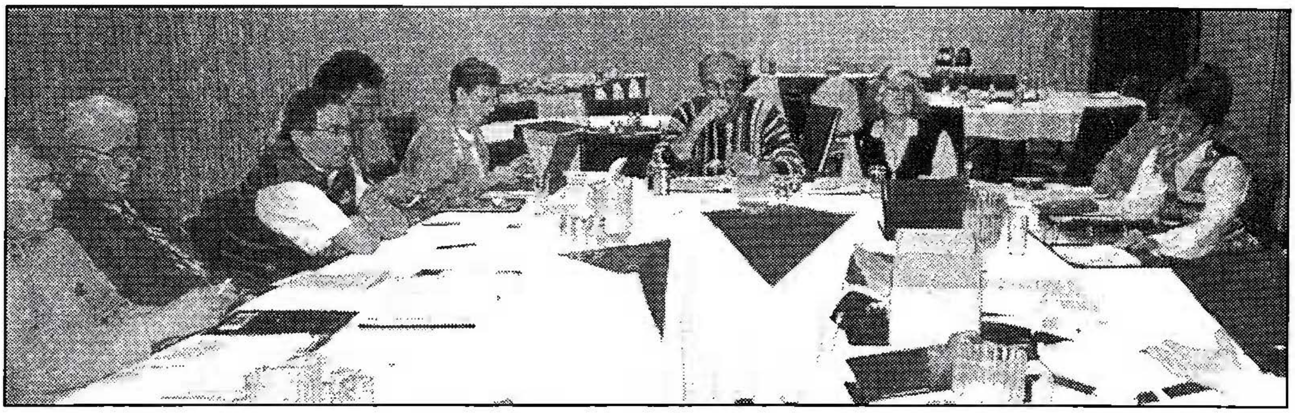
Part B: Multiple-Choice Practice Questions

1	D
2	A
3	B
4	D
5	C
6	A
7	D

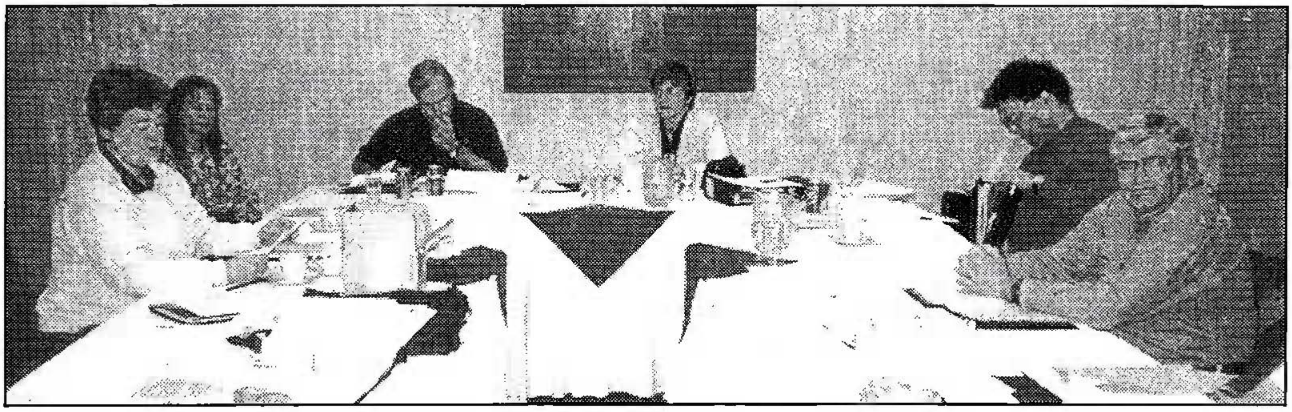
Your MCATA Executive at Work

The MCATA executive meets several times during the year. Meetings commence Friday evening and continue all day Saturday. Agenda are filled with many challenging items that keep the executive focused not only on its immediate tasks but also on its overall mission, which is providing leadership to encourage the continuing enhancement of teaching, learning and understanding mathematics. General members are encouraged and invited to take an active role in achieving our mission. Please contact any executive member with your ideas, proposals and suggestions. The following pictures were taken at the executive meeting September 18–19, 1998.

Note: The following MCATA executive members were present at the meeting but are not shown in the pictures: Bob Michie, 1998 conference chair; Rick Johnson, director; and Carol Henderson, PEC liaison.



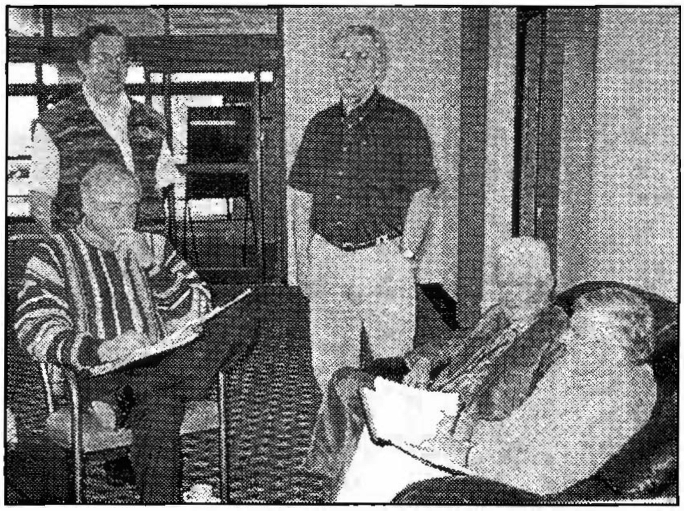
(l-r) Dick Pawloff, webmaster; Art Jorgensen, newsletter editor; Dale Burnett, faculty of education representative; Doug Weisbeck, treasurer; Donna Chanasyk, secretary; David Jeary, ATA staff advisor; Cynthia Ballheim, president; Florence Glanfield, past president; Geri Lorway, vice president.



(l-r) Elaine Manzer, director; Sandra Unrau, director; Graham Keogh, director; Lorraine Taylor, director; Daryl Chichak, Alberta Education representative and membership director; Dick Pawloff, webmaster.

(l-r) Lorraine Taylor, Betty Morris (vice president), Daryl Chichak and Donna Chanasyk discuss public relations issues facing MCATA.

Two new members join the executive! (l-r) Geri Lorway, principal of Iron River School, and Lorraine Taylor, Grade 9 mathematics teacher at Gilbert Patterson Community School in Lethbridge.



(l-r) Dale Burnett, David Jeary (seated), Klaus Puhlmann (journal editor and 1999 conference chair), Art Jorgensen and Dick Pawloff discuss publication issues.

(l-r) Geri Lorway, Elaine Manzer, Cynthia Ballheim and Graham Keogh discuss professional development issues for mathematics teachers.

12th Semi-Annual Alberta Mathematics Leaders' Symposium

The Semi-Annual Alberta Mathematics Leaders' Symposium has become a major professional development activity for mathematics teachers in Alberta. A record crowd attended this 12th symposium, which was held October 22, 1998, in Calgary just prior to the NCTM Canadian Regional Conference.

The symposium is organized jointly by MCATA and Alberta Education, and it deals with a wide range of topics relevant to the Alberta mathematics curriculum. The program for this symposium included a keynote address by Rita Janes, Canadian director of the National Council of Supervisors of Mathematics (NCSM); presentations on classroom assessment; Alberta Education updates on pure/applied mathematics, resource development and curriculum implementation timelines; a presentation by Nola Aitken, University of Lethbridge, on preservice teachers' mathematics ability; a panel discussion on professional development; and sharing sessions.

The following pictures have attempted to capture images from this symposium.

Rita Janes, NCSM Canadian director, delivers her keynote address.

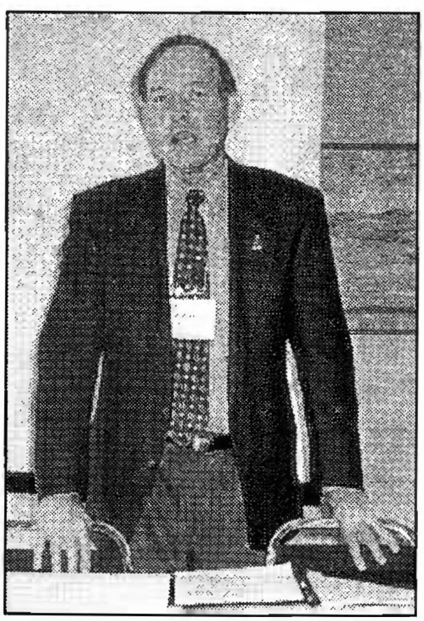
Rita Janes, NCSM Canadian director, and Betty Morris, symposium cochair and MCATA vice president.

Symposium delegates were attentive to the speakers.

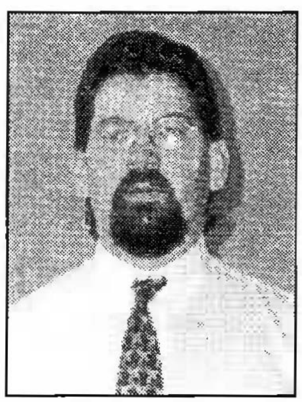
(l-r) Debbie Duvall, Alberta Education, and Betty Morris, MCATA vice-president, shared leadership duties for the symposium.

Kathy McCabe and Ron Zukowski, Alberta Education, present "Assessment in the Mathematics Classroom."

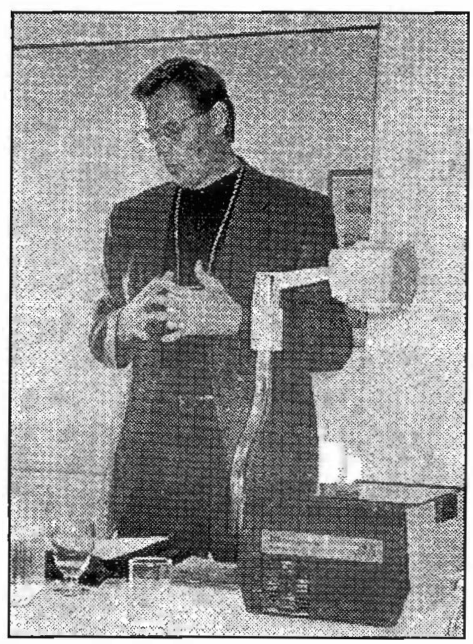
Technology in the mathematics classroom was the focus of a panel discussion.



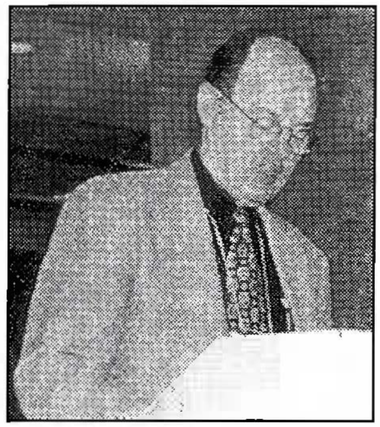
Graham Keogh, math teacher, Notre Dame School, Red Deer



Trevor Meister, math/technology consultant, Edmonton Public Schools



Carl Hauserman, principal, George Davison School, Medicine Hat



Hugh Sanders, Alberta Education, presents the Alberta Education update on mathematics.

Cynthia Ballheim, MCATA president, gave some closing remarks.

Nola Aitken, University of Lethbridge, presents "Preservice Teachers' Mathematics Ability: Is it Good Enough to Teach Elementary and Junior High School Mathematics?"

"How can we meet our professional development needs?" was the question at the centre of this panel discussion.

Wendy Fox, director of curriculum, Lethbridge School District No. 51

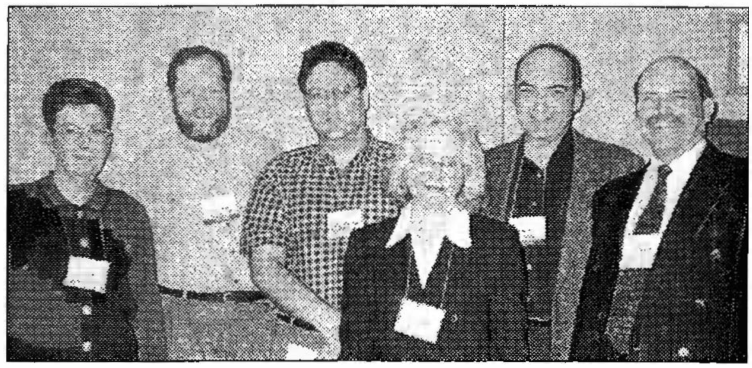
Louise Beerman, executive director, Southern Alberta Professional Development Consortium

Dorothy Negropontes, curriculum and instruction coordinator, Chinooks' Edge Regional Division No. 5

1998 NCTM Canadian Regional Conference

"Mathematics Education: Living the Challenge," was the theme of the 1998 NCTM Canadian Regional Conference, held October 23–24 in Calgary. The following is a potpourri of images from the conference.

George Ditto, conference cochair, opened the conference.



Some friends of MCATA, with MCATA President Cynthia Ballheim in the foreground.

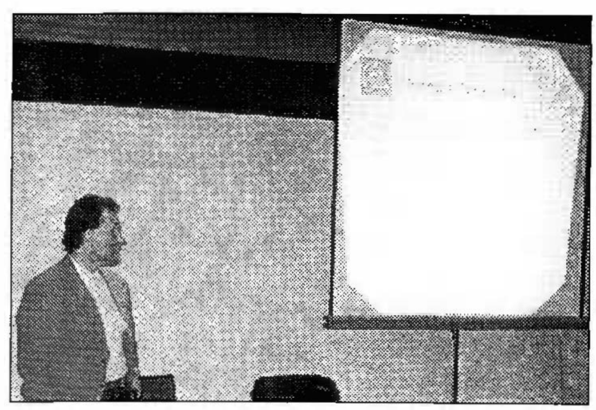
Glenda Lappan, NCTM president, gave the opening address.

Bob Michie, conference cochair, led the opening session.

Art Jorgensen, MCATA newsletter editor, makes last-minute preparations for his presentation.

Cynthia Ballheim, MCATA president, announces the recipients of the Friends of MCATA recognition.

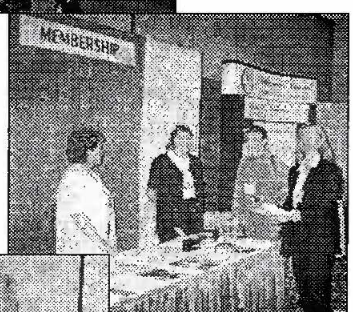
Sue Ditchburn delivering her keynote address, "44.722^2 and Beyond: Issues in Mathematics Teaching and Learning."



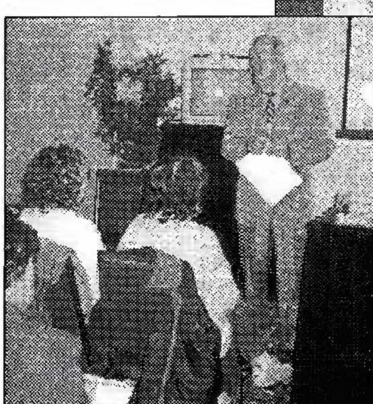
Ron Lancaster talking about "Revising the NCTM Standards for Year 2000 and Beyond."

(l-r) Bobbie Hanson and Elaine McInnes dealt with statistics and probability at the primary level.

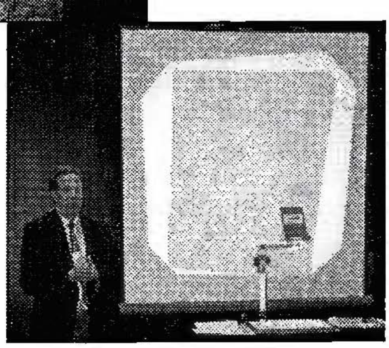
Daryl Chichak works with teachers on the use of manipulatives in the classroom.



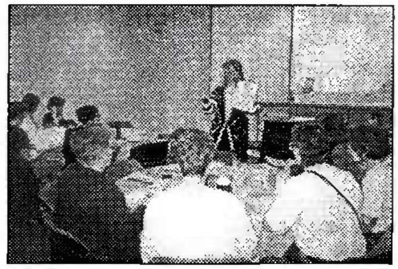
(l-r) Sherry Raymond, Karen Smith, Sandra Unrau and Sandra Lesveque check out the MCATA booth.



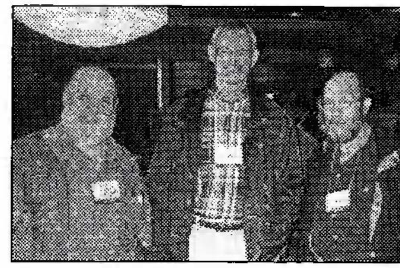
Werner Liedtke spoke about the importance of fostering the development of number sense.



Ralph Connelly presented "Probability Panorama: Flip, Spin and Roll Your Way to the Understanding of Concepts of Chance."

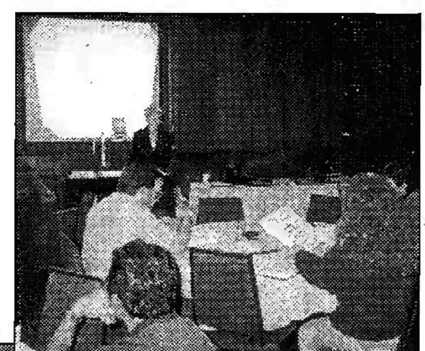


Shannon Lorenzo-Rivero kept teachers motivated with a session on mathematizing art at the elementary level.

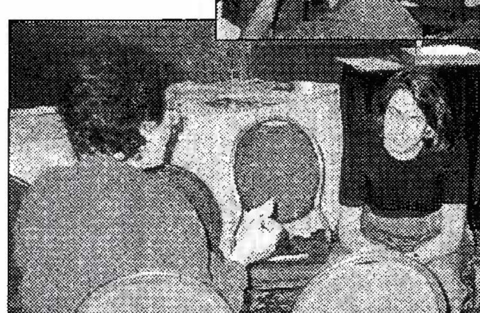


(l-r) Gary Nichols and Rick Armstrong, mathematics teachers from Grande Yellowhead Regional Division No. 35, and Ken Joseph, mathematics teacher from the Calgary Board of Education.

Ian deGroot presented "Mathematics Vignettes: Applications that Inspire Classroom Lessons."



Marie Hauk spoke about topological concepts and networks that are now part of the Western Canadian Protocol.



Nancy Moore offered a hands-on session on assessment in the integrated curriculum.

In this section, we will share your points of view on teaching mathematics and your responses to anything contained in this journal. We appreciate your interest and value the views of those who write.

In the following article, Nicholas Pyke expresses his views on the mathematics reform in Great Britain.

Lone Study "Disaster" for Maths: The Government's Numeracy Task Force Has Unveiled the Final Version of Its Rescue Plan

Nicholas Pyke

Letting children work on their own has been a disaster for mathematics teaching, according to the task force charged with devising a national maths strategy. Even now, some children spend four-fifths of a lesson working by themselves, said Professor David Reynolds, chair of the British government's numeracy task force.

The task force published the final version of its scheme to rescue primary school mathematics, a strategy with whole-class, "interactive" instruction at its core. The £60-million [Can$144-million] national numeracy strategy will begin in September 1999, one year later than its counterpart literacy strategy. The numeracy program features what Stephen Byers, the school standards minister, described as "tried and tested methods in a modern context." It includes a daily numeracy hour, a strong emphasis on mental arithmetic and times tables, and a training program for teachers.

Ministers have promised that by the end of a first Labour term in office, three-quarters of all 11 year olds will be reaching level 4 in the national curriculum maths tests. By 2007, says Labour, all pupils will be hitting this target.

"We are clear about what went wrong," said Professor Reynolds, speaking at the launch. "Methods of teaching introduced in the 1970s and 1980s had deleterious effects on maths in particular," he said. "All the research agrees that the one thing that badly affects performance in maths is letting children work on their own."

The final numeracy document shows few changes from the interim, consultation version previously published. Its recommendations include the following:

- A daily numeracy lesson 45–60 minutes long, depending on pupils' ages
- Training for every primary maths teacher on effective methods, including live demonstrations
- 300 local numeracy experts to advise schools
- A three-day training course in summer 1999 for headteachers, maths coordinators, and one other teacher from every school, plus a governor
- Intensive support for up to 60 percent of primaries
- Numeracy targets for 2002 to be agreed with each Local Education Authority
- A column to be added to primary league tables that notes the achievements of pupils who do not reach level 4

The money for the numeracy strategy will be announced as part of the forthcoming comprehensive spending review.

STUDENT CORNER

Mathematics as communication is an important curriculum standard, hence the mathematics curriculum emphasizes the continued development of language and symbolism to communicate mathematical ideas. Communication includes regular opportunities to discuss mathematical ideas and explain strategies and solutions using words, mathematical symbols, diagrams and graphs. While all students need extensive experience to express mathematical ideas orally and in writing, some students may have the desire—or should be encouraged by teachers—to publish their work in journals.

delta-K *invites students to share their work with others beyond their classroom. Such submissions could include, for example, papers on a particular mathematical topic, an elegant solution to a mathematical problem, posing interesting problems, an interesting discovery, a mathematical proof, a mathematical challenge, an alternative solution to a familiar problem, poetry about mathematics or anything that is deemed to be of mathematical interest.*

Teachers are encouraged to review students' work prior to submission. Please attach a dated statement that permission is granted to the Mathematics Council of the Alberta Teachers' Association to publish [insert title] in one of its issues of delta-K. *The student author must sign this statement (or the parents in the case of the student's being under 18 years of age), indicate the student's grade level, and provide an address and telephone number.*

No submissions were received for this issue. We look forward to receiving your submissions for the next issue.

The Missing Dollar

Three men stayed overnight at a hotel and paid $60 in advance for the room that all three shared. The next day, after the men had just left the hotel, the clerk discovered that the room was only $55 for the night. He ran after the men to return the $5. When he reached them, he gave each of the men $1 and kept $2 for himself.

Now each of the men had only paid a share of $19 for the room for a total of $57 (3 × $19). Add the $2 that was kept by the clerk, and the total becomes $59. Where is the missing dollar?

NCTM Standards in Action

Klaus Puhlmann

Learning to reason mathematically is fundamental to doing mathematics. The curriculum standards for school mathematics for Kindergarten to Grade 12 include mathematical reasoning and proof as one of five standards that describe mathematical processes through which students should acquire and use their mathematical knowledge. At each level, students should study mathematics in ways that include opportunities for mathematical reasoning and the construction of proofs.

At the primary level, the curriculum and evaluation standards suggest that the study of mathematics should emphasize reasoning with a focus on

- drawing logical conclusions about mathematics;
- using models, known facts, properties and relationships to explain students' thinking;
- justifying students' answers and solution processes;
- using patterns and relationships to analyze mathematical situations; and
- making students believe that mathematics makes sense. (NCTM 1989)

However, the standards do not suggest that formal reasoning strategies be taught at the elementary level. Instead, at this level, mathematical reasoning should centre on informal thinking, conjecturing and validating that help students to see that mathematics makes sense. It typically involves questions such as, Why do you think that is the correct answer? or Do you think that you would get the same answer if you added the parts in a different order? Enhancing students' confidence in their ability to reason and justify their thinking is critical, because as they move through the grades, students will begin to see that mathematics is not simply memorizing rules and procedures but that it makes sense, is logical and is enjoyable. It is also critical that the teacher nurture a climate in which the students have a genuine respect and support for one another's ideas. Statements made by both teacher and students should be open to question, reaction and elaboration from others. Students should be constantly challenged and encouraged to justify their solutions, thinking processes and conjectures in many ways. The use of manipulatives and models is an effective way of engaging students as active participants in the learning process. Models and manipulatives also provide students with concrete objects to gain a better understanding of the mathematical ideas and concepts.

At the middle school level, the curriculum and evaluation standards suggest that the study of mathematics should emphasize reasoning with a focus on

- recognizing and applying deductive and inductive reasoning;
- understanding and applying reasoning processes, with special attention to spatial reasoning and reasoning with proportions and graphs;
- making and evaluating mathematical conjectures and arguments;
- validating their own thinking; and
- appreciating the pervasive use and power of reasoning as a part of mathematics. (NCTM 1989)

Students should be provided with opportunities to explain their own reasoning and such explanations should be followed with questions: Why? What if...? Can you give me a counterexample? Can you give me an example of...? Do you see a pattern? Is it always true? Sometimes true? Never true? How do you know? Such questions prompt students to validate and value their own thinking. Having students identify patterns provides them with a powerful problem-solving strategy. It is also the essence of inductive reasoning. These patterns, in turn, can lead to conjectures about the problem. Students at these grade levels should be exposed to problem situations that are challenging but within reach. This may also lead to the use of computers for specific problems.

Students should be introduced to many kinds of mathematical reasoning. They can use reasoning to illustrate when something always, sometimes or never works. Situations involving counterexamples are also useful and important. Throughout the grades at this level, students should also develop the ability to reason proportionally, a process that requires a great deal of time for its development.

Reasoning must pervade all mathematical activities if students are to develop the ability to conjecture and to demonstrate the logical validity of conjectures. Teachers need to be mindful that at this level students still need many concrete materials to support their reasoning; this is especially true for spatial reasoning. Whether it is the use of technology or the presentation of challenging mathematical situations, students need the freedom to explore, conjecture, validate and convince others if they are to develop the ability to mathematically reason.

At the Grades 9–12 level, the mathematics curriculum should include numerous and varied experiences that reinforce and extend logical reasoning skills with an emphasis on

- making and testing conjectures;
- formulating counterexamples;
- following logical arguments;
- judging the validity of arguments;
- constructing simple valid arguments;

and, for students intending to go on to postsecondary studies,

- constructing proof of mathematical assertions, including indirect proofs and proofs by mathematical induction. (NCTM 1989)

For students at this level, inductive and deductive reasoning are required individually and in concert in all areas of mathematics. The goal is for students to experience both forms of reasoning in mathematics and in situations outside mathematics. For example, conjecturing by generalizing from a pattern of observations made in particular cases (inductive reasoning) and then testing the conjecture by constructing either a logical verification or a counterexample (deductive reasoning) are important mathematical experiences for students.

A second goal of this standard is to expand the role of reasoning, which is currently addressed in geometry only but which needs to be emphasized in all mathematics courses for allstudents. In addition, students planning postsecondary studies need to learn more formal methods of proofs.

The third goal of this standard is to give increased attention to proof by mathematical induction, the most prominent proof technique in discrete mathematics. In Grades 9–12, as the depth and complexity of content is increased, this emphasis on the interplay between conjecturing and inductive reasoning and the importance of deductive verification should be maintained. Furthermore, it is most appropriate that students see the application of various forms of reasoning to areas outside mathematics. The potential for transfer between mathematical reasoning and the logic needed to resolve issues in everyday life can be enhanced by explicitly subjecting assertions about daily affairs to analysis in terms of the underlying principles of reasoning.

Assessment of students' ability to reason mathematically has a critical part in ensuring that students have actually understood the different types of reasoning. To determine students' understanding and use of different types of reasoning, assessment must be focused on how students use all types of reasoning appropriate for their grade level. Such assessments of students' ability to reason mathematically should provide evidence that they can

- use inductive reasoning to recognize patterns and form conjectures;
- use reasoning to develop plausible arguments for mathematical statements;
- use proportional and spatial reasoning to solve problems;
- use deductive reasoning to verify conclusions, judge the validity of arguments and construct valid arguments;
- analyze situations to determine common properties and structures; and
- appreciate the axiomatic nature of mathematics. (NCTM 1989)

Teachers need to be aware that while all aspects of reasoning can be used at any grade level, some aspects of reasoning might be more appropriate than others at a given grade level. "Reasoning" and "proof" should not be taught in isolation, but rather, reasoning and proof must be a consistent part of students' mathematical experiences from Kindergarten to Grade 12.

The three articles that follow provide excellent examples of how reasoning can be developed through consistent use in the classroom.

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Learning to Reason from Lewis Carroll

Andrea Rothbart

Lewis Carroll, the author of *Alice in Wonderland*, was not only a writer but a mathematician. In particular, he devised scores of charming logic puzzles, similar to this one, which was devised by the author:

1. If I can work a logic problem, then anyone can.
2. I do not recommend solving problems that I cannot do.
3. None of the problems that I develop are boring.
4. The only logic problems that I do *not* recommend solving are those that are boring.

What logical conclusion can you draw by using *all four* of these premises?

I have posed this question to many classes; invariably, students come up with several different responses. The dilemma becomes, which responses are, in fact, logical conclusions of all the premises and which are not? How can we decide?

Mathematical logic to the rescue!

We use a powerful technique commonly applied to "empirical" problems. We translate the problem into a mathematical one, solve the mathematical problem, and then translate our answer back into the context of the original problem. Of course, our answer is only as good as the fit between our mathematical model and the problem.

In particular, we translate each of the foregoing four premises into symbolic sentences. Then we manipulate these symbolic sentences using "rules of inference," which are discussed later. After deriving a symbolically expressed answer, we translate this answer back into ordinary English (see Figure 1).

As you will soon discover, the tough part is translating from English sentences to symbolic sentences. By comparison, the mathematics involved seems like a piece of cake.

Translating English Sentences to Symbolic Sentences

All the premises can be expressed as sentences of the form

If (blah), *then* (stuff).

Both "blah" and "stuff" are complete sentences. In fact, the first premise, "If I can work a logic problem, then anyone can," is already expressed as an if-then sentence. But how can we express the second premise, "I do not recommend solving problems that I cannot do," in if-then form? This question is linguistic, not mathematical. What do the words in the sentence mean, and how can we express this meaning in an if-then sentence? To figure out this problem, we need only to rely on our understanding of how ordinary English is used.

Here are some possibilities. Which, if any, of the following sentences do you think makes the same assertion as "I do not recommend solving problems that I cannot do"?

(A) If I can solve a problem, then I do not recommend it.
(B) If I cannot solve a problem, then I do not recommend it.
(C) If I can solve a problem, then I recommend it.
(D) If I cannot solve a problem, then I recommend it.

Okay, now, stomp your feet if you think that (B) is the correct answer. Good for you! Many people initially interpret the sentence to mean (C). This mistake is common—and comes from the extensive experience we all have in using language imprecisely. Ordinary communication is not a science. We do not all use language in the same way. At least with *spoken*

Figure 1
The Translation Process

(English) (tough)		(mathematics) (easy)	(English) (pretty easy)
LEWIS CARROLL PUZZLE ⇒	SYMBOLIC SENTENCES ⇒	SOLUTION ⇒	ENGLISH

language, we can question one another about our meanings and use body language and so on to help us communicate. But in *written* language, we have only the words, and so differing interpretations invariably arise.

However, we cannot afford ambiguity when we communicate mathematical ideas. When we do mathematics, therefore, we need to use language with great precision. In particular, the sentence "I do not recommend solving problems that I cannot do" describes the writer's response to problems that she cannot solve. The sentence makes no assertion about her response to problems that she can solve. She may not recommend those, either! Hence, (B) is a correct restatement of the premise and (C) is not. Incidentally, it is also correct to restate the premise as "If I recommend a problem, then I can solve it." I discuss the equivalence of this sentence with (B) later.

Next, let us consider the third premise: "None of the problems that I develop are boring." Try expressing it as an if-then sentence before reading on.

I agree with you if you wrote either

- If I develop a problem, then it is not boring

or

- If a problem is boring, then I did not develop it.

In general, when Carroll and I and mathematicians in general use the syntax

None of (junk) are (stuff),

we mean

If (junk), then not (stuff).

This sentence is the same as

If (stuff), then not (junk).

Consider premise 4, "The only logic problems that I do not recommend solving are those that are boring." This premise does not assert that I do not recommend any boring problems, but it does say that I do recommend the nonboring ones. Maybe I recommend all the neat problems and also some of the boring ones. Then, it is still the case that the *only* problems that I do not recommend are those that are boring. In other words, premise 4 can be expressed as

If I do not recommend a problem,
then it is boring

or equivalently, as

If a problem is not boring,
then I recommend it.

We now have the four premises expressed as if-then sentences. But English is bulky, and so we are going to abbreviate these sentences by using the following dictionary.

Dictionary: B: The problem is boring.
D: The problem was developed by me.
R: I recommend solving the problem.
W: I can work the problem.
A: Anyone can work the problem.

The English sentence "If I can work a problem, then anyone can" is abbreviated to

1. If W, then A.
   (Equivalently, if not A, then not W.)

The other premises are abbreviated as follows:

2. If not W, then not R.
   (Equivalently, if R, then W.)
3. If D, then not B.
   (Equivalently, if B, then not D.)
4. If not R, then B.
   (Equivalently, if not B, then R.)

Finally, we use the symbol "~" for "not" and the symbol "→" for "if-then" and further abbreviate

If not W, then not R

to

$\sim W \rightarrow \sim R$.

So the four premises are now expressed as symbolic sentences as follows:

1. $W \rightarrow A$
2. $\sim W \rightarrow \sim R$
3. $D \rightarrow \sim B$
4. $\sim R \rightarrow B$

Once we abbreviate, we can forget to what English sentences the W, A, R and B refer. We solve the problem using the abbreviations and then refer to our dictionary to translate our answer into ordinary English.

Solving by Using Rules of Inference

At the beginning of the article, I asked for a logical conclusion of the premises 1, 2, 3 and 4. Did you think that I was implying that only one conclusion was possible? If so, I do apologize for misleading you. But I never really stated that restriction explicitly, you know. In fact, infinitely many correct conclusions exist. For example, you could string the four premises together by inserting the word *and* between each two consecutive premises—not very interesting, but it works. However, Lewis Carroll had in mind only one conclusion, and it is derived from using the three rules of inference that I am about to describe.

Please forget our premises for the moment while I digress into an explanation of the rules of inference that we shall use to solve this puzzle as well as the Lewis Carroll puzzles printed later in this article. Actually, I was using the first rule of inference when I claimed that two ways can be found to translate each of our premises into if-then sentences:

Inference rule 1: A → B is logically equivalent to ~B → ~A.

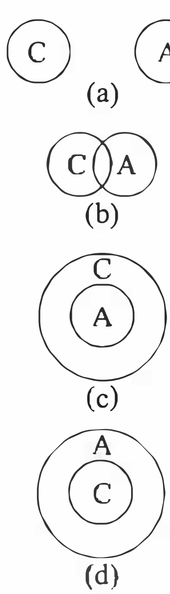
This rule of inference, called *contrapositive* (CP), asserts that any sentence of the form A → B can be replaced with the sentence ~B → ~A; conversely, ~B → ~A can be replaced with A → B. To convince yourself, consider the following assertions about my pet:

C: It is a cat.
A: It is an animal.

On a piece of paper, draw a circle and imagine that all the cats in the world are inside that circle. So, in particular, any cats that I may have are in that circle. Next, draw a circle containing all the animals in the world. If the circles look like those in Figure 2a, then cats are not animals. And if your circles look like those in Figure 2b, then some cats are not animals. And if your circles look like the ones in Figure 2c, then all animals are cats. So your circles should look like the circles in Figure 2d.

Figure 2d illustrates the assertion that all cats are animals, that is, that if it is a cat, then it is an animal. It is clear from the picture that if something is not an animal, that is, outside the circle of animals, it cannot be a cat. Indeed the picture illustrates that the sentences "If it is a cat, then it is an animal" and "If it is not an animal, then it is not a cat" make precisely the same assertion. That is, C → A is logically equivalent to ~A → ~C.

Figure 2
Illustration of Four Possible Relationships of Two Circles A and C



Inference rule 2: ~~A is logically equivalent to A.

This rule of inference, called ***double negation*** (DN), asserts that any sentence of the form "~~A" can be replaced with the sentence "A" and conversely that sentence "A" can be replaced with the sentence "~~A." To be convinced that this rule of inference is valid, consider any pair of English sentences of the form A and ~~A, say, the sentences that follow:

A: My father's first name is Harold.
~~ A: It is not true that my father's first name is not Harold.

Both sentences make the same assertion, and you can freely replace one with the other.

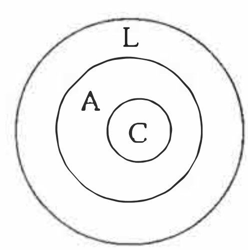
Inference rule 3: If A → B and B → C, then A → C.

The third, and final, rule of inference used in solving the Lewis Carroll puzzles is called *transitivity* (TR). Philosophers call it the "hypothetical syllogism." Let us refer back to my pet:

If it is a cat, then it is an animal. C → A.
If it is an animal, then it is a life form. A → L.

From these two premises we can infer the following (see Figure 3):

Figure 3
Illustrating the Transitivity Rule of Inference: If All Cats Are Animals and All Animals Are Life Forms, Then All Cats Are Life Forms.



If it is a cat, then it is a life form. C → L.

We are now ready to deduce a conclusion from our four premises:

1. W → A
2. ~W → ~R
3. D → ~B
4. ~R → B

Here is one of several ways of deriving our answer:

<u>Claim</u>	<u>Reason</u>
1. W → A	Premise
2. ~W → ~R	Premise
3. D → ~B	Premise

4. $\sim R \to B$	Premise
5. $B \to W$	CP (2)
6. $R \to W$	TR (5) and (1)
7. $\sim B \to R$	CP (4)
8. $\sim B \to A$	TR (7) and (6)
9. $D \to A$	TR (3) and (8)

Note that our final sentence was deduced by using all four premises. We used premise (1) and (the contrapositive of) premise (2) to infer (6), and we combined (the contrapositive of) premise (4) with (6) to infer (8). And then we combined premise (3) with (8) to infer our conclusion, $D \to A$.

Translating Back into English

Now we consult our dictionary. The letter D stands for "The problem was devised by me." The letter A stands for "Anyone can work the problem." Hence, we translate $D \to A$ to "If the problem was devised by me, then anyone can work the problem" or, in the lingo that human beings actually use, "Anyone can solve the problems I devise."

Let us solve a problem that actually was created by Lewis Carroll.

1. Babies are illogical.
2. Nobody is despised who can manage a crocodile.
3. Illogical persons are despised.

What can you deduce by using all three premises?

To solve by using the procedure we have developed here, we first restate each premise as an implication, that is, as an if-then sentence. Then we devise a dictionary and translate each English sentence into a symbolic sentence.

Premises 1 and 3 are fairly easy to express symbolically. Premise 1 states that if you are a baby, then you are illogical; premise 3 states that if you are illogical, then you are despised. Using the following dictionary, we obtain this group of sentences:

Dictionary: B: This person is a baby. I: This person is illogical. D: This person is despised. C: This person can manage a crocodile.

1. $B \to I$
3. $I \to D$

But premise 2 is likely to bewilder some of us. Does it assert that if you cannot manage a crocodile then you are despised? Or does it assert that if you are despised then you cannot manage a crocodile?

Note that these two assertions are very different. For example, "If it is a cat, then it is an animal" states something quite different from "If it is an animal, then it is a cat." The sentences $C \to A$ and $A \to C$ are called *converses*, and if you abbreviate a sentence as $C \to A$ when it is intended to mean $A \to C$, well, you will not get the same answers as the rest of us—unless you make several mistakes that somehow turn out to negate one another!

Anyway, let us go back to "Nobody is despised who can manage a crocodile." When I cannot clearly see how to proceed, I try to think of an English sentence with the same syntax whose meaning is clear to me. This strategy is an excellent approach: Remember it! Here I might think, "Nobody graduates from college who fails English." Surely I am not saying that anyone who passes English graduates, that is, "If you do not fail English, then you graduate," but rather, I am saying the converse: "If you graduate, then you did not fail English."

"Nobody graduates who fails English"	is equivalent to	"If you graduate, then you did not fail English."

$$G \to \sim E$$

Therefore,

"Nobody is despised who can manage a crocodile"	is equivalent to	"If you are despised, then you cannot manage a crocodile."

$$D \to \sim C$$

Incidentally, if you restated premise 2 as "If you can manage a crocodile, then you are not despised," you have created a sentence that is logically equivalent to "If you are despised, then you cannot manage a crocodile." The sentence $C \to \sim D$ is the contrapositive of $D \to \sim C$, and so either symbolic sentence accurately reflects premise 2.

Our deduction follows:

Claim	Reason
1. $B \to I$	Premise
2. $D \to \sim C$	Premise
3. $I \to D$	Premise
4. $B \to D$	TR (1) and (3)
5. $B \to \sim C$	TR (4) and (2)

We have used all three premises to deduce $B \to \sim C$. Translating back into ordinary English, we get, "If you are a baby, then you cannot manage crocodiles" or, in everyday lingo, "Babies cannot manage crocodiles."

Try your hand at the following puzzles devised by Lewis Carroll. The answers appear in the appendix.

(1)

1. My saucepans are the only things I have that are made of tin.
2. I find all of your presents very useful.
3. None of my saucepans are of the slightest use.

Dictionary: S: It is my saucepan; T: It is made of tin; P: It is your present; U: It is useful.

(2)

1. No potatoes of mine that are new have been boiled.
2. All of my potatoes in this dish are fit to eat.
3. No unboiled potatoes of mine are fit to eat.

Dictionary: B: My potato is boiled; E: My potato is edible; D: My potato is in this dish; N: My potato is new.

(3)

1. No ducks waltz.
2. No officers ever decline to waltz.
3. All of my poultry are ducks.

Dictionary: D: She is a duck; P: She is poultry; O: She is an officer; W: She is willing to waltz.

(4)

1. Everyone who is sane can do logic.
2. No lunatics are fit to serve on a jury.
3. None of your sons can do logic.

Dictionary: A: He can do logic; J: He is fit to serve on a jury; S: He is sane; C: He is your son.

(5)

1. Nobody who really appreciates Beethoven fails to keep silent while the *Moonlight Sonata* is being played.
2. Guinea pigs are hopelessly ignorant of music.
3. No one who is hopelessly ignorant of music ever keeps silent while the *Moonlight Sonata* is being played.

Dictionary: G: She is a guinea pig; I: She is hopelessly ignorant of music; S: She keeps silent while *Moonlight Sonata* is being played; A: She really appreciates Beethoven.

(6)

1. No goods in this shop that have been bought and paid for are still on sale.
2. None of the goods may be carried away unless labeled sold.
3. None of the goods are labeled sold unless they have been bought and paid for.

Dictionary: C: These goods in the shop may be carried away; B: These goods in the shop are bought and paid for; S: These goods in the shop have been labeled sold; O: These goods in the shop are on sale.

(7)

1. No boys under 12 are admitted to this school as boarders.
2. All of the industrious boys have red hair.
3. None of the day boys (nonboarders) learn Greek.
4. None but those boys under 12 are idle.

Dictionary: B: This boy is a boarder; I: This boy is industrious; G: This boy learns Greek; R: This boy has red hair; T: This boy is under 12.

(8)

1. Things sold in the street are of no great value.
2. Nothing but rubbish can be had for a song.
3. Eggs of the great auk are very valuable.
4. It is only what is sold in the streets that is really rubbish.

Dictionary: H: It may be had for a song; E: It is an egg of the great auk; R: It is rubbish; S: It is sold in the streets; V: It is very valuable.

(9)

1. No kitten that loves fish is unteachable.
2. No kitten without a tail will play with a gorilla.
3. Kittens with whiskers always love fish.
4. No teachable kitten has green eyes.
5. No kittens have tails unless they have whiskers.

Dictionary: E: This kitten has green eyes; F: This kitten loves fish; T: This kitten has a tail; U: This kitten is unteachable; W: This kitten has whiskers; G: This kitten is willing to play with a gorilla.

Teaching Notes

These puzzles not only give our students another example of the power of mathematics in solving problems but help them develop a greater sensitivity to language and reasoning. Students also find that these puzzles are entertaining, if we teachers do it right. What does not work is to try to lecture our students on correct procedures for translating from English to symbolic sentences. Instead, introduce a couple of puzzles to the class as a whole and allow time for students to debate how to restate the premises as if-then sentences. If your students are finding a premise particularly tricky to restate, suggest that they use the strategy of finding another sentence with the same syntax whose meaning *is* clear to them. Restating premises into if-then sentences gives my students—and sometimes their teacher!—the most difficulty, partly because some have not yet learned to distinguish between a statement of the form "If A, then C" and its converse "If C, then A." Draw Venn diagrams—the circles I used earlier in this article—and use such easy-to-understand sentences as "If something is a cat, then it is an animal" versus "If something is an animal, then it is a cat" to illustrate the difference. Emphasize the idea that when we state "If (stuff), then (junk)" we are not addressing what happens when stuff does not occur. For example, suppose that a father says to his son, "If you finish your homework by 8:00, then we will go to the movies." Father is *not* saying what will happen if the son does not finish by 8:00. Perhaps the father *means* no movie, but that is not what he said. Sorry, Dad!

Sometimes I have a student who insists that when he says "X," he really means something that to the listener is quite different. It is important to acknowledge that in ordinary discourse, we all speak somewhat loosely and that it is a manifestation of human intelligence to listen for what is implicitly, as well as explicitly, stated. (Okay, Dad, so I did understand what you meant!) However, I also point out that to solve these puzzles––and, in general, to think mathematically—we all need to be flexible enough to learn how language is used precisely by Carroll. Otherwise, we can never complete the step of translating into symbolic sentences, let alone go beyond it.

Developing a facility with expressing the contrapositive of an implication may take a little practice. But in my experience, once the premises have been expressed symbolically, few students have difficulty learning how to string them together to derive a conclusion and to restate the conclusion in ordinary English.

After doing a couple of examples with the whole class, invite them to work in small groups on several more puzzles. If possible, provide a facility with soundproof walls.

You can find more Lewis Carroll puzzles in several books, including *The Complete Works of Lewis Carroll* (New York: Random House, 1939). Better yet, ask your students to make up Lewis Carroll-like puzzles. Again, I have them work together in small groups and tell them that I relish such words as *none* and *only*, but most of all, I relish puzzles that *work*. So I suggest that they try out their puzzles on one another before giving them to me. Of course, checking a pile of these puzzles can be very time-consuming, so I have developed a "fast and dirty" approach. It is dirty because it is a sloppy use of the implication sign; but it works, so I do it anyway. After translating the English sentences into symbolic ones, I string them together horizontally. For example, suppose that the premises are $B \rightarrow \sim C$, $\sim A \rightarrow \sim D$, $\sim B \rightarrow D$, and $E \rightarrow C$. Write down any one of them, and then start hooking the other premises onto it. Say that I start with $B \rightarrow \sim C$. Hook onto the right of it the contrapositive of $E \rightarrow C$, getting $B \rightarrow \sim C \rightarrow \sim E$. Next hook onto the left the contrapositive of $\sim B \rightarrow D$, obtaining $\sim D \rightarrow B \rightarrow \sim C \rightarrow \sim E$. Continue stringing on the premises until you have exhausted them; $\sim A \rightarrow \sim D \rightarrow B \rightarrow \sim C \sim\rightarrow \sim E$, so the conclusion is $\sim A \rightarrow \sim E$.

Appendix
Answers to Puzzles

1. Your presents are not made of tin.
2. None of my potatoes in this dish are new.
3. None of my poultry are officers.
4. None of your sons are fit to serve on a jury.
5. Guinea pigs never really appreciate Beethoven.
6. No goods in this shop that are still on sale may be carried away.
7. Only red-haired boys learn Greek in this school.
8. An egg of the great auk cannot be had for a song.
9. Kittens with green eyes will not play with gorillas.

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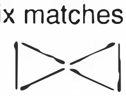
Match Triangles

Six matches form two equilateral triangles.



Move three matches in a different position so that four equilateral triangles are formed.

Middle School Students' Reasoning About Geometric Situations

William M. Carroll

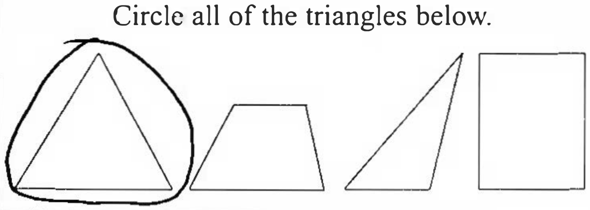
Along with increased emphasis on reasoning, communication and problem solving, the National Council of Teachers of Mathematics (1989, 1991, 1995) has called for a change in assessment techniques. In contrast to short-answer questions, assessments that elicit writing, diagrams and other representations offer better windows into students' understandings and misconceptions about mathematics. This article describes some short geometry tasks that go beyond the simple recognition of figures and properties. Because they have been field-tested with students using various mathematics curricula, we have collected hundreds of student responses to these questions that seem to represent a good range of students' geometric thinking and development. From these responses, along with ideas about the development of geometric thinking (Fuys, Geddes and Tischler 1988), we have developed scoring rubics to go along with many of these questions. The rubrics and questions might be useful to middle school teachers who are developing short open-ended questions that encourage and assess students' thinking.

Questions that require written reasoning or student-generated illustrations are useful for several reasons. First, although short-answer questions are easy to correct, they provide limited information about students' thinking. For example. the cause of the mistake in Figure 1 is unclear. Did the student have a narrow working definition for *triangle*, miss the second triangle or simply misread the question? Without this information, it is difficult for the teacher to plan relevant instruction.

Figure 1

Single-Answer Question That Does Not Illustrate a Student's Reasoning

Circle all of the triangles below.



Second, questions that require writing, drawing or other representation encourage and require more complex thinking. The question in Figure 1 would reveal more about the student's thinking if the direction "Explain why the figures not circled are not triangles" was added. In constructing an explanation, students are more likely to reason about the more relevant properties of the figure and, as they do so, to access and integrate previous knowledge. Responses to a question like this one are fairly easy to interpret, and they can assist the teacher in identifying misconceptions that their class may hold.

Third, these types of activities and assessment more closely resemble the activities that we value in the mathematics classroom. The assessment requires a student to use and integrate information actively. Many open-ended questions can also encourage multiple ways of thinking about the problem.

Three activities that we have used to assess more complex geometric thinking are described here. Some samples of students' work and rubrics that we have developed are also provided. They are well suited for Grades 5–8.

Task 1: Properties of Triangles

The first question asks students to reason about whether a triangle can be constructed with two right angles:

> Sheila said, "I can draw a triangle with two right angles." Do you agree with Sheila? Explain your answer.

Even though the question itself is fairly straightforward, the range of student responses is quite wide. Some students skip the question or answer without using geometric language or apparent reasoning. For example, a response like "I disagree with Sheila because you can't do it," provides no evidence of geometric understanding. At the other end, students use their knowledge of triangles and angles to provide an explanation that amounts to an informal proof: "No. Because the sum of the angles in a triangle is 180 degrees. And two right angles make 180 degrees.

So there would be no third angle for the triangle." A few students noted that two sides would be parallel and so the figure could not be a triangle, or they drew several counterexamples. Of course, a triangle could be constructed with two right angles on a sphere or on other non-Euclidean surfaces. Although no students responded, "Perhaps this would be possible on another type of surface that isn't flat," such a response would be acceptable and classified as a high-level response. In between these two examples, many of the Grades 5–6 students showed fair understanding but were limited in their ability to explain. Another group attempted to answer, but its members were limited by their misconceptions, for example, "There is one right and one left angle in a triangle, no matter how you draw it."

Some of the students clearly lacked an understanding of *right angle*, thinking that it referred to orientation. Most classes knew the basic definition but would benefit from explorations of figures that could be made with right angles. To deepen their understanding and facility, these students might use geoboards or dot paper to attempt to construct different types of triangles, prompted by such questions as "Which of the following triangles can you make? Look for a pattern, and explain what you found." These findings could be recorded and discussed, and reasoning about the relationship between properties could be emphasized.

In conjunction with these tests, we also conducted individual interviews. Interestingly, although many of the students "knew" earlier in the interview that the sum of the interior angles of a triangle was 180 degrees, few used this information spontaneously when this problem was posed. Instead, most attempted to draw such a figure, then stated that it was impossible. They were unable to use their factual knowledge without working concretely or visually. Such questions afford an opportunity to integrate geometric ideas that are otherwise loosely connected, that is, to generalize knowledge to more abstract understandings.

Table 1 describes five levels of response that we found and some illustrative responses. We think that the levels correspond fairly well to the van Hiele model of geometry, which describes a progression in thinking from recognition without reasoning to analyzing properties separately. Note that students can achieve the higher levels by various approaches to the question.

Table 1

Rubric and Sample Responses to Sheila's Triangle

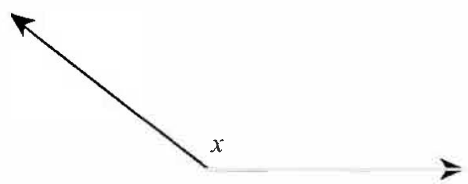
Level	Description and Sample of Students' Responses
0	No response or off task; geometric language is not used: "I disagree with Sheila because you can't do it."
1	Incorrect response, but some reasoning is attempted: "Yes, because all triangles have a right angle and a left angle." "Yes, you make one at the top and one at the bottom." Partially correct response, but reasoning is weak: "No, because all triangles have right angles."
2	Correct response, but reasoning is not complete: "No, because you can only put 1 right angle in a triangle." "No, it would have to be a square or a rectangle."
3	Correct response and good reasoning. Explanation goes beyond level 2 but relies on concrete or visual understanding rather than on abstract knowledge of properties: "Because if you put 2 right angles together, you already have 3 sides, and the sides are not closed." "No, because if you draw 2 right angles ⊔ and try to connect them, you get a square or a rectangle. Two right angles is already 3 sides."
4	Exemplary response. Student used knowledge of triangles and angles: "Triangles have 3 angles and 180°. If there are 2 right angles, then it would equal 180°. But that is only 2 angles." "How could you possibly have 2 right angles equaling 180° when you have 2/3 of a triangle done?" "You would have 2 parallel sides."

Task 2: Estimating the Measure of Angles

The second task was developed to assess students' knowledge of angular measurement, especially their use of such benchmark angles as 90 degrees, 180 degrees, and 45 degrees, to estimate the size of given angles. For example, a student who is familiar with common angles would recognize that the angle is greater than 90 degrees. We are also interested in whether students feel comfortable giving an estimate rather than an exact answer. One form of this task is illustrated in Figure 2.

Figure 2

Angle Estimation



Estimate the measurement of angle x in degrees.
Angle x is about _____degrees.
Explain or show how you got your estimate.

Because students were asked to explain their reasoning or the method they used, the range of students' reasoning, and of errors in reasoning, was apparent. Students could rely on visualizing the benchmark angles for a fairly complete answer. The largest group of Grades 5–6 students fell in this category: "It's a little more than 90 degrees but less than 180 degrees." Frequently, students drew in a right angle or stated that they pictured a right angle fitted into the obtuse angle.

Although this response was completely satisfactory, some students went further, using more precise estimates. A typical response was, "I made the angle 90 degrees and looked at the remaining part and saw that it was about half of 90 degrees, which is 45 degrees. I added 90 degrees and got 135 degrees."

Many students were successful on this task. Large numbers of middle school students, especially those who have had geometry experiences, have a good picture of benchmark angles and are able to use them successfully. This use of benchmark angles seems like a real-life skill that can promote estimation skills. The rubric that we have developed with four levels of responses, is shown in Table 2.

Task 3: Hidden Geometry Figures

The third task involves more problem solving. Like the first task, Sheila's triangle, this one requires students to consider geometric properties of polygons—

Table 2

Rubric and Sample Responses to Angle Estimation

Level	Description and Sample of Students' Responses
0	No response or off task.
1	Answer is not between 90 degrees and 180 degrees, but some attempt to explain is made: "70°, because the angle is less than a straight line." Answer is between 90 degrees and 180 degrees, but the student does not provide good reasoning: "I looked at it and I knew it was about 120°."
2	Student gives answer between 90 degrees and 180 degrees and includes use of benchmark angles: "I knew it was bigger than a right angle, but not 180°." "120°. It looks a little more than 90° but less than 90° away from 180°."
3	Student uses more precise benchmarks, perhaps in two steps. Estimate is within 15 degrees of exact answer: "The angle is about 90 and a half, so I divided 90 in half and add it to 90°." "130°. I drew a right angle and then counted up by 10 degrees."

parallel sides or types of angles. If two of the three sides showing are parallel, is it possible for the hidden figure to be a triangle? (See Figure 3.) What properties distinguish a square or a trapezoid from other figures? These questions require more than naming figures; they require reasoning about what makes the figures unique, combined with using the problem-solving strategies needed for sorting the figures. For example, three of the figures in Figure 3 can be trapezoids, and all could be hexagons; but only the second can be a triangle, and only the fourth can be a square. Note that the figures are not the standard geometric figures generally shown in books, that is, a regular hexagon or a triangle with the base at the bottom.

These problems can lead to nice discussions of geometric properties. In some classes observed, students worked in small groups, actively drawing and discussing the properties of the given polygons. Some examples of students' work are shown in Figure 4. A scoring rubric for the assessment of the activity, along with response levels, is shown in Table 3. The responses in Figure 4 correspond to those levels, with only the last response including correct drawings and names.

The three rubrics illustrated in this article vary in levels from five (Table 1) to three (Table 3). The complexity of the rubric should mirror both the complexity of the task and the purpose the teacher has in mind. Often a three-point rubric is sufficient for teachers who want to assess the range of student understanding for the purpose of planning instruction.

Another purpose for developing rubrics is to include students in the assessment process. Often students are not clear about what differentiates an excellent response from a poor response. A clearly stated rubric, along with some examples, can clarify

Figure 3
Hidden Figures Task

Gina drew some shapes: a triangle, a square, a trapezoid and a hexagon.
She covered most of each figure, as shown below.
Can you tell which figure is which? Write the name below each figure.
Then try to draw the rest of the figure.

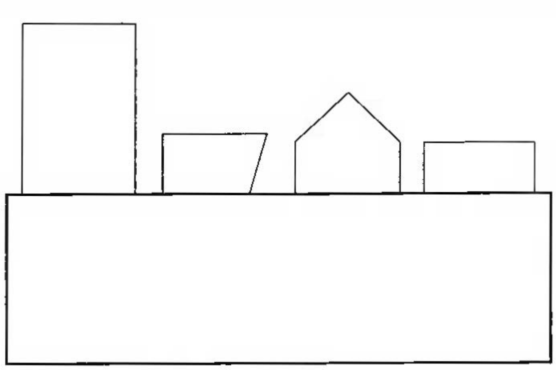
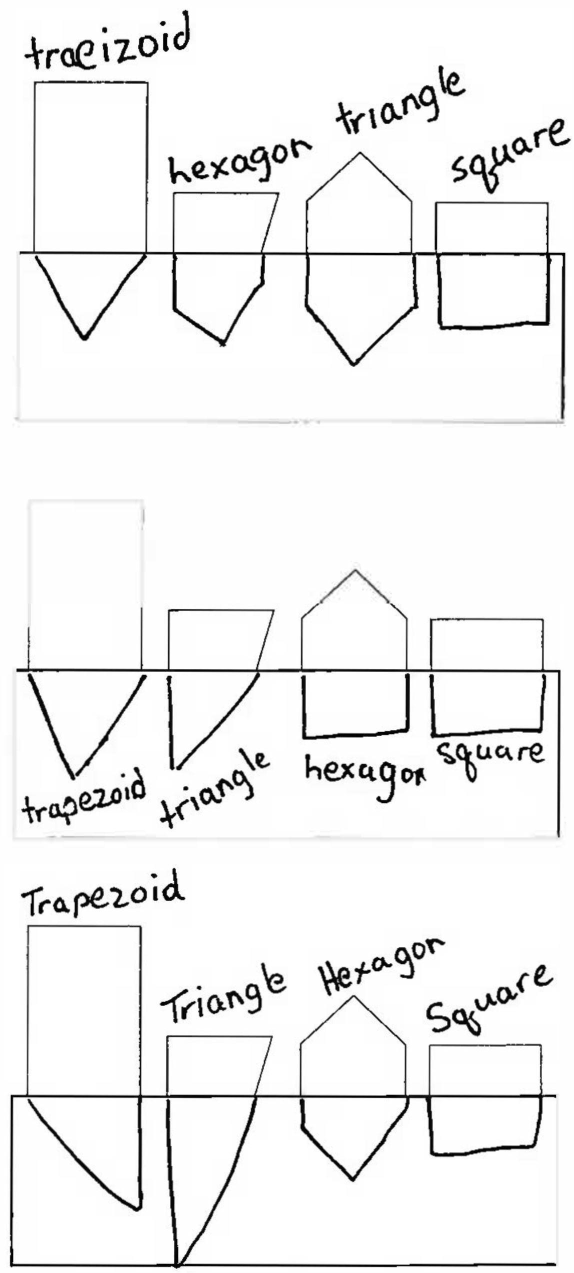


Figure 4
Responses to the Hidden-Figures Question



these standards for the students. Teachers have indicated that this process is ongoing. Early in the year, students are unclear about how to express their reasoning or what constitutes a good explanation, but with experiences and exemplars, they improve their responses.

An additional example of hidden geometry figures is illustrated in Figure 5. This form of the question asks students to include an explanation about their reasoning. The reader is invited to consider the possible responses that students might give and how a rubric might be developed to assess these responses. Then test it with students to see how well their responses fit the rubric and what adjustments are necessary in it.

Figure 5

Another Hidden-Figure Task

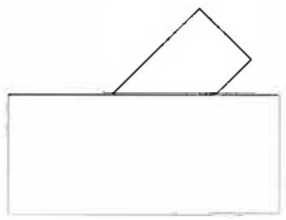
A figure is partly hidden. Which of the following might it be? Circle all the possible answers.

rectangle

triangle

trapezoid

square



Choose one of the figures that you did not circle, and explain why you did not choose it.

Conclusion

Research has shown that many secondary school students in the United States are ill-prepared for formal geometry classes (Senk 1989). Often, junior high and senior high school students lack experiences in reasoning about geometric properties. To prepare students for more formal thinking in the secondary school, geometry activities in the middle school must go beyond simple visual exercises. Rigorous proofs are not necessary at this age, but students should be able to use ideas about geometry to construct informal arguments, which helps them better understand the structure of geometry. These arguments might involve oral or written responses.

Teachers have been quite positive about the types of questions illustrated in this article. They can be used as individual or group activities or assessments or both. Because the questions emphasize reasoning, problem solving and communication, teachers have reported that these types of questions help them implement the NCTM (1989, 1995) standards in their classrooms. Students are also often enthusiastic about engaging in these types of activities, as opposed to simple classification and vocabulary activities.

As the examples in this article illustrate, a good deal of information about students' mathematical knowledge can be gathered from fairly short activities and assessments that involve reasoning. Although longer projects and tasks are also needed, these short activities can make reasoning a regular part of the classroom in a manageable fashion. With some practice, such questions and rubrics are easily developed, especially when teachers work collaboratively. Often, single-response questions can be used as the base from which the question is expanded

Table 3

Rubric for Hidden-Figures Question

Level	Description and Sample of Students' Responses
0-Little progress	Student incorrectly names and draws most of the geometric figures. Response shows little understanding of geometric shapes and their properties. (First response in Figure 4)
1-Shows progress	Student correctly names all geometric figures. However, some drawings show incorrect figure. (Second response in Figure 4)
2-Good understanding	Student correctly names and draws all geometric figures. Drawings illustrate correct properties of the figure. (Third response in Figure 4)

and students are asked to explain their thinking in words or drawing.

Having students draw, explain and elaborate on their answers has several benefits. They must apply knowledge more fully when reasoning is required. In the process of explaining their reasoning, they more fully integrate previous knowledge and learning occurs. As suggested by the assessment standards document (NCTM 1995), assessment and learning are not separate processes. They can be used as activities to develop and discuss reasoning or assessments. These types of questions are also more like the mathematics we expect people to need as a life skill. Real problem situations require planning, reasoning and communication.

Teachers who have attempted to construct reasoning questions and rubrics often report that the process gives them a better insight into their students' thinking. Considering the range of possible responses and misconceptions is helpful in planning instruction and activities.

Perhaps more important, when more open, more complex assessment tasks are used, students' thinking is more clearly revealed and information that is crucial to planning individual and class instruction can be gathered. As our results indicate, middle school students are quite capable of developing good reasoning about geometric situations when they have had substantial experiences in geometry throughout elementary school. However, many students fail to go beyond a simple visualization of geometric figures. Challenging students to apply their knowledge in situations that require application, explanation and illustration is one step toward improving geometric reasoning.

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Length of the Belt

A drive belt runs around three pulleys that are in a triangular arrangement to one another. The distances between the pulley shafts are 1.5 m, 2.0 m and 2.5 m, respectively. All three pulleys have a radius of 50 cm. How long is the drive belt?

Ideas for Developing Students' Reasoning: A Hungarian Perspective

Anita Szombathelyi and Tibor Szarvas

As the end of the 20th century approaches, we start to realize again the significance of proof in mathematics education. The NCTM's (1989) *Curriculum and Evaluation Standards for School Mathematics* cautions against the tendency to completely abandon proofs and focus only on the end results and formulas. In this article, we re-emphasize the importance of proofs in teaching by sharing some of our experiences as students and teachers in Hungary, in addition to our experiences as graduate teaching assistants at an American university. We offer examples and ideas that might help educators develop students' mathematical reasoning skills.

The temptation has always existed to de-emphasize the role of proofs in mathematics teaching, because they are considered too time-consuming for a class period and because of the growing perception that mathematics is primarily needed for applications. However, we believe that danger lurks in this perception. Removing all proofs from calculus, for example, as the Hungarian-born Pólya (1957, 219) wrote, may "reduce the calculus to the level of the cookbook. The cookbook gives a detailed description of ingredients and procedures but no proofs for its prescriptions or reasons for its recipes."

Indeed, students in Kindergarten through Grade 12 are expected to be able to observe patterns and relationships, to use them to make conjectures and to construct arguments by drawing logical conclusions. They are encouraged to use deductive and inductive reasoning, formulate counterexamples and indirect proofs, and use mathematical induction—just as in the typical Hungarian K–12 curriculum. We should start promoting the use of reasoning, introduce critical thinking at an early age and reiterate its importance at each grade level. Teaching students to look for more than one way to establish the validity of the same statement helps them make connections among different mathematical concepts and procedures.

We also believe that it is important to prepare mathematics teachers to emphasize proofs in their lessons. In Hungary, prospective high school teachers are required to present to their professors several theorems covered in their courses, along with complete proofs. Besides developing confidence and ability, this exercise helps teachers more readily incorporate mathematical reasoning in their own instruction.

We next share a series of examples and ideas that can help develop logical thinking and reasoning ability—examples and ideas that we have used successfully in Hungary and in the United States. We then move from such intuitive aspects of proofs as making conjectures to more formal types. We use some classical examples, along with real-life problems that are easily accessible to students.

Making Conjectures

Beginning at an early age, students are frequently asked to observe patterns and relationships. They should be encouraged to make conjectures about the particular case of a problem and to test these conjectures. During that process, they might devise a proof for the general case. They can start with simple problems and, using the same techniques and reasoning, gradually proceed to more abstract levels.

Example 1

Calculate $2^3 \cdot 2^2$, $2^5 \cdot 2$, $2^4 \cdot 2^2$.

$$a^n \cdot a^m = (\overbrace{a \cdot a \cdot \ldots \cdot a}^{n})(\overbrace{a \cdot a \cdot \ldots a}^{m}) = a \cdot a \cdot \ldots \cdot a \cdot a \cdot a \cdot \ldots \cdot a = a^{n+m}$$

Example 2

Examine the differences of consecutive square numbers.

Students will quickly notice the pattern: $2^2 - 1^2 = 3$, $3^2 - 2^2 = 5$, $4^2 - 3^2 = 7$, ... In general, for n and $n + 1$, $(n + 1)^2 - n^2 = 2n + 1$. After students arrive at a conjecture, they can easily prove why this general statement is always true.

Using basic examples, we can increase the opportunities for our students to give complete proofs. Making a conjecture is an achievement, but students should recognize that it is not yet a proof. They should be encouraged to furnish a more formal argument.

It should be clear to them that proving one particular case is not equivalent to proving the statement in general. For many students, this concept is difficult to comprehend.

True-or-False Questions

Children are introduced to true-or-false questions when they are very young. We think that they must always give reasons for their answers. Otherwise, true-or-false problems are nothing but an easier version of multiple-choice questions, and then we really do not assess students' mathematical thinking. A rigorous, formal proof is desirable for a true statement, whereas a counterexample should be provided for a false statement. Students use counterexamples as solutions to different problems at a very early age, but they do not have a word for it—think of such simple statements as "Every flower is pink." This method may avoid the dilemma pointed out by Galbraith (1995, 416), "the fact that a single exception disproves a generalization is not accepted by students"; they "do not understand what constitutes a counterexample, namely, an instance that satisfies the conditions but not the conclusion of the statement."

Example 3

Prove or disprove: Straight lines that have no point in common are parallel.

Example 4

True or false? Give reasons. If $f'(c) = 0$ for a function f, then f has a maximum or minimum at $x = c$.

Such examples as these have two parts. First, students must decide whether they think that the statement is true, and then they must either prove it or find a counterexample, that is, an example that satisfies the given conditions but does not satisfy the conclusion of the statement. Students who try to learn by simply memorizing everything can find these problems to be very challenging. Recalling formulas and theorems without really understanding them will not help students find counterexamples. These types of problems promote mathematical thinking and reasoning, not guessing or working from memory.

Examining Flawed Arguments

In these problems, students are asked to find errors in the argument. It is useful to introduce students to proofs that have hidden errors or logical inconsistencies and ask them to go through the argument step-by-step, searching for the flaws and correcting them if possible. This exercise is an excellent opportunity to discuss common mistakes, different methods of proofs and so on.

Example 5

Can you find an error? Start with the equation

$$x + y = -z$$

Multiply both sides by 4 and then by 5, and exchange the sides of the second equation:

$$4x + 4y = -4z$$
$$-5z = 5x + 5y$$

Adding the two equations yields

$$4x + 4y - 5z = 5x + 5y - 4z.$$

Adding $9z$ to both sides, we obtain

$$4x + 4y + 4z = 5x + 5y + 5z.$$

Then

$$4(x + y + z) = 5(x + y + z),$$

Which implies that

$$4 = 5.$$

Proving by Contradiction

Proving by contradiction also builds reasoning skills. Moreover, these types of proofs build on students' interests in making logical arguments that are not necessarily straightforward to others. We start with given conditions, assume the conclusion to be false and arrive at a contradiction.

Example 6

Ten teams are playing in a championship in which each team meets every other team exactly once. So far, 11 games have been played. Prove that one team has played at least three games.

Proof: Let us assume the contrary. Each team therefore has played at most two games; that is, at most $(10 \cdot 2)/2 = 10$ games have been played, in contradiction with the wording of the problem. Therefore, a team must have already played at least three games.

Example 7

Prove that the sum of a rational number and an irrational number is irrational. Q is the set of rationals; I, the set of irrationals; and Z, the set of integers.

Proof: Let $x \in Q$ and $y \in I$. Then $x = p/q$. For some $p, q \in Z (q \neq 0)$. Assume that $x + y \in Q$. Then $x + y = p/q + y = r/s$ for some $r, s \in Z$ $(s \neq 0)$. So $y = r/s - p/q = (rq - ps)/sq \in Q$, which contradicts the hypothesis.

We think that using mathematical symbols to rewrite expressions and whole sentences is another area that needs more emphasis. Students, especially those who plan to attend college, need to acquire a basic familiarity with symbols and notation.

Working Backward

This method of proof should be used very carefully, always paying attention to working with equivalent statements.

Example 8

Prove that for two positive numbers a and b, the arithmetic mean $(a + b)/2$ is always greater than or equal to their geometric mean $\sqrt{ab}$. Furthermore, equality occurs if and only if $a = b$.

Proof 1:

$$\frac{a+b}{2} \geq \sqrt{ab},$$
$$a + b \geq 2\sqrt{ab},$$
$$a - 2\sqrt{ab} + b \geq 0,$$
$$(\sqrt{a} - \sqrt{b})^2 \geq 0,$$

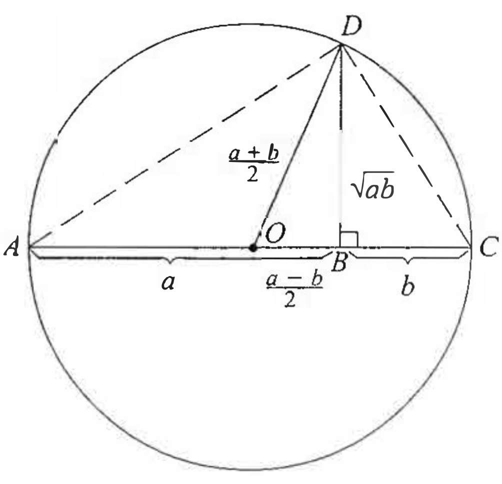
which is always true. Furthermore,

$$(\sqrt{a} - \sqrt{b})^2 = 0,$$
$$\sqrt{a} = \sqrt{b},$$
$$a = b.$$

Another way to solve this problem is by using a geometric construction, as in proof 2.

Proof 2: See Figure 1. We used Thales' theorem and the Pythagorean theorem, along with the fact that the length of the hypotenuse of any right triangle is always greater than the length of any of its legs.

Figure 1
Geometric Proof of Example 8



Proving by Mathematical Induction

This method is often deemed uninteresting because it repeats the same technique for different problems. We do not believe that proofs by induction can be skipped because they are "too simple." We had to learn and use induction in high school in Hungary, and when we became teachers, we expected our students to do the same. We think that expecting students to learn to use mathematical induction is justified because students analyze and work with such different ingredients of mathematical arguments as hypotheses and conclusions. Also, when proving a statement by induction, students learn to use symbols and how to express themselves in the language of mathematics.

Example 9

Prove that

$$s_n = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

This proof is an example of a classical application of mathematical induction. The important point is to conjecture the sum of the first n positive integral numbers. Too often, students accept the formula as a fact without a formal proof.

Proof: For $n = 1$, the statement is trivial:

$$1 = \frac{1(1+1)}{2}$$

To carry out the inductive step, let k be an arbitrary positive integer, and suppose that

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}.$$

To show that the assertion holds true for $n = k + 1$, we use the induction hypothesis along with some algebra:

$$\begin{aligned} 1 + 2 + \cdots + k + k + 1 &= (1 + 2 + \cdots + k) + k + 1 \\ &= \frac{k(k+1)}{2} + k + 1 \\ &= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2} \end{aligned}$$

The principle of mathematical induction guarantees that the statement is true for all positive integers n, and the proof is complete.

In addition to mathematical induction, several other methods of proving this theorem exist. In school, we were always encouraged to find more than one way to solve a problem. Students should learn that several ways can be used to arrive at a conclusion. Students can come up with different processes, along with their advantages and disadvantages.

A way to prove example 9 not using mathematical induction is the following:

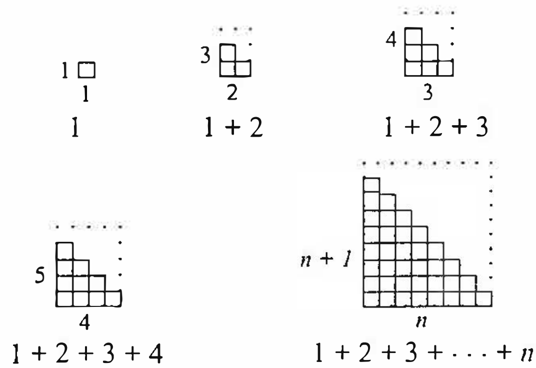
Proof: $s_n = 1 + 2 + 3 + \cdots + (n-1) + n,$

$$\begin{aligned} s_n &= n + (n-1) + (n-2) + \cdots + 2 + 1 \\ 2s_n &= (n+1) + (n+1) + (n+1) + \cdots + (n+1) \\ 2s_n &= n(n+1) \\ s_n &= \frac{n(n+1)}{2}. \end{aligned}$$

Proof: See Figure 2.

Figure 2

Proof 3 of Example 9



Deductive Method

The deductive method is widely used to establish the validity of theorems, propositions and corollaries. Previously proved results, definitions and the given hypotheses are applied in a straightforward manner to reach the desired conclusion.

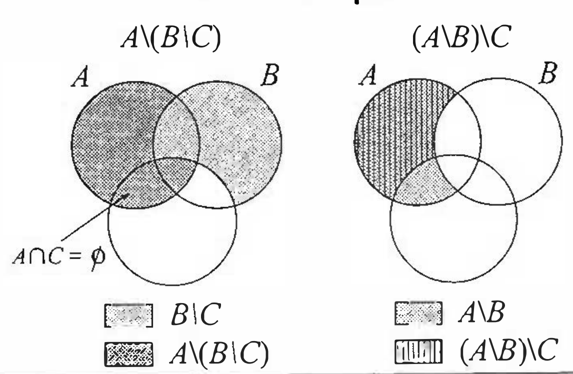
Example 10

Prove that if $A \cap C = \phi$, then $A\backslash(B\backslash C) = (A\backslash B)\backslash C$. (Note that in this article $B\backslash C$ should be taken to mean all points B not in C.)

Proof: See Figure 3.

Figure 3

Proof of Example 10



Example 11

Theorem. A circle can be inscribed in a quadrilateral if and only if the sums of the lengths of the opposite sides of the quadrilateral are equal.

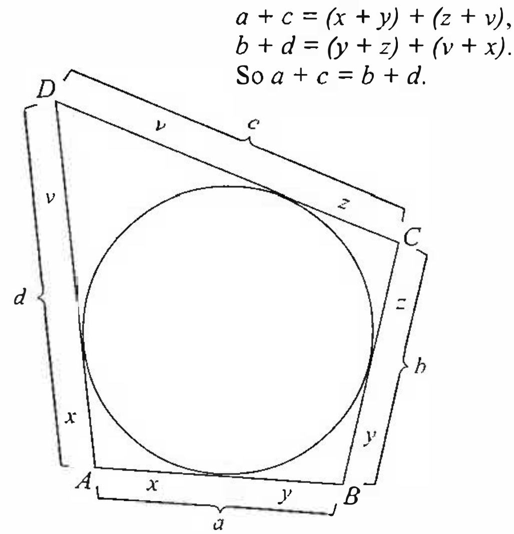
Proof: Part 1: "Necessity"

Hypothesis. The given quadrilateral $ABCD$ has a circle inscribed in it.

Conclusion. $a + c = b + d$ (see Figure 4). Here we used the theorem that the segments that are tangent to a circle from an outside point have equal lengths.

Figure 4

Part 1 of Example 11



Part 2: "Sufficiency"

Hypothesis. $a + b = b + d$ in $ABCD$ (see Figure 5).

Conclusion: ABCD is a quadrilateral in which a circle can be inscribed.

We prove the second part by contradiction. Assume that $a + c = b + d$ in $ABCD$ but that no circle can be inscribed in it. Construct a circle inside the quadrilateral that touches three of the four sides (see Figure 5). Draw a tangent to the circle from point C. Let E be the point that is the intersection of the tangent line and side AB; and let $x = AE$, $y = EC$. Then $EB + EC = BC$. But $EB + EC > BC$ by the triangle inequality, showing that the indirect assumption was false.

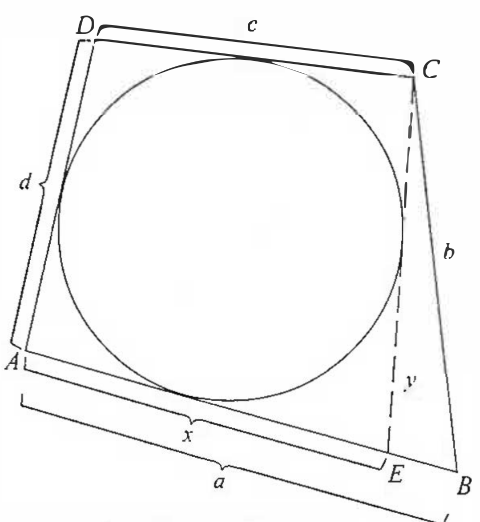
The theorem in example 11 requires a combination of different proof techniques. It is an excellent opportunity to introduce students to discovery learning, conjecture making, testing and proving. They can construct a quadrilateral with an inscribed circle either with a ruler and compass or by using geometric software. Excellent geometric software packages include The Geometer's Sketchpad (Jackiw 1995) and Cabri II (Bellemain and Laborde 1995). Technology can be combined with cooperative learning to get high results in the classroom.

As the next step, students measure the lengths of opposite sides and try to find some relationship between them. The more data they have, the easier to conjecture—they can record one another's findings. The teacher assists as students set up at least the "necessity" part of the theorem and try to prove it. This straightforward deductive proof requires no tricks or fancy techniques. The other direction—if $a + c = b + d$ in a quadrilateral, then a circle can be inscribed in it—needs some thought but can easily be proved using the indirect method.

Geometric constructions are important in gaining insight. They help students develop mathematical thinking and understanding and offer excellent opportunities for exploration, recognizing patterns, making and testing conjectures, and fun, of course.

Figure 5

Part 2 of Example 11



$a + c = b + d$ by hypothesis,
$x + c = y + d$ by part 1,
$a - x = b - y$.
So $EB = BC - EC$ or $EB + EC = C$.
But $EB + EC > BC$
by the triangle inequality.

More generally, geometric proofs have an important place in our mathematics instruction. "The study of the elements of plane geometry yields still the best opportunity to acquire the idea of rigorous proof. . . . If general education intends to bestow on the student the ideas of intuitive evidence andlogical reasoning, it must reserve a place for geometric proof" (Pólya 1957, 215, 217).

These examples illustrate the types of exercises that can help students develop good reasoning skills and the confidence and ability to prove statements. Of course, every single statement need not be prepared along with its proof in the mathematics classroom, but with all the additional content that has been introduced into the high school curriculum to meet the needs of the 21st century, proofs can permeate the curriculum in much the same way that problem solving now does. We, as teachers, should always decide what is important but must emphasize the reasoning aspect of instruction. We must find ways to encourage our students to justify their responses and structure their arguments using mathematical notations. They will then have the tools necessary for formal proofs. This process should begin early, for it will empower students to make mathematical connections later on. They will be freed from the frustrations of a multitude of seemingly unrelated rules and techniques. Last, but most important, they will see the real beauty of mathematics.

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A Collection of Connections for Junior High Western Canadian Protocol Mathematics

Sol E. Sigurdson, Thomas E. Kieren, Terri-Lynn McLeod and Brenda Healing

We have put together "A Collection of Connections" that consists of 12 uses of junior high school mathematics. These activities support the communication and connections strands of the Western Canadian Protocol mathematics curriculum. In using them, teachers may adapt them extensively. They can serve as a basis of one to three mathematics periods. We have also found that teachers need to plan if they intend to incorporate them into their teaching units. They can be used as end-of-unit activities or as focal activities in the development of a unit. We would encourage teachers to use the activities as a means of bringing mathematical skills to life. These contexts provide an opportunity to enhance a student's view of mathematics. As one Grade 9 girl said at the conclusion of one activity: "That just proves that mathematics is everywhere."

The following are samples from the number and algebra strand.

Number (Ratios and Scale Drawings)
Church Windows
Church Windows Student Activities
Algebra
Rating the Bouncing of Balls
Rating the Bouncing of Balls Student Activities

Church Windows

Intent of the Lesson

This practical problem can be solved using only scale drawings and simple ratios. Students studying trigonometry should compare the scale drawing solution to the trigonometric solution. Solving the problem in two ways can be an important learning experience.

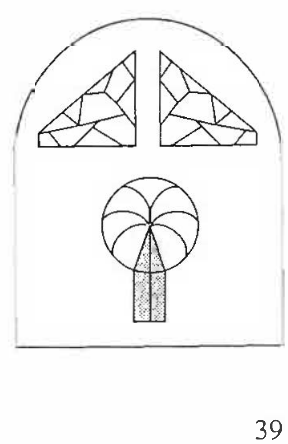
General Question

Architects are working on designing a church, whose main feature is a 9 m-wide "window wall." The windows in the wall are going to be stained glass. Because this wall faces east, the design will cast interesting patterns of light on the altar during morning service. Stained glass is an old tradition in many religions. However, sold by the square metre it is very expensive. The best price that the architects have been able to find is $3,000 per square metre. The total glass budget for the church is $270,000 and no more than a fifth of this amount can be used for stained glass. Architects want to know if the design below falls within the budget. The design consists of two windows that make a Gothic design at the top of the wall and a stylized candle with a glow in the lower part. In addition to the total area and the cost of the windows, the carpenter will have to know their dimensions, namely, how long each side is and the size of any angles.

The design is given below with the additional information:

The two windows in the upper part of the wall have a top angle of 49°. The length of the base of these windows is 3 m and the short vertical side is 0.4 m.

In the candle window the glow circle has a diameter of 3 m. The candle is 1 m wide, and the distance from the centre of the glow to the bottom of the candle is the same as the diameter of the glow.



Discussion Questions

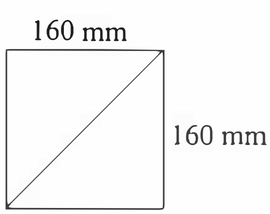
- Why is stained glass so expensive? (Because it is custom-made with lead filling.)
- What does *Gothic* mean in architecture? (*Gothic* is a style of architecture, featuring such things as pointed arches, common in Europe in the 12^{th}–16^{th} centuries.)
- Why do candles figure so prominently in many religions?
- Why are places of worship often oriented toward the east?

Preliminary Activity

Activity 1

Although building problems of this type can be solved using advanced mathematics, namely, trigonometry, they can also be solved by drawing to scale on grid paper. Most grid paper does not measure exactly to the millimetre. Because of this, a ruler can be used which measures to the nearest millimetre. The grid paper is, therefore, only useful in keeping the drawing at right angles.

Suppose we had a square window, each side measuring 1.6 m. What would be the length of the lead strip forming the diagonal? Our knowledge of the Pythagorean relationship would tell us that the length is given by $(1.6)^2 + (1.6)^2 = L^2$, or L = 2.27 m (approximately). Using the procedure of a scale drawing, a convenient scale should be chosen and the square can then be drawn. if we choose 1 mm = 10 mm (for a scale of 1:10) then 160 mm represents 1,600 mm (1.6 m).



By drawing the square as carefully as possible, the diagonal measures 226 mm. Because our scale was 1 mm = 10 mm, the answer for the length of the diagonal is 2,260 mm (2.26 m). This result is very close to the result achieved using the Pythagorean relationship (2.27 m). This activity shows that this scale measurement system for finding lengths is fairly accurate. In making the drawing, the thickness of even a pencil mark can make a difference of a millimetre or two. (When cutting a board, a carpenter must take into account the width of the saw blade.)

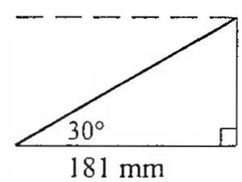
Discussion Questions

- What should be considered in choosing a scale? (It should be large enough for accuracy, but should still fit on the page.)
- Why do we make the drawing as large as possible?
- Why does the thickness of the pencil line lead to inaccuracies? (It is 1 mm wide.)
- What might be done to minimize this? (Use the procedure of measuring to the inside of the line.)
- When we solve problems by scale drawing, what mathematical concept are we using? (Ratios.)

Activity 2

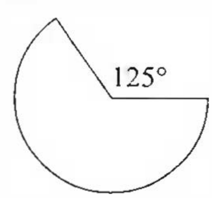
What are the dimensions of a rectangle whose longer side is 3.62 m and whose diagonal makes a 30° angle with the longer side? Choosing an appropriate scale when making the drawing is important. For one side to be roughly 180 mm long, the scale of choice will be 1 mm = 20 mm. Our rectangle will be 3,620 ÷ 20 =181 mm long. Draw this on the square paper first, and then construct an angle of 30° to this line.

The shorter side measures 106 mm. Converting back to the original scale, the side of the rectangle will be 106 mm × 20 which is 2,120 mm (2.12 m). Using trigonometry the answer is 2.12 m. Again our scale drawing method gives very close answers.

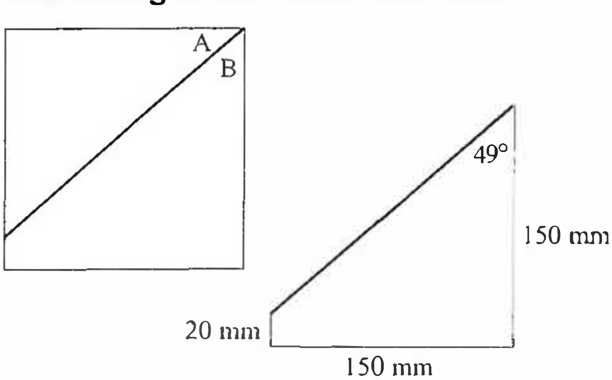


Activity 3

The stained glass pattern on the door can be thought of as a fraction of a circle. For example, the area of a quarter circle can be determined by $\frac{1}{4}\pi r^2$. If the angle in the pattern is 125°, the area of the pattern is $235/360\ \pi r^2$.



Answering the General Question



The Trapezoidal Windows

The student is asked to solve the problem by drawing a fairly large rectangle on the grid paper. The figure can be drawn starting with the 3-m side of the rectangle and angle A equal to 41°.

A convenient scale is 1 mm = 20 mm. The 3-m side becomes 150 mm. The angle at A is drawn. The 0.4-m side is 20 mm. The remainder of the rectangle can now be drawn.

The remaining side is measured as 150 mm in the representative drawing or as 3,000 mm on the window. Using these values, the area of both trapezoidal windows can be found. To calculate the area, it is necessary to divide the window into two parts: a rectangle and a triangle. The base of the triangle would be 3.0 m and the height would be 2.6 m (since 3.0 m − 0.4 m = 2.6 m). The following calculations give the values of the areas for these sections:

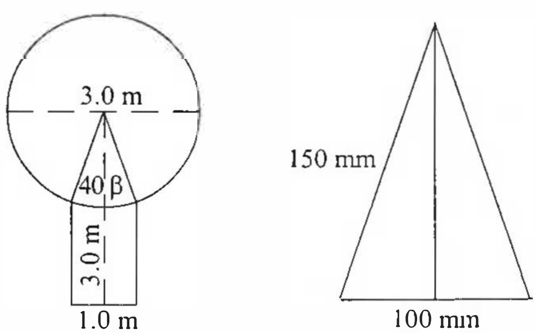
Area of Rectangle = 0.4 m × 3.0 m
= 1.2 m^2

Area of Triangle = 0.5(3.0 m)(2.6 m)
= 3.9 m^2

Therefore, the area of both trapezoidal windows is 2(1.2 + 3.9) m^2 = 10.2 m^2

The Candle

Again the problem is solved with a scale drawing. Using a scale of 1:10, the radius of the glow is 3,000 ÷ 10 = 300 mm. The distance from the centre to the bottom of the candle will be 300 mm and the width of the candle will be 100 mm. Once the drawing is complete, measurements can be made. The angular measurement of the glow is 320°. Therefore the area of the glow is $320/360\,\pi r^2$, where r = 1.5 m. The area of the glow is 2π m^2 or 6.28 m^2.



The area of the candle can be determined as a rectangle with a triangle on top. The dimensions of these figures can both be determined through measurement. Ruler measurement yields a rectangle of 100 mm by 155 mm while the triangle has a base of 100 mm and a height of 300 − 155 = 145 mm.

In using scale drawings for this activity, students do not need to reproduce the complete window designs; they only need to make those scale drawings that help in the calculations. Without using trigonometry, scale drawing is the only means of solving this problem.

The areas of these figures are given:

Area of Triangle = 0.5 (1.0 m)(1.45 m)
= 0.725 m^2

Area of Rectangle = (1.55 m)(1.0 m)
= 1.55 m^2

The total area of the candle is then 0.725 + 1.55 = 2.275 m^2.

In calculating areas, students can be reminded that they can calculate the area in the scale figure or convert to actual dimensions of the window and find the area. The latter is less confusing.

The students can use the calculated areas to solve for the overall cost of the stained glass window. They will find that they will be over budget if all the designs are made from stained glass. Perhaps suggest that the candle be made of wood instead. Calculations of the cost of the stained glass are as follows:

Cost of Candle = (2.275 m^2)($3,000/$m^2$)
= $6,825

Cost of Glow = (6.28 m^2)($3,000/$m^2$)
= $18,840

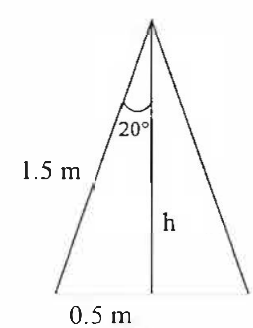
Cost of Trapezoidal Window = (10.2 m^2)($3,000/$m^2$)
= $30,600

The total cost of the windows is then $56,265.

Materials

Grid paper (paper with 0.5-cm markings is useful), a ruler with millimetre markings and a protractor will all be very useful.

Modifications



This lesson can also be taught as a review or extension of trigonometry rather than by scale drawings. If trigonometry is to be used, the height of the triangle can be calculated using the Pythagorean theorem: $h^2 = (1.5\text{ m})^2 - (0.5\text{ m})^2$
h = 1.4 m

Trigonometric functions can also be used to solve for the value of h as follows:

$\cos 20° = \frac{h}{1.5 \text{ m}}$ or $\tan 20° = \frac{0.5 \text{ m}}{h}$

$h = 1.4 \text{ m}$ $h = 1.4 \text{ m}$

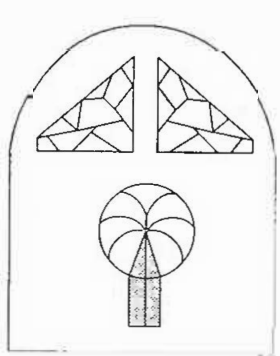
Church Windows Student Activities

General Question

Architects are working on designing a church whose main feature is a 9 m-wide "window wall." The windows in the wall are going to be stained glass. As this is an east facing wall, the design will cast interesting patterns of light on the altar during morning service. Stained glass is an old tradition in many religions. However, sold by the square metre it is very expensive. The best price that the architects have been able to find is $3,000 per square metre. The total glass budget for the church is $270,000 and no more than a fifth of this amount can be used for stained glass. Architects want to know if the design below falls within the budget. The design consists of two windows that make a Gothic design at the top of the wall and a stylized candle with a glow in the lower part. In addition to the total area and the cost of the windows, the carpenter will have to know their dimensions, namely, how long each side is and the size of any angles.

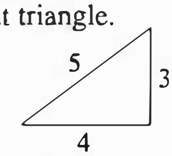
The design is given below with the additional information:

> The two windows in the upper part of the wall have a top angle of 49°. The length of the base of these windows is 3 m and the short vertical side is 0.4 m.
>
> In the candle window, below, the glow circle has a diameter of 3 m. The candle is 1 m wide, and the distance from the centre of the glow to the bottom of the candle is the same as the diameter of the glow.



Activities

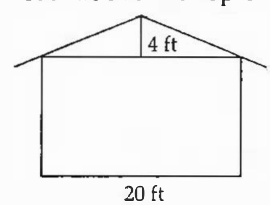
1. (a) Carpenters make a right angle by drawing a 3, 4, 5 right triangle.



Draw the triangle to scale and determine the measure of each angle.

(b) What are the angle measures of the 6, 8, 10 right triangle? Explain.

2. (a) A building is 20 feet wide and the peak of the roof is 4 feet above the top of the sides.

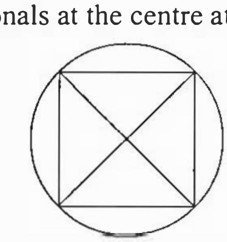


How long must the rafters be if they need to stick out 2 feet from the building?

(b) What is the angle of the peak of the roof?

(c) The building code requires that any roof rise has a ratio of 1:4. This means the roof must go 1 foot up for every 4 feet across. Is this roof steep enough? What is its ratio?

3. (a) What is the area of a square inside a circle 1 m in diameter?
(A square can be drawn to scale by making the diagonals at the centre at right angles.)

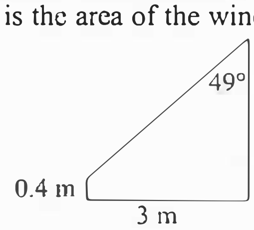


(b) What is the ratio of the area of the square compared to the area of the circle? Does this answer seem reasonable?

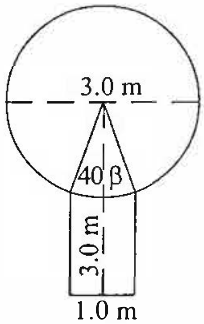
(c) Find the area of the square using the Pythagorean relationship. Is this answer similar to the answer in 3a?

4. (a) The angle at the top of this window is 49°. The base is 3 m and the short vertical side is 0.4 m. What are the dimensions of the window?

(b) What is the area of the window?

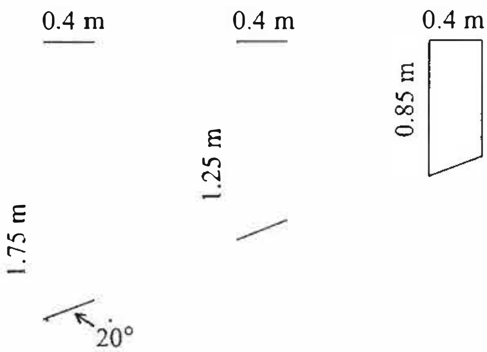


5. (a) In the candle window below, the circle, or the glow, has a diameter of 3 m. The candle is 1 m wide, and the distance from the centre of the glow to the bottom of the candle is the same as the diameter of the glow. What is the area of the glow? (Do not include the area of the glow that the candle takes up.)



(b) What is the area of the candle?

6. Suppose that instead of a candle, the following design was used in the window. Find the area of each of the strips, given the following diagram:



Rating the Bounce of Balls

Intent of the Lesson

Regardless of the height from which a ball is dropped, it will always bounce the same fraction of the original height. A series of linear equations is developed based on the bounce of several balls. The mathematics involved is fractions, variables, graphs of linear equations, the slope of a line and the line of best fit.

General Question

Everyone knows that different balls bounce differently. How high will a ball bounce? The question is asked by a consumer rating group that wants to rate a variety of balls to tell its consumers how much bounce any given ball will have. In this activity, a system for rating the "bounce" of a variety of the common balls currently sold will need to be developed. The use of the unscientific word "bounce" is informal and intuitive. Bounce will turn out to be the coefficient of elasticity, a scientific concept defining a property of elastic materials. A much more informal method of rating of the bounce in balls will be examined in this lesson.

Initially, the teacher can take a volleyball and bounce it. Be sure to hold it high and simply drop it. The class can guess what fraction of its original height it bounces to. It can then be held at a lower height, and the fraction guessed again. The question is how to make a solid estimate of the bounce of any ball. A rating system will definitely be needed that will describe the bounce from a whole range of heights.

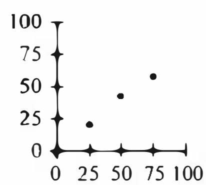
There are 10 balls that should be rated. Each will be tested and a way of labeling the balls according to their bounce will need to be developed.

Discussion Questions

- On what does the bounce of a ball depend? (On how it is made: hard rubber, full of air, less air; on the surface of the ball: completely spherical or with tiny indentations, like a golf ball; on the surface of the floor: soft or hard.)
- How high will a ball bounce? (Depends on how hard it is thrown or on its fall height.)
- In comparing balls, how can conditions be kept the same? (Drop them on the same floor or from the same height.)
- Does the speed of the fall depend on the weight of the ball? (A large ball will encounter air resistance but heavy balls fall at the same speed as light balls.)
- Does the ball bounce the same from different heights? (This will be one thing to find out in the trials. Do some balls not bounce very well from low heights? The "perfect ball" will bounce the same fraction at different heights.)

Preliminary Activity

Given three points on a graph which are connected by a straight line, how can the equation for the line be deciphered? First of all, it is very easy because the points lie in a straight line. Second, it becomes even easier if the line goes through the origin. Example of points: (25, 20), (50, 40), (75, 60).



These points lie in a straight line that goes through the origin. If each x value is multiplied by 0.8, the result is the y value. For example, $0.8 \times 25 = 20$; this shows the relationship between the y and x values is

$y = 0.8x$

Another set of points is (25, 15), (50, 30), (75, 45). By experimenting, the multiplier is seen to be 0.6. Therefore, the relationship is

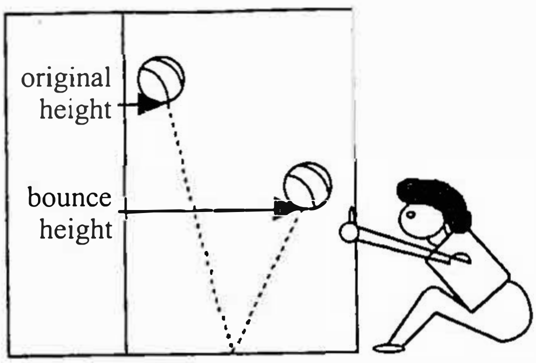
$y = 0.6x$

Whenever there is a set of points in a straight line going through the origin, there is always a simple relationship where y is some number times x.

Answering the General Question

Making the Measurements

Have a look at a volleyball. One way of investigating this is to drop the ball and take measurements of the *original height* and the *bounce height*. Measurements should be taken from the bottom of the ball. A discussion of this point will demonstrate that when the original height is zero, the desired bounce height should also be zero. In the case of the volleyball, if measured from the top, our smallest original height would then be 25 cm.



The Trials

An investigation of the bounce for different heights will be conducted because the consumer rating service wants to give the reader a complete picture of how each ball bounces. In these trials, students should take five different original heights in addition to the zero height. Experimenting with the volleyball makes it clear that measurements are not very accurate for small heights. The teacher should suggest five trials with heights ranging from the lowest of 0.6 m to the highest at 2 m. For each original height, there is a corresponding bounce height. The students can record their data in a table such as the one below, which is a typical record of one ball's trials.

Each trial will result in two values; these can be thought of as variables. The values can then be plotted on a graph. Now the mathematical problem starts. The problem for each group is to do the trials, make the graph and see if they can tell from the graph what the relationship between the original height and the bounce height is.

The Graph

Two suggestions for making a graph are

1. Put the original height on the x-axis and the bounce height on the y-axis.
2. Use grid paper but make the y-axis a scale of 2 larger than the x-axis.

Teaching Suggestions for the Trials

Student groups can work at different stations around the classroom. A hard floor is essential because most balls do not bounce very well. Each group needs a different ball, as well as adding machine tape on which to mark original heights and bounce heights. They should mark the initial bounce height, then repeat the trial to make sure the mark is correct. Once the marks are made accurately, the paper may be laid on the floor and measured precisely. Students might use a system of marking the original height with one colored pen and bounce height with another color (or a pencil). They will need to experiment with a few drops of the ball to get their markings more accurate. One person should hold the ball, a second should make sure it is at the right original height and the third should be ready to note its bounce height.

Trial number	Original height	Bounce height	Bounce height based on the line of best fit	Line of best fit equation
1	0	0	0	
2	0.6	0.2	0.2	$y = 0.3x$
3	0.9	0.3	0.3	
4	1.2	0.4	0.4	
5	1.6	0.5	0.53	
6	2.0	0.7	0.66	

After each trial has been completed at the different heights, the students can begin making the graph. Each student should make his or her own graph to ensure that each has a record of the group's findings for future reference.

The teacher may want to specify the original heights. This is not necessary but adds a measure of control to the trials. We have noted that, unless the teacher gives firm guidelines, students are not good at carrying out accurate measurements. Accuracy is not essential for the purposes of this lesson, but it is still important.

Observations from the Graph

When the graphs are completed, a discussion can reveal that the graph is described as linear. Students should talk about error in measurements; they should then make a "line of best fit" in accordance with these measurements. This, of course, is an approximation though it still must go through the origin (0, 0); if the original height is zero, the bounce height will be zero. Once they have a line, students should rename the points of the bounce height (the fourth column of our table). The students can now use their calculators to find the multiplier of the x-value (original height) to give the y-value (bounce height). This may require some teacher guidance because it is really a trial-and-error process. Following the discovery of the multiplier, every bounce height should be checked against its original height. A two-decimal place number (designated as B) should be used. This will result in the equation $y = Bx$, where y is bounce height and x is original height for the line of best fit. A discussion of the equation will reveal that the numerical coeffecient, B, is the bounce of the ball.

After students have the graphs of their points, straight lines and B values, all 10 lines can be plotted on the same graph on the blackboard, overhead projector or computer, if available. All lines should go through the origin, and they should all have a different slope.

Discussion Questions

- What does mean if the line is straight? (The ball bounces the same fraction for all heights.)
- Which graph has the points closest to the line of best fit? What does this mean? (A "perfect" ball situation and careful measurement.)
- What does it mean when a point falls far from the line? (It probably means an error in measurement.)
- In comparing all of these lines, which one represents the ball with the most bounce? (The line with the steepest slope.)
- Why are none of the B values greater than one? (A ball cannot bounce higher than its original height.)
- What number can be used for labeling the balls? (Use rounded B value.)

Lesson Conclusion

Each group can choose one of its graphs to be displayed. The original points and the line of best fit should be indicated in addition to the B value for their ball. If the points are close to the line, it means that the measurements were taken accurately. The fact that all these points lie on a straight line shows that the ball's bounce is the same fraction regardless of the height it is dropped from. For the consumer group, each ball could be labeled with a two-digit number, given by B × 100. Therefore, a ball with a label of 24 will bounce 0.24 of its original height. Question 5 in the student activities deals with consumer labels.

The discussion should also include balls of special note such as volleyballs and basketballs. These balls should have a standard bounce. In fact, if two basketballs and two volleyballs are in the trial, their bounce can be compared. Would a ball that had more bounce be better in either sport? In baseball, for example, the "superball" favors the batter.

Students may also be interested to know that the numerical coefficient which is being called "bounce" is actually called the "coefficient of elasticity" in science. Mathematically speaking, however, this coefficient is the slope of the line. It is evident that for every metre by which the original height increases, the bounce height increases B metres. It should also be noted that none of the lines of best fit has a larger slope than B = 1. If B were greater than one, it would mean that the ball was bouncing higher than its original height, which is not possible. If the x and y-axes have the same scale, then no line is at an angle greater than 45°.

Students may be asked to attach their adding machine tape to their graph. This is a kind of trial record to show an observer how accurately the trial was done. It is noteworthy that the science involved in this activity is in making nice drops and accurate measurements. Scientists would also be interested in explaining why the graph of the ball's bounce height falls in a straight line. Mathematicians, on the other hand, would be curious mainly about the relationship being developed.

Materials

Several differing balls, adding machine tape and grid paper are essential items for this activity. In preparation for this class, the teacher should either borrow some balls from the gymnasium or have the students bring 10 different balls. Some students will have super-bounce balls, while others will bring volleyballs, basketballs, tennis balls or racket balls. Students should also be encouraged to bring any balls that are especially bouncy. Perhaps, if more than 10 balls are brought, the teacher can select balls that have some range of bounce.

Modifications

If students are not familiar with finding linear relationships or if they are not familiar with graphing, the fraction for the five different heights can be averaged. This average will be the bounce of the ball.

Rating the Bounce of Balls Student Activities

General Question

Everyone knows that different balls bounce differently. How high will a ball bounce? A consumer group wants to rate how high a variety of balls will bounce. Can you develop a system for rating the bounce of a variety of balls? The bounce from different heights will have to be considered.

Activities

1. Use your calculator and experiment to find the multiplier (the multiplier times the first number will give you the second number) for these pairs of numbers.
   (a) (2, 5), (4, 10), (6,15)
   (b) (20, 1.2), (40, 2.4), (60, 3.6)
2. In the following sets of numbers, one of the sets does not quite fit the pattern. What is the pattern (that is, what is the multiplier)? Which set does not fit the pattern?
   (a) (3, 0.75), (5, 1.25), (6, 1.45), (8, 2)
   (b) (0.2, 0.3), (0.3, 0.5), (0.5, 0.75), (0.6, 0.9)
3. Write the four different relations from questions 1 and 2 as
   second number (S) = multiplier (M) × first number (F),
   or S = M × F
   For example, the relation for question 1a is
   S = 2.5 × F
4. Given below are the measurements for a trial of bounces:
   (a) For each trial, find the multiplier that the original height must be multiplied by to get the bounce height.
   (b) Examine the column of multipliers and choose a reasonable average multiplier. Using this average multiplier, fill in new bounce heights in the appropriate column. What is the relation now between the new bounce height and the original height?
   (c) What is the difference between the new bounce height and the experimental bounce height for each trial? Fill in the final column of the table with the differences? Why is there a difference?
   (d) Plot the original heights and bounce heights on a graph. Then, plot the new bounce heights on the same graph. Indicate any discrepancies.
5. (a) A consumer rating group has given you a ball that has a bounce rating of 45. Make a graph using similar original heights as in your trial and the expected bounce heights.
   (b) Using the graph and your knowledge of bounce heights, determine how much higher this ball will bounce for each increase of 1 m in the original height.

Authors' Note: Those readers interested in the entire volume of "A Collection of Connections," may contact Sol E. Sigurdson, Faculty of Education, Department of Secondary Education, University of Alberta, Edmonton T6G 2G5; phone (780) 492-0753.

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Trial	Original height	Bounce height	Multipliers	New bounce height based on average multiplier	Differences bounce ht. and new bounce ht.
1	0	0			
2	0.6	0.24			
3	0.9	0.36			
4	1.2	0.50			
5	1.6	0.65			
6	2.0	0.78			

Calendar Math

Art Jorgensen

Here are math exercises for the month of September.

1. When you write the numerals from 1 to 100, how many times do you write the digit 3?
2. A mathematics contest has only three questions for which 2 points, 3 points and 4 points can be given, respectively. What scores are possible?
3. Susan has 6 blouses and 5 pair of slacks. How many different combinations of a blouse and a pair of slacks could she wear?
4. Mr. Johnson wants to enclose his dog in a square pen with a perimeter of 200 m. If the posts are 10 m apart, how many posts will he need?
5. Tom can mow a lawn in 4 hours. Bill can mow the same lawn in 3 hours. If they work together, how long will it take them to mow the lawn?
6. If the sides of a square were 2 cm longer, the area would contain 64 more square centimetres. What are the lengths of the sides of the original square?
7. Find 2 numbers whose sum is 11 and whose product is 24.
8. In a parking lot there are 10 vehicles which are either motorcycles or cars. Altogether there are 32 wheels. How many cars and how many motorcycles are in the parking lot?
9. If you cut a circle with 3 straight lines, what is the fewest number of pieces you will have? What is the largest number of pieces you will have?
10. If circles are worth 2 points, triangles are worth 3 points and squares are worth 5 points, draw a figure that has a total point value of 49 points. (There are several possible solutions.)
11. Using pennies, nickels and dimes, how many ways can you make change for a quarter?
12. A hen lays 4 eggs one week and 5 eggs the next week. If this pattern is repeated, how many eggs would the hen lay in one year?
13. If Bimba gives one half of his candies to Domenic and one third of his candies to Maria, what fraction of his candies are left for himself?
14. To go to school Daniel has to walk 3 blocks north, then 4 blocks west. If Daniel could walk directly to the school from his house, how many blocks would he have to walk?
15. Tim's birthday is April 19. Sarah's birthday is 23 days later. On what date is Sarah's birthday?
16. What are the fewest number of coins I can use to pay for a 99¢ hamburger?
17. What numbers come next in the following sequence? 1, 2, 3,__, __, __.
18. To fly from Saskatoon to Toronto takes 3 hours. If a plane leaves Saskatoon at 11:30 a.m., what time will it arrive in Toronto?
19. Jane says that she is 3 years younger than her brother and 2 years older than her sister. If the sum of their ages is 37 years, how old is each child?
20. Becky bought a hamburger for $1.59 and a milkshake for $2.37. How much change did she get from a $5 bill?
21. A triangle has a perimeter of 27.9 cm. If the lengths of two of the sides are 7.8 cm and 12.3 cm, respectively, what is the length of the third side?
22. Complete this pattern. 2, 2, 4, 6, 10, 16,__, __, __, 110.
23. If vowels are worth 5 points and consonants are worth 10 points, how much is the word *Mississippi* worth?
24. A perfect score in 5-pin bowling is 450 points. Wesley scored 219 points. How many points fewer than a perfect score did he score?
25. Teresa was asked to write a specific digit number. When she wrote her number, she reversed the unit and tens digits. The number she wrote was 62 larger than the number she was asked to write. What number was she asked to write?
26. Miss Olsen bought candy bars for each of her 24 students. However, after handing out the candy bars she found that she still had one third the original number. How many students were absent?
27. Ben Franklin went to a party where 10 people were present. Everybody shook hands with

everybody else. How many handshakes were there altogether?

28. Sue began a walking program. She walked 1 km on the first day, 2 km on the second day and 3 km on the third day. If she continued the same pattern for a week, what was the total distance she walked?
29. Create a word problem that has an answer of 12. Doing this can result in a variety of interesting problems, especially with practice.
30. A telephone company charges 15¢ for the first minute and 9¢ for each additional minute. If the charge for Anita's call was 96¢, how long did Anita talk?

Answers

1. 20
2. 0, 1, 2, 3, 4, 5, 6, 7, 8, 9
3. 30
4. 21
5. 1 5/7 hours
6. 15 cm
7. 3 and 8
8. 6 cars, 4 motorcycles
9. (a) 4 (b) 7
10. Several possible solutions
11. 9
12. 234
13. 1/6
14. 5 blocks
15. May 12
16. 9 (3 × 25¢, 2 × 10¢, 4 × 1¢)
17. 4, 5, 6
18. 4:30 p.m.
19. Jane's sister is 10 years old, Jane is 12 and her brother is 15.
20. $1.04
21. 7.8 cm
22. 26, 42, 68
23. 90
24. 231
25. 19
26. 8
27. 45
28. 28 km
29. Answer will vary
30. 10 minutes

The Locker Problem: An Explanation Using the Properties of Primes

Murray L. Lauber

The Problem

A small-town high school, we'll call it Hamlet High, has exactly 100 students and exactly 100 lockers, one for each student. After a lesson on combinations, the Math 30 class at Hamlet High decided to conduct an experiment. With the authority that comes from seniority, they lined up all the students in the hall after school, each student next to his or her locker. Then, at their bidding, student 1 opened every locker, student 2 closed every second locker, student 3 changed the state of every third locker and so on, for what the Math 30 class hoped would be all 100 students. But after student 25, amidst rumblings of mutiny, changed the state of every 25th locker, the students finally revolted and headed for Hamlet's pizza place. The Math 30 class, left with an unfinished experiment, might have abandoned it. But after studying and arguing over the incomplete results, they concluded that they could predict a priori which lockers would be open without having to coerce a single student's cooperation. Moreover, they were audacious enough to declare that the pattern they had found could be extended to 1,000 mutinous students, or even to 10,000 such students.

What pattern did the Math 30 class discover? What is its mathematical basis?

A Solution

The Math 30 class concluded that, at the end of the experiment, the lockers whose numbers were perfect squares, that is, lockers 1, 4, 9, 16, 25, 39, 49, 64, 81 and 100, would have been open and the remainder closed. Alan constructed a computer program with a 100 × 100 array to mimic the experiment, but Theresa, who, along with a great many mathematicians, was of the opinion that a computer program did not constitute a proof, went further. She discovered that perfect squares have an odd number of factors, whereas other counting numbers have an even number of factors. The insight behind her discovery came from examining the factors of the numbers 36 and 40.

36 has factors 1, 2, 3, 4, 6, 9, 12, 18 and 36. Thus locker number 36 would have its state changed by students 1, 2, 3, 4, 6, 9, 12, 18 and 36 and not by any other students. Locker 36, having had its state changed an odd number of times, would end up in a state opposite to its initially closed state—that is, open.

Theresa pursued the analysis further. The prime factorization of 36 is

$$36 = 2^2 \cdot 3^2$$

She constructed the factors of 36 from this prime factorization as follows:

$2^0 \cdot 3^0 = 1$	$2^0 \cdot 3^1 = 3$	$2^0 \cdot 3^2 = 9$
$2^1 \cdot 3^0 = 2$	$2^1 \cdot 3^1 = 6$	$2^1 \cdot 3^2 = 18$
$2^2 \cdot 3^0 = 4$	$2^2 \cdot 3^1 = 12$	$2^2 \cdot 3^2 = 36$

These factors of 36 were obtained systematically by using combinations of 2s and 3s: no 2s, one 2, or two 2s in combination with either no 3s, one 3, or two 3s.

Theresa then determined the prime factorization of 40:

$$40 = 2^3 \cdot 5^1$$

From this prime factorization, she constructed the factors of 40 as follows:

$2^0 \cdot 5^0 = 1$	$2^0 \cdot 5^1 = 5$
$2^1 \cdot 5^0 = 2$	$2^1 \cdot 5^1 = 10$
$2^2 \cdot 5^0 = 4$	$2^2 \cdot 5^1 = 20$
$2^3 \cdot 5^0 = 8$	$2^3 \cdot 5^1 = 40$

These factors are the possible combinations of 2s and 5s: no 2s, one 2, two 2s, or three 2sin combination with no 5s or one 5. Since there are 8 such factors, locker 40 would have its state changed an even number of times and would be in the same state in which it started—that is, closed.

The Mathematician's Solution

Theresa ended her initial investigation here but later generalized the solution as follows.

As with 36, so with any perfect square. Suppose that N is a perfect square. Then its prime factors each occur an even number of times. So let

$$N = p_1^{2k_1}, p_2^{2k_2} \dots p_t^{2k_t}$$

where each p_i is a prime and each k_i is a counting number (making $2k_i$ an even counting number). Then the number of factors of N is $(2k_1 + 1)(2k_2 + 1) \dots (2k_t + 1)$. This is a product of odd numbers and therefore odd. Thus locker N will have its state changed an odd number of times. Having started out closed, it will end up open.

As with 40, so with any counting number that is not a perfect square. Suppose that M is such a number. Then at least one of the prime factors of M occurs an odd number of times. Suppose that prime factor is p_j and that it occurs $2k_j - 1$ times. Let

$$M = p_1^{k_1}, p_2^{k_2} \dots p_j^{2k_j - 1} \dots p_t^{k_t}$$

where each p_i is prime and each K_i a counting number. Then the number of factors of M is $(k_1 + 1)(k_2 + 1) \dots (2k_j) \dots (k_t + 1)$. This product has an even factor, $2k_j$ (and perhaps others as well) and is therefore even. Thus locker M has its state changed an even number of times and ends up in its starting state, namely closed.

Reference

Stewart, B. M. *Theory of Numbers*. 2d ed. New York: Macmillan, 1965.

Any book on number theory should contain some treatment of the properties of primes. The Stewart book, pages 62–68, is one that I happen to have in my office library.

The Window

A window, whose four sides are each 1 m long, is to be divided into eight individual panes using cross pieces. Each pane is to be 50 cm × 50 cm. Can this problem be solved?

Value of Sheep

Two shepherds owning a flock of sheep agree to divide its value: A takes 72 sheep and B takes 92 sheep and pays A $350. What is the value of a sheep?

The Ancient Problem of Trisecting an Angle

Sandra M. Pulver

The problem of trisecting an angle dates back to the ancient Greeks, and as early as the 5th century BCE, Greek and Muslim geometers devoted much time to this puzzle. This problem is one of the Three Famous Problems, which also include doubling the cube and squaring the circle. These three great construction problems of geometry could not be solved using an unmarked straightedge and compass stone, the only implements sanctioned by the ancient Greeks. But it was not until the 19th century that advances in the algebra of the real number system allowed us to make instruments which made possible these constructions that were impossible with the straightedge and compass alone.

This problem is certainly the simplest one of the three famous problems to comprehend, and because the bisection of an angle presented no difficulty to the geometers of antiquity, there was no reason to suspect that its trisection might prove impossible.

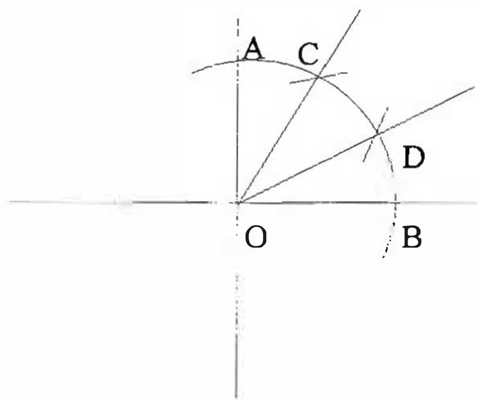
The multisection of a line segment with Euclidean tools is a simple matter, and it may be that the ancient Greeks were led to the trisection problem in an effort to solve the analogous problem of mutisection an angle. Or perhaps, more likely, the problem arose in efforts to construct a regular nine-sided polygon, where the trisection of a 60° angle is required.

The angle trisection problem is not entirely unsolvable using the classical method of compass and straightedge. Even the Greeks knew this, but they were searching for a generalized construction (such as one of the angle bisection) that could be used to trisect any angle.

Actually, there are an infinite number of angles that can be trisected. Among this group are angles whose degree measure equals $360/n$ where n is an integer not evenly divisible by three.

That is, a 90° angle can be trisected because n in $360/n$ is four, which is not evenly divisible by three. Figure 1 is a trisected 90° angle.

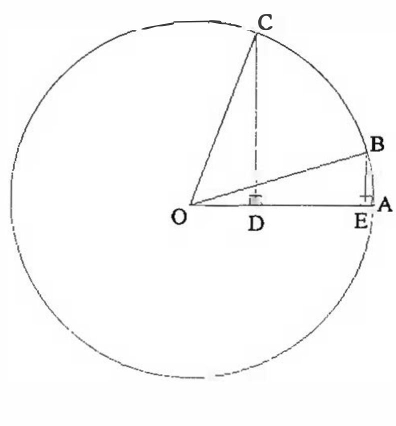
Figure 1
Trisection of a 90° Angle, ∡AOB



The trisection of the 90° angle can be done quite simply using the following method:

Construct a 90° angle. Then, draw arc AB. Without changing the size of the compass opening, place the compass at point B and draw an arc intersecting arc AB at point C. Line OC is the line trisecting right angle AOB. Line OD also trisects right angle AOB using the same method outlined above, and placing the compass at point A. Line OD also bisects 60° angle COB.

Figure 2



However, there are, of course, an infinite number of angles that cannot be trisected by means of compass and straightedge. These are angles whose degree measures are equal to $360/n$, where n is an integer divisible by three; for example, a 60° angle cannot be trisected because n, in $360/n$, would be six, which is divisible by three. To prove that general angle trisections are impossible with just an unmarked straightedge and compass, we use the special case of a 60° angle.

In Figure 2, suppose ∡ COA = 60° and ∡ BOA = 20°. For the proof we make use of the following trigonometric identity:

$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

Let $3\theta = 60°$ and let $x = 2\cos\theta = 2\cos 20°$. Then

$$\cos 60° = \frac{x^3}{2} - \frac{3x}{2}$$

$$2\left(\frac{1}{2}\right) = \left(\frac{x^3}{2} - \frac{3x}{2}\right)2$$

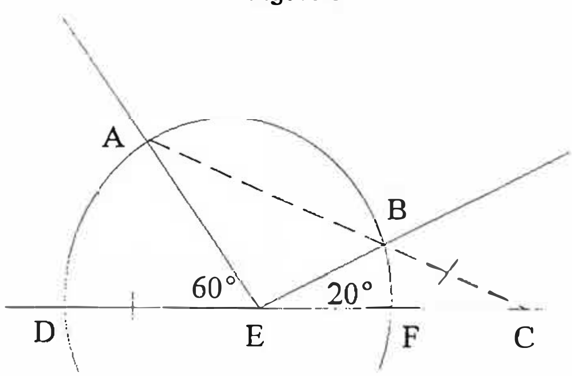
$$1 = x^3 - 3x$$

$$x3 - 3x - 1 = 0$$

This cubic equation is irreducible. Thus, its roots cannot be constructed with a straightedge and compass. From this, we can conclude that the construction of trisecting the general angle cannot be performed by the use of the straightedge and compass alone.

One ingenious method for trisecting angles was presented by Archimedes who used the marking of two points on a straightedge to mark off a line segment (that is, not using the classical rules of compass and straightedge). Figure 3 is a trisected 60° angle.

Figure 3

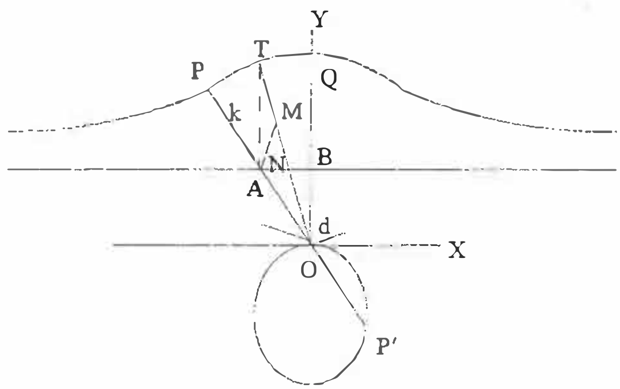


Archimedes' method is the following: Draw angle AED. Draw a semicircle through the angle whose radius is the same length of DE. Extend DE to the right. With the compass open to the length of radius DE, hold the legs of compass against the straightedge and hold the straightedge so it passes through point A. Adjust the straightedge till the points marked by the compass intersect points B and C (where BC is equal to DE). Arc BF is one third arc AD. Since central angles are congruent in degree measure to their intercepted arcs, ∡ BEF is one third the degree measure of ∡ AED.

Another method of trisecting an angle is by using the Conchoid of Nicomedes (c. 1800 BCE). To define a conchoid, we refer to Figure 4.

Figure 4

Conchoid of Nicomedes



Nicomedes took a fixed point O, which is d distant from a fixed line AB, and drew OX parallel to AB and OY perpendicular to OX. He then took any line OA through O and on OA made AP = AP' = k, a constant. Then the locus of points P and P' is a conchoid. The equation of the curve is

$(x^2 + y^2)(x - d)^2 - k^2x^2 = 0.$

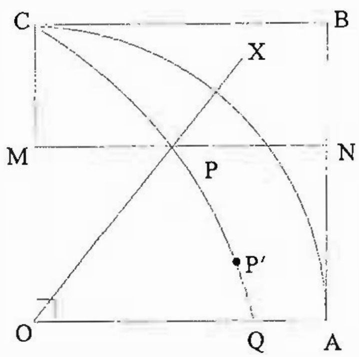
To trisect a given angle, let ∡ YOA be the angle to be trisected. From point A, construct AB perpendicular to OY. From point O as pole, with AB as a fixed straight line, 2(AO) as a constant distance, construct a conchoid to meet OA produced at P and to cut OY at Q. At A, construct a perpendicular to AB meeting the curve at T. Draw OT and let it cut AB at N. Let M be the midpoint of NT.

Then MT = MN = MA.
But NT = 2(OA) by construction of the conchoid.
Hence MA = OA.
Hence ∢AOM = ∢AMO = 2 ∢ATM = 2 ∢TOQ.
That is, ∢AOM = $^2/_3$ ∢YOA, and ∢TOQ = $^1/_3$ ∢YOA.

Hippias of Elis (b. 460 BCE) wrestled with this problem and, realizing the inadequacy of ruler-and-compass method, resorted to other devices. These involved the use of curves other than the circle, and the one employed by Hippias was the quadratrix, so called because it serves equally well for the problem of quadrature (squaring the circle) as for the dividing of an angle into three, or indeed, any number of equal parts.

The quadratrix of Hippias may be defined as follows: Let the radius OX of a circle rotate uniformly about the centre O from OC to OA, at right angles to OC. At the same time, let a line MN parallel to OA move uniformly parallel to itself from CB to OA. The locus of the intersection P of OX and MN is the quadratrix (Figure 5).

Figure 5
Quadratrix of Hippias



In the trisection of an angle, X is any point in the quadrant AC. As the radius OX revolves at a uniform rate from OC to OA, MN always remains parallel to OA. Then if MN is one nth of the way from CB to OA, the locus of point P, the intersection of OX and MN, is one nth of the way from OC around to OA. If, therefore, we make CM = $^1/_3$ (CO), MN will cut CQ at a point P such that OP will trisect the right angle. In the same way, by trisecting OM we can find a point P' on CQ such that OP' will trisect angle AOX, and so for any other angle. This method evidently applies to the multisection as well as to the trisection of an angle.

In this article I have cited only three of the most ancient methods. There are many other ways to trisect an angle by using other techniques such as the "Tomahawk" and the Mira.

Other ingenious mathematicians of recent times have developed original methods to trisect angles. Leo Moser from the University of Alberta trisected angles with the use of an ordinary watch. He said that if the minute hand passed over an arc equal to four times the measure of the angle to be trisected, the hour hand would move through an arc exactly one third the measure of the given angle to be trisected. Alfred Kempe, a London lawyer, developed a linkage method of folding parallelograms so that the two opposite sides cross.

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Five-Digit Prime Numbers

Is it possible to find a five-digit prime number whose sum of the digits is 21?

Did You Know That? Against All Odds, Lotto 6/49!

Ian deGroot

The growth of lotteries in Canada has provided a source of great math connections and interesting problems involving probability and combinations that are well worth sharing with students.

Did you ever wonder how they calculate the odds of winning any of the five prizes in the Lotto 6/49? I spent many an evening with my calculator trying to make sense of the many combinations. I chanced upon an article in the November 4, 1996, *Maclean's* that discussed the probability of winning the Lotto and which in turn led me to an Internet site compiled by Fred Hoppe, a professor of mathematics and statistics at McMaster University in Ontario: http://www.the-web.net/Lotto.

I could easily calculate the odds of winning the top prize, the jackpot, but I must admit that I was baffled as to how to calculate the odds of winning the lesser prizes. So what follows was gleaned from Professor Hoppe's explanations.

Jackpot (All Six Winning Numbers Selected from 49 Possibilities)

There are a total of 13,983,816 different groups of six numbers that could be drawn from the set of numbers 1, 2, ..., 49. How do we get that? There are 49 possibilities for the first number drawn, following which there are 48 possibilities for the second number, 47 for the third, 46 for the fourth, 45 for the fifth, and 44 for the sixth. Multiply (49)(48)...(44) to get 10,068,347,520.

Each possible group of six numbers (combination) can be drawn in different ways depending on which number in the group was drawn first, which was drawn second and so on. There are 6 choices for the first, 5 for the second, 4 for the third, 3 for the fourth, 2 for the fifth, and 1 for the sixth. Multiply these numbers out to get 720. We then divide 10,068,347,520 by 720 to arrive at 13,983,816 as the number of groups of six numbers (different picks).

This is the same as calculating the number of combinations of 49 numbers taken six at a time, ${}_{49}C_6$. Since all numbers are assumed to be equally likely and since the probability of some number being drawn must be one, it follows that each pick of six numbers has a probability of $1 \div 13{,}983{,}816 = 0.00000007151$. That is the probability that you will win the jackpot in the Lotto 6/49!

To use a concrete example, this can be considered to be roughly the same probability as obtaining 24 heads (or tails) in succession when flipping a fair coin, 2^{24}.

Second Prize (Five Winning Numbers + Bonus)

To win the second prize in the 6/49, the pick of six must include five winning numbers plus the bonus. Because five of the six winning numbers must be picked, this means that one of the winning numbers must be excluded. There are six possibilities for the choice of the excluded number, ${}_6C_1$; therefore, there are six ways for a pick of six to win the second place prize. Therefore the probability is $6 \div 13{,}983{,}816 = 0.0000004291$. Invert this number to get the odds of 2,330,459:1. This is roughly the same probability as obtaining 21 heads (or tails) in succession when flipping a fair coin! That is about 2^{21}.

Third Prize (Five Winning Numbers Only Selected)

As with the second prize, there are six ways for a pick of six to include exactly five of the six drawn numbers, ${}_6C_5$. The remaining number must be one of the 42 numbers left after the six winning numbers and the bonus number have been excluded. Therefore there are a total of $6 \times 42 = 252$ ways for a pick of six to win the third prize. To get this probability divide $252 \div 13{,}983{,}816$ to get 0.00001802, which when inverted gives odds of about 55,493:1. Again, this is about the same as the probability of obtaining 16 heads (or tails) in succession when flipping a fair coin. That is about 2^{16}.

Fourth Prize (Four Winning Numbers Selected)

There are 15 ways to include four of the six winning numbers, ${}_6C_4$, and 903 ways to include two of the 43 nonwinning numbers, ${}_{43}C_2$, for a total of 15 × 903 = 13,545 ways for a pick of six to win the third prize, which works out to a probability of 13,545 ÷ 13,983,816 = 0.0009686, which is odds against of about 1,032:1 or about the same as tossing 10 heads in a row with a coin.

Fifth Prize (Three Winning Numbers Selected)

There are 20 ways to include three of the six winning numbers, ${}_6C_3$, and 12,341 ways to include three of the 43 nonwinning numbers, ${}_{43}C_3$, for a total of 20 × 12,341 = 246,820 ways for a pick of six to win the fourth prize. This is a probability of 246,820 ÷ 13,983,816 = 0.01765, which gives odds against of about 57:1. Again, this is about the same as tossing 6 heads (or tails) in a row, 2^6.

I also discovered that a British actuary, Jim Roberts, used the same odds as Lotto 6/49 to compute the life expectancy of men and women. Here is what he calculated:

> A 20-year-old man is as likely to die in the next 40 minutes as to win the jackpot. A 40-year-old man is as likely to die in the next 20 minutes. A 60-year-old man is as likely to die in the next 2 minutes and an 80-year-old man in the next 20 seconds!

It seems that things are somewhat brighter for females:

> A 20-year-old woman has an equal chance of dying in the next 107 minutes as winning the jackpot, while an 80-year-old could last 30 seconds!

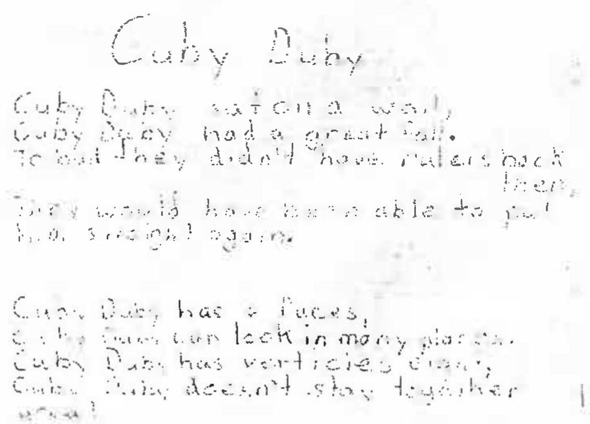
So now you know!

The Brooches of the Princess

Princess Astrid has a number of valuable brooches in her safe. Every brooch has the same number of diamonds on it. If one knew the number of diamonds in the safe, one would be able to determine the number of brooches and the number of diamonds on each brooch. Let me tell you that there are between 200 and 300 diamonds in the safe. How many brooches with how many diamonds on each brooch does the princess own?

An Angle on Multiple Intelligences in Geometry

Ellen Meller



Could this little song have sprung from a math class? Why would a math teacher request her students to write such a song?

Two girls in my Grades 4–5 class wrote this song during a math lesson to describe the attributes of a cube. But this was not a traditional math lesson; this lesson was one of several multiple-intelligences lessons that I used in an attempt to reach the energetic, often behaviorally challenged learners in my classroom. Not only did I have some children who exhibited off-task and sometimes violent behavior but also the range of learners' abilities in my class spanned eight years. I was looking for some strategies to make learning captivating, challenging and enjoyable.

Teaching in the 1990s means dealing with many issues, such as students who exhibit disruptive behaviors, large class sizes, a wide range of learning abilities and an overloaded and ever-changing curriculum. Today, teachers work with children who spend most of their free time in front of the television or playing computer games. Computer games present a new stimulus every 15 seconds, so it is little wonder that children are captivated by the myriad electronic and computerized devices around them. Students expect to be entertained and stimulated in school the same way as they are by electronic toys. Some children have a difficult time focusing their attention on what they need to learn. Many teachers, including me, sense this challenge and try desperately to capture students' attention.

I lay awake night after night thinking about the challenges of being a teacher in the 1990s. I was frustrated with trying to cope with all that was going on inside and outside my classroom. I struggled to find a solution. Then, as if in answer to my questions and concerns, I discovered an article in my mailbox outlining the multiple intelligences approach to teaching. As I read, I wondered if the multiple intelligences strategies would help me deal with the issues that kept me awake. I was also hopeful that the multiple intelligences approach would help improve student learning and motivation. Finally, after months of researching the subject, I decided that the multiple intelligences technique was worth a try. I developed a geometry unit using multiple intelligences to find out for myself if the theory lived up to its promises.

What Is the Multiple Intelligences Approach?

The multiple intelligences approach is a teaching tool that allows students multiple options for taking in information, making sense of ideas and expressing what they learn. The multiple-intelligences approach recognizes that students have learning strengths and weaknesses. It also acknowledges that these strengths can be further developed and that the weaknesses, to at least some extent, alleviated. Teaching with multiple intelligences allows teachers to meet a wide range of student needs because information is presented in various ways. Both teachers and students can choose activities within the curriculum that will help the students learn best while strengthening minor intelligence areas. This concept is not new to most teachers. Teachers have been using a variety of strategies for years. However, multiple intelligences provides a structured approach to adding variety in the classroom. It is a medium to reach children that corresponds with their learning strengths while giving them an opportunity to expand their list of strengths. Multiple intelligences also offers students a chance to challenge themselves in less dominant learning modes.

The multiple intelligences theory was first articulated by Howard Gardner, a neuropsychologist, who worked with brain-injured and savant individuals in the 1980s. Gardner (1983) concluded that each person has seven distinct intelligences:

- Visual/Spatial—the ability to create visual or spatial representations mentally or concretely
- Mathematical/Logical—the ability to use inductive and deductive reasoning, and to recognize and manipulate abstract patterns and relationships
- Bodily/Kinesthetic—the ability to use the body to solve problems or create products and convey ideas and emotions
- Verbal/Linguistic—the ability to read, write and work with words
- Musical—the ability to use music; this includes a sensitivity to pitch, timbre and rhythm of sounds
- Interpersonal—the ability to work with others and understand them
- Intrapersonal—to be deeply aware of one's own feelings, intentions and goals

Recently, Gardner has articulated an eighth intelligence and labeled it the *naturalist intelligence*, the ability to see similarities and differences in one's environment.

Multiple Intelligences Geometry Unit

The multiple intelligences geometry unit that I developed consisted of seven lessons. Each lesson (except the first) revolved around an objective drawn from the *Program of Studies* (Alberta Education 1993). The first lesson was designed to provide an overview of the multiple intelligences theory (see Figure 1). The remaining six lessons explored geometry concepts using a different intelligence in each lesson.

Lesson 1 (Introduction)

✓ Ask the students what they think it means to be smart.

✓ List the student replies on an overhead projector. (I was surprised to find that they knew there were many and varied kinds of intelligence other than linguistic and mathematical.)

✓ Show the students a chart listing all the intelligences, the characteristics of each and a famous person who displayed each intelligence (Figure 1).

✓ Discuss each of the intelligences. Ask the students to think about where their strongest and weakest intelligences might be.

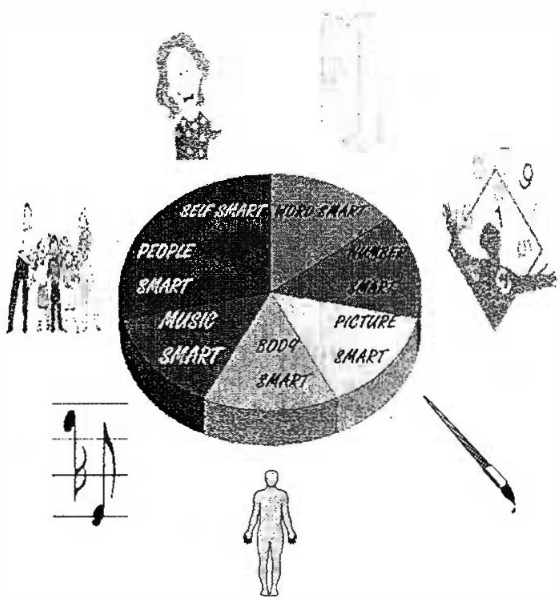
✓ Show the students pictures and overheads of the famous people and what these people accomplished.

✓ Display a pie graph of the intelligences (Figure 2) that illustrates how each person holds all the intelligences within them but how some intelligences are stronger than others.

Figure 1
How Are You Smart?

Smart Area	Are You Good At...?	Do You Like To...?	Famous People
Word Smart	reading, writing, telling stories	read, write, talk, memorize	T. S. Eliot, Abraham Lincoln
Number Smart	math, problem solving, patterns	solve problems, work with numbers	Albert Einstein
Picture Smart	reading maps and charts, drawing	draw, design, look at pictures	Picasso, Frank Lloyd Wright
Body Smart	sports, dancing, crafts	play sports, dance, move around	Charlie Chaplin, Michael Jordan
Music Smart	singing, remembering songs	sing, hum, listen to music	Mozart
People Smart	understanding people, resolving arguments	be with friends, talk to people	Mother Teresa
Self Smart	recognizing your strengths, setting goals	work alone, think alone	Sigmund Freud

Figure 2



Lesson 2 (Mathematical/Logical)

Objective: *Students will use the terms* line, segment, ray *and* angle.

✓ Show and discuss pictures of cities and buildings where geometric shapes can be seen.
✓ Look around the classroom to find geometric shapes.
✓ On a sheet of chart paper, list the shapes and objects found, using terms such as *angle* and *line segment*. Discuss these terms—*ray, angle, line segment, vertex, edge* and *face*—as they apply to the shapes and objects.
✓ Go outside and find geometric shapes in the neighborhood. Add these items to the list.

Lesson 3 (Musical Intelligence)

Objective: *The students will distinguish two-dimensional figures from three-dimensional (3-D) objects by naming the 3-D objects as prisms or pyramids.*

✓ Show the children some two-dimensional figures and three-dimensional objects, especially those that are pyramids and prisms.
✓ Discuss the attributes of each geometric shape. Put these on large chart paper so children can see them throughout the lesson
✓ Ask the students to write a rap or simple song or verse about a geometric shape of their choice.
✓ Play some rap songs and simple tunes on the tape player to help generate ideas.
✓ Share the songs and raps with the entire class. Be sure they can classify the shapes and name the prisms and pyramids

Lesson 4 (Visual/Spatial)

Objective: *The students will construct three-dimensional objects, as well as name cylinders, cones and spheres.*

✓ Discuss the meaning of the term *three-dimensional*. Point out that geometric shapes contain vertices, angles, line segments and so on.
✓ Display a number of three-dimensional objects.
✓ Give each child several toothpicks and a bunch of miniature marshmallows and ask students to create a three-dimensional geometric shape of their choice.
✓ Have each child showcase the shape to the class and describe its characteristics.

Lesson 5 (Interpersonal)

Objective: *The students will classify and name two-dimensional shapes.*

✓ Have the students sit back to back, one with a clipboard holding an 8½ × 11" sheet of blank paper and the other with a slip of paper bearing the name of a two-dimensional shape. Use the following names of shapes: circle, triangle, rectangle, pentagon, hexagon, octagon.
✓ Ask the student with the name of the shape to, without looking at the paper, tell the student with the paper how to draw the shape. They are not allowed to mention the name of the shape itself or objects that obviously represent the shape.
✓ Ask the students to switch roles using another shape.

Lesson 6 (Bodily/Kinesthetic)

Objective: *The students will construct three-dimensional objects.*

✓ Allow the students to choose a group that they would like to work with. These can be groups of two to four. This activity involves touching so the children must be comfortable with the others in their group.
✓ Ask the groups to create a geometric shape of their choice with their bodies. For example, four children could hold out their arms and legs to create a cube.
✓ Have the groups demonstrate to the entire class how they made their shape and explain why the shape is three-dimensional. Make sure they use terms such as *line segments, angles, vertices, faces, edges* and *rays*.

Lesson 7 (Verbal/Linguistic)

Objective: *The students will describe the essential attributes of prisms, pyramids, cones, cylinders and spheres.*

✓ Discuss what a riddle is and share some with the class.

✓ Have the names of the three-dimensional objects listed in the objective above written on pieces of paper and give these to the children.

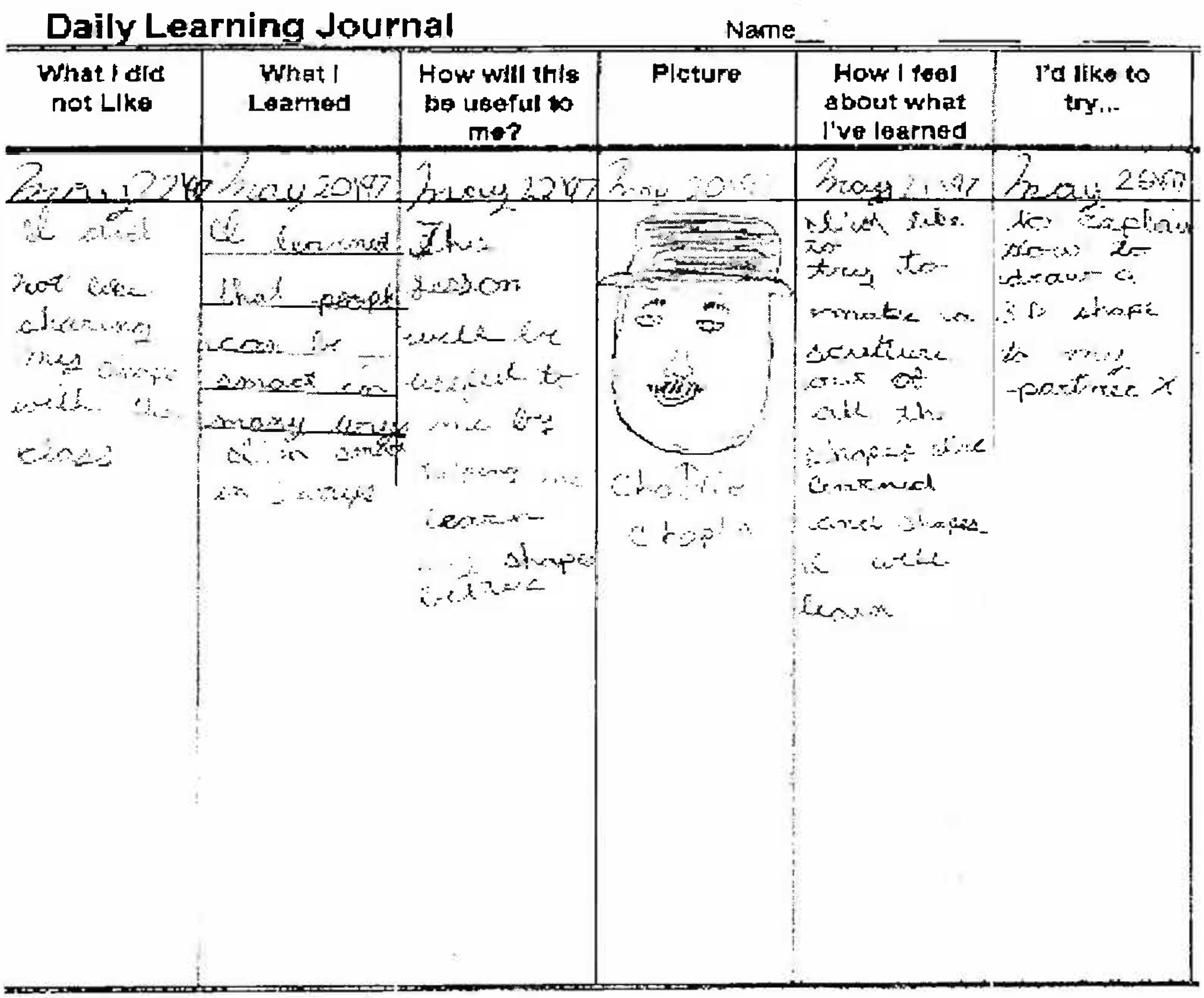
✓ Ask students to work alone or in pairs to create a riddle about the object on the piece of paper.

✓ Remind students that they need to use terms such as *vertices, edges, faces* and *angles*.

✓ Ask students to share their riddles and have the rest of the class try to solve them.

Each class lasted about 45 minutes and ended with time to write in a learning log (see Figure 3). This learning log asked the children to think about their learning in a deeper way (intrapersonal intelligence). The *Alberta Program of Studies for K–9 Mathematics* (Alberta Education 1996) requires that students communicate what they have learned in math, and this fits in well with the multiple intelligences approach. I prepared the learning log as a chart or graphic organizer. At the top of each column was a question to focus students' ideas.

Figure 3



Daily Learning Journal

Name

What I did not Like	What I Learned	How will this be useful to me?	Picture	How I feel about what I've learned	I'd like to try...

When I taught these lessons, I asked the children to fill in one or two columns a day responding to a variety of questions such as what they learned, how they felt about the lesson, what they liked and did not like about the lesson, how this lesson could be applied to other areas of their lives and what they would like to learn more about. The child chose to fill out the log using either words or pictures.

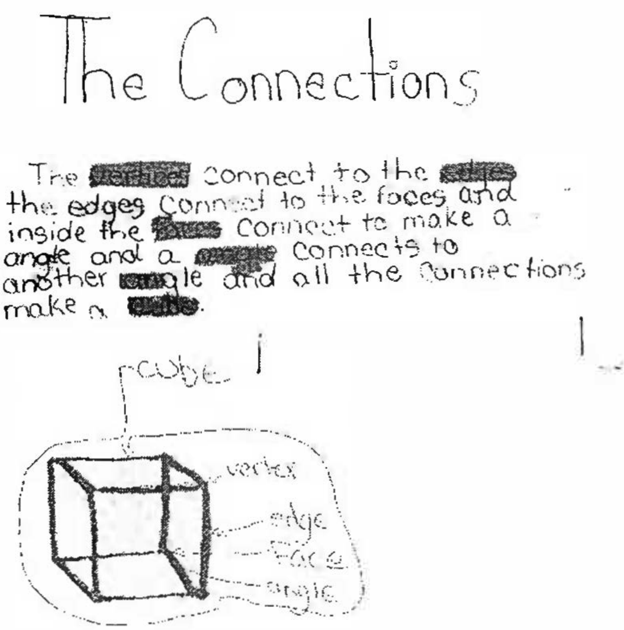
By the end of the last lesson, students were aware of the intelligences and how they could be used and also were motivated and enthusiastic about their learning. Students who normally exhibited behavior problems were on-task and ready to learn in this way. Most students said they wanted to try more of the strategies. The log notes demonstrated a good knowledge of geometric terms and an ability to classify three-dimensional objects and two-dimensional shapes.

This approach is not the only way to teach using multiple intelligences. Many teachers use centres and/or integrate several subjects within one theme. In a centres approach, several concepts are taught at each centre so the children learn not only in their dominant intelligences but develop their minor intelligences as well. Math integrates well with music, science, language arts and even physical education. So, for example, a problem-solving theme could involve the students writing their own problems using a code language (verbal/linguistic), adding some musical effects to bring the story to life or putting the story to music (musical) and designing a gymnastics routine that demonstrates how problems such as balance and use of space can be solved (bodily/kinesthetic).

What Are the Benefits of Multiple Intelligences ?

One main benefit of using the multiple-intelligences approach is that the same concept is taught in many ways, although not every concept needs to be taught using all the intelligences. Students can then assimilate the material in a way that they can understand. If one activity does not get the idea across, another activity will. Other benefits that teachers have found in their students include increased student leadership and responsibility for their own learning, improved behavior, improved cooperation, ability to work multimodally in student presentations and better retention of material.

The multiple-intelligences approach helped me individualize my math teaching. The students discovered what their dominant intelligences were, participated more in their learning and felt better about learning. I witnessed improved on-task behavior: students who normally worked well worked even better, and students who were often off-task were both on-task and enjoying themselves. In general, my students demonstrated a love for learning and an eagerness to learn.



The multiple-intelligences approach offers teachers and students multiple ways of teaching and learning math concepts. Some students have a negative outlook toward math but the multiple-intelligences approach can help change that attitude. Multiple intelligences is a tool that can help teachers reach more learners in the math classroom. As we reach more learners, students will experience more success in being able to make sense of the world of math.

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Tiling Pattern Possibilities: A Geometric Classroom Activity

Bonnie H. Litwiller and David R. Duncan

The NCTM curriculum standards call for integrating geometry at all levels of the curriculum. Teachers are always looking for interesting settings which can be used to aid in this integration. We shall present a situation involving floor tiles.

We recently saw two floor tile patterns in two unrelated buildings. In each case, identical copies of an original 8 × 8 square were used to cover the entire floor. These patterns are displayed in Figures 1 and 2. An obvious question is, Are these all the same pattern? A cursory investigation suggests that they are not. For instance, the upper left-hand corners are contained in, respectively, 2 × 1 and 1 × 2 rectangles. However, a count of the number of different sized rectangles in each arrangement yields the results shown in Table 1 for each of Figure 1 and Figure 2.

Figure 1

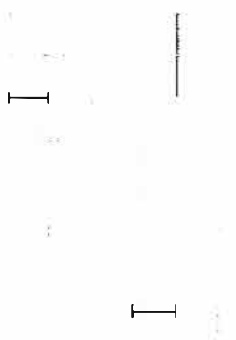


Figure 2

Table 1

Size of Rectangle	Number in Each Figure
1 × 1	14
2 × 2	6
2 × 1 or 1 × 2	13

The number of square units resulting from this distribution of rectangles is

14(1) + 6(4) + 13(2) = 64. This result led us to consider whether or not a rigid transformation can be applied to Figure 1 to yield Figure 2.

In fact, such a rigid transformation does exist. By trial and error, we found that the following steps will transform the pattern of Figure 1 into the pattern of Figure 2.

1. Rotate Figure 1 clockwise 90° about its centre.
2. Reflect the results of step 1 about the horizontal axis which passes through the midpoints of the opposite sides of the square.

The results of these two steps yield a pattern identical to that of Figure 2. Since a rotation is the combination of two flips about intersecting lines, the total transformation can be accomplished by three flips. Draw them.

You may note that a single reflection, about the lower-left to upper-right diagonal of Figure 1, will also transform the pattern of Figure 1 into the pattern of Figure 2. How can this result be reconciled with the previous rotation-reflection sequence?

Some students may note that although the transformation of Figure 1 into Figure 2 works mathematically, it would yield an upside-down tile. Would your students feel that this is an acceptable transformation?

In how many other ways could Figure 1 be transformed into other patterns by a series of rotations and/or flips? Recall that there are four possible rotations (0°, 90°, 180°, 270°); four possible flips (vertical at the midpoints of the opposite sides, horizontal at the midpoints of the opposite sides, and through the two diagonals); and two possible orderings of rotations and flips.

This is a nice group activity. Give each student a copy of Figures 1 and 2 and let the class experiment.

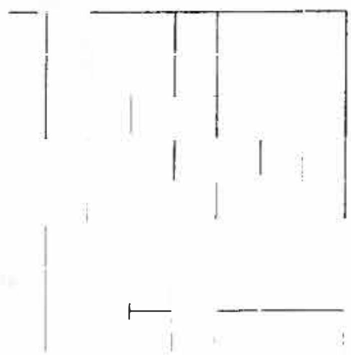
Extensions

1. Arrange the rectangle of Figure 1 to achieve patterns that are not the result of applying a rigid transformation.

   For example, Figure 3 is achieved by interchanging the 2 × 1 rectangle in the upper left-hand corner with the two 1 × 1 squares directly beneath it. In how many other distinct ways can the 33 original rectangles of Figure 1 be arranged into an 8 × 8 square?
2. Use combinations of other numbers of the original three types of rectangles to make new patterns on the 8 × 8 square.
3. Use rectangles of other sizes to make the 8 × 8 square.
4. Change the size of the 8 × 8 square.

We believe that mathematical situations constantly occur in nonconventional places. Teachers and students must be alert for them and share them in their classrooms.

Figure 3



Two Drivers

At 8:00 a.m., one driver starts his 782-km journey from A to B. He drives his car at 125 km/h. At 8:40 a.m., a second driver starts at B and drives toward A at 115 km/h. How far apart are the two cars 15 minutes before they meet on the highway?

The Problem with Factorials!

Klaus Puhlmann

One Saturday, my friend was preparing a presentation for an important conference when his daughter, Susan, stepped into his office. She appeared bored, began to play with his old typewriter, touched everything else in sight and muttered some incomprehensible gobbledygook. My friend asked her what she had done in school the day before. She replied that she had learned something about "factorial of a number." "What is that?" he asked. Susan replied that "the factorial of a positive integer is the product of all positive integers from 1 up to and including the given number." She added, "For example, the symbol for factorial is !, so that 4!, which is read '4 factorial,' means $1 \times 2 \times 3 \times 4 = 24$."

At this point, my friend presented Susan with a challenging problem in the hope that it would keep her occupied for a long time, so that he could complete his presentation. My friend made Susan a proposition: "When you add the factorials of all numbers from 1 to 100, that sum is a very large number. If you can find the last digit of that sum I will give you $5." His hope to be free from disruption by Susan was short-lived. In less than two minutes, Susan shouted out a number and claimed that this was the last digit of that sum. My friend, not knowing the answer himself, thought that Susan simply made up her answer. Well, my friend was not prepared to accept this answer without checking it, nor was he going to make it easy for Susan. He put his presentation aside and began to use his calculator. He soon realized that the calculator was not much help. Suddenly, he remembered a shortcut, which his daughter had probably used, and determined that her answer was correct.

At this point, I invite you, the reader, to participate. What is the last digit of the sum of all factorials from 1 to 100? Please submit your solution and solution process to the editor (see the executive list on the back inside cover for address).

The Sum of the Digits

Take the sum of the digits for each number from 1 to 1,000,000 and then find the sum of these sums.

Are There an Infinite Number of Twin Primes?

Klaus Puhlmann

Twin primes are pairs of prime numbers—such as (3,5), (5,7), (17,19)—that differ by 2. It appears at first glance that twin primes occur among the smaller numbers only, but that is not the case. For example, when one examines the numbers between 800 and 900, one finds the twin primes 821/823, 827/829, 857/859 and 881/883. Even among much larger numbers, there is evidence of twin primes: 9890641/9890643. In fact, even among numbers as large as 11,000 digits, twin primes have been found. "The largest known twin primes are 1,706,595 $\times 2^{11,235} - 1$ and 1,706,595 $\times 2^{11,235} + 1$, which were found on August 6, 1989, by a team in Santa Clara, California" (*Guinness Book of World Records* 1997, 137).

However, to date no one has presented a mathematical proof that the sequence of twin primes is either finite or infinite.

Two Ferries

Two ferries are traveling continuously but in opposite directions at constant speed across a wide river without any time loss as they turn around. In the morning, they start at the same time from opposite sides of the river and meet for the first time 800 m from the south shore. As they continue back and forth, they meet for the second time 400 m away from the north shore. How wide is the river?

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