READER REFLECTIONS

In this section, we will share your points of view on teaching mathematics and your responses to anything contained in this journal. We appreciate your interest and value the views of those who write. In the following article, "Men as Trees Walking," Ed Barbeau from the University of Toronto offers his opinion about the Principles and Standards of School Mathematics.

Men as Trees Walking

Ed Barbeau

This article is based on a plenary talk given May 29, 1999, at the summer meeting of the Canadian Mathematical Society in St. John's, Newfoundland.

"Men as trees walking"—this was the phrase that came into my mind as I examined the draft *Principles and Standards of School Mathematics* (NCTM 1998) recently issued by the National Council of Teachers of Mathematics. The phrase is from the Bible (Mark 8:22–26), and describes a curious miracle—a blind man touched by Christ could at first see only "men as trees walking" and required a second touch in order to see clearly.

One has the same feeling about the *Principles and Standards*; while it is generally praiseworthy, it is hard to bring the document into a clear focus. It contains a mixture of recommendations about topics, processes, teaching styles and general philosophy, but, as a lecturer of first-year university students, I did not get a clear sense of what I would be able to count on from the students in front of me. It seems that there is such a plethora of ideas put forward that perhaps a second healing touch is needed to tease out the main threads.

The central issue became clear to me one evening as I watched Rob Buckman on TV Ontario present a documentary on alternative medicine. Many people have become alienated from traditional medicine because they find it too reductionist and narrowly focused. Whatever the specifics, alternative medical regimes are attractive because they are holistic, proceed from a broader worldview and, significantly, involve the patient in the diagnosis and choice of treatment. Cases that might seem identical to traditional doctors might receive quite different treatments in different environments. Surely something like this is behind the pressure for reform in education. Teachers and students are reacting against a curriculum that seems to be reduced to a list of topics and processes, against an imposed canon robbing students and teachers of their autonomy. Whatever the details might be, we want students to be intimately involved in an educational process that cares about who they are and what characteristics they bring to the mathematics class.

But the charge that traditional education (as traditional medicine) has consistently lacked the human touch is far too stereotypical; it has resulted in a number of ghosts that have haunted recent educational reform. Let us raise a few of them.

Ghost 1: failure, streaming, elitism. There is no doubt that school was a brutal experience for many students in the past, but the fact remains that success in mathematics depends on a certain level of ability and application. To deny students the opportunity to fail is also to deny them the opportunity to enjoy success. This is not a call to ignore students who are floundering. Instead, we must create conditions that do not neglect the imperatives of learning mathematics and the possibility for achievement and that provide adequate support for students to move on in confidence. Many students now are uncertain about what they can do or should know; even good students are denied the chance to demonstrate their capabilities. While there is much that can be done to make mathematics more generally accessible, it needs to be recognized that the subject is often difficult and beyond the ability or interest of some segment of the population.

Ghost 2: the syllabus, list of prescribed topics, facts and procedures. The charge here is that having a list is too confining and leads to an emaciated mathematics of disembodied set of facts and routines.

It is not clear why this should necessarily be the case. In fact, the criticism seems to be misplaced; it should be directed at matters of design. In the hands of a teacher familiar with mathematics, the syllabus can serve as a set of markers and goals that frame what will happen in the class. An uncertain or ignorant teacher will hold to the list without any regard to connections and deeper understanding.

Ghost 3: drill, rote learning and memorization. Memorization was an important part of traditional education, and many cultures put a great deal of emphasis on what children should remember. Children seem to have good memories, and it seems foolish not to take advantage of this. But they need to be taught to evaluate what is worth memorizing, how mastery can be reached and how they can exploit the coherence of mathematics to leverage their knowledge of a few facts into fluency in a larger domain. The most able children can do this naturally; others need to have the issue explicitly addressed, so there is an underlying equity issue for students without natural talent.

Ghost 4: arithmetic. This is a word that seems to have fallen upon hard times in the curriculum stakes. But it is through arithmetic that most ordinary citizens see the connection between mathematics and the world, and lack of numeracy can present a severe handicap. Arithmetic has come to symbolize mindless memorization and manipulation. We need to detach it from this calumny and exalt it to the level of mathematical richness that it deserves.

Ghost 5: paper-and-pencil algorithms. This is also sometimes seen as drudgery, and the advent of modern technology has provided detractors of traditional calculation with a pretext for summarily discarding it. The issue is really how traditional arithmetic should be handled in a modern curriculum. Perhaps we need to see it more as additional means of accustoming students to working with figures or as examples of algorithms. Long division seems to be particularly suitable for showing how one can move from the idea of tallying a continued subtraction to a mechanical algorithm that is fast and accurate; one can point out that it was a human invention and replaced a method that was decidedly inferior (the "scratch method"). What has changed is that the importance of paper-and-pencil methods as practical techniques is reduced and we now have alternatives for reaching children encountering difficulties.

Ghost 6: word problems. Traditional word problems are criticized as being artificial, but one can argue that this is precisely the point of most word problems. They provide an imaginary situation in which certain points about interpretation and formulation can be made. The question again is not whether we should retain word problems, but how appropriate they are to the situation and how we plan to move beyond them.

Ghost 7: authoritarianism. This ghost has two manifestations, depending on whether you are referring to the teacher, who, we are told, should be the "guide on the side" rather than the "sage on the stage," or to the subject itself in which pupils are oppressed by the tyranny of "one right answer." Without disputing the advantages of the more open and friendly classrooms that modern students enjoy, it remains the case that a teacher's effectiveness depends on what she knows and that sometimes students need to submerge their egos and pay attention to what she has to say. In the same way, it seems mischievous to deny the power of mathematical certainty, especially given the diverse ways in which one might think about concepts and approach problems. (One might say that the work of Gödel and Lakatos, however lauded by serious mathematicians, have had a particularly pernicious effect on some mathematics educators seduced by the vamps of relativism.) There are many ways to encourage the individuality of pupils without permitting them to believe that black is white.

All of these ghosts are the traces of essential components of mathematical education in the past which must be part of the future as well. Children are going to either succeed or fail at any worthwhile task, and the question is how humanely the failure is handled and whether pupils are held back or advanced for frivolous reasons. Mathematics is a hierarchical subject, and we need to spell out what students need to master at each stage; the issue of the syllabus is one of design and focus. We cannot deny the need for practice; the question is whether the student has the strategies and perspective to learn and memorize efficiently and effectively. Arithmetic and the standard algorisms are as important as ever; we need to be sure that they are put in the proper context and conceptual framework.

These ghosts are accompanied by a number of sirens who drive a lot of educational reform. Like the ghosts, the sirens also speak to important aspects of the mathematics education. Here are some of them.

Siren 1: problem solving and investigation. This siren calls us away from the ghost of drill and of dry and unilluminating exercises. There is nothing wrong with wanting our children to solve problems and explore mathematical situations, but we can run into serious distortions if we do not take the trouble to ground children mathematically and psychologically.

Any problem we present to children should be carefully analyzed for its mathematical content and appropriateness. Strategic decisions need to be made as to what mathematics has to be presented beforehand as background and what can be brought out in the analysis of the problem. Let me give an example that I have used with students and prospective teachers.

Let ABCD be a unit square and let E and F be the respective midpoints of the sides BC and CD. The three line segments AE, AF and BF partition the square into five regions. It is required to determine the area of each region.

There is actually quite a lot in this. At what point should this example be introduced---before or after the pupils see the area formula (half-base-times-height) for triangles? This will govern how they might approach the problem. If the students are to try a structural rather than a formulaic approach, they might need some understanding of isometries and might need to understand that areas of nonoverlapping sets add, that areas are invariant under rigid transformations and that they vary as the square of the factor of a dilatation. Should some of this be discussed ahead of time, or can it emerge as the problem is covered? If the students exploit the similarity of the two subtriangles of ABE, they may need to know that AE is perpendicular to BF; what tools can they be expected to deploy? Will students who assign letters to the five regions have the necessary algebraic understanding to proceed, or is this a nice vehicle to introduce them to this approach? Finally, will they be able to negotiate the fractions? Are the fractions appropriately expressed in vulgar or in decimal form? Why? When we take all this into consideration, this example could take quite a bit of time to do properly---has this been anticipated by the teacher?

The problem-solving approach to the curriculum has a great deal to be said for it, but the teacher must be able to envisage what might happen and, importantly, what can possibly go wrong. It is risky, and we should be sure that teachers are equipped to accept the risks.

Siren 2: relevance; real life. Mathematics has been seen as alienating because of the artificiality of what we ask students to do. To counter this, we should make reference to students' daily lives and concern ourselves with what they will need to succeed in their later careers. But what passes for relevance often raises the question, "For whom?" Often children are subject to tedious arithmetic problems dressed up with clowns or are brought into the world of adult concerns in the name of relevance. Play is a part of the world of a child, and mathematics gives lots of opportunity for this. I have never been aware of any nice number, geometric, topological or combinational novelties being beyond the pale for the children I have taught.

Siren 3: patterns. No mathematician can gainsay the importance of a sense of structure in doing mathematics, and the ability to recognize pattern is an important part of this. What has been forgotten in a lot of modern educational reform is that the power of patterning is seen in its use in analyzing and generalizing mathematical situations. Much of what passes for patterning is a kind of teacher "guess-my-rule" game and rather ad hoc examples. There is no excuse for this. It is hard to progress through the curriculum without finding natural opportunities to exploit patterns, and teachers need to be alert to this and make them explicit to their pupils at the right time. One opportunity that seems to be neglected is giving students the mathematical voice to describe and analyze patterns.

Siren 4: data analysis. We do not have to look very far to realize how much of our daily lives is governed by a flood of data of one sort or another. So there is certainly a duty to help pupils become number-wise and to interpret what they read astutely. But the danger is that we do not just cover a lot of canned techniques and introduce jargon, which may in some cases stand in the way of good discussion and analysis.

Siren 5: technology. First, let me say that I agree that through modern technology we are undergoing changes that are at least as profound as those introduced by the invention of the printing press and the Industrial Revolution. But any revolution, no matter how pervasive, is not completely disengaged from the past. Many of the issues raised by technology are old ones, and the charge of misuse and mindlessness can be and has been leveled against Arabic numeration, algebraic symbolism, logarithms, slide rules, mechanical adding machines and numerical algorithms of all types whether executed on paper or on a bench with pebbles. Certainly, the modern computer has greatly expanded both the range of the mathematics we *can* do as well as the mathematics we *want to* do: our curriculum should reflect this. But we do not need idolatry. Technology is a part of our environment, as are books and pens, and a general purpose of education is to produce students who can understand and work within their environment. There are core mathematical issues that are independent of technology, but can be greatly informed by our use of technology—it is in this spirit that we should embrace the use of calculators and computers in our classrooms.

All of the sirens speak to the important aspects of the modern mathematics curriculum, but they all involve subtle issues and risk of being trivialized. The big question is whether we will have teachers in front of our children who will handle the issues in a sensitive, moderate and intelligent way.

In designing a curriculum, we need to keep both the ghosts and sirens in mind, for each of them not only speaks to important goals but also carries a warning of distortion and counterproductivity. There are a number of factors that we need to be cognizant of.

1. For a body of mathematics, I believe that there are three stages a learner must pass through: *initiation, formalization* and *consolidation*. In the stage of initiation, the learner encounters the ideas in a somewhat haphazard way, feeling her way about, exploring; there has to be some motivation, some reason that the person is interested in the material at hand. At the formalization stage, the ideas are drawn together and organized; at this point, the pupil should learn proper concepts, processes and conventions. There may indeed be a lot of artificiality; the payoff should be a growth in the knowledge and mathematical power of the learner. In the consolidation stage the learner reflects upon what has gone before, contextualizing it, detecting relationships and connecting it to other material. It may be only here that the learner truly appreciates the reasons for what she has been taught before. Traditional education has emphasized the second of these stages while modern practice seems to focus on the first and third without the buttress of the second stage. A good curriculum allows for all three stages to occur. The first signifies to the pupil that the subject matter *could* belong to them, the second provides the tools to *allow* it to belong to them and the third ensures that it *does* belong to them.

2. Have a strong focus and a slender core for each course in the curriculum. Make sure that context is established, the purpose of the material becomes clear, there is enough depth to support its assimilation and allow students to develop the necessary skills and understanding to proceed.

3. Have one or two attainable goals for each year; plan to achieve them and *move on*. This means that the sterile spiraling that now occurs in schooling should cease, but it does not preclude returning to previous material to inform and consolidate it.

4. Have enforceable entry requirements for secondary and tertiary courses. No teacher should be asked to deal with a student who does not have reasonable prerequisites for the material to be studied. There is an important pedagogical purpose in requiring students to review and mentally organize the work of several months or a year; it is this process that makes purposeful curriculum progress possible.

5. Change pace. The diversity of the mathematical enterprise should be reflected in our curriculum, whether it be with respect to subject matter, level of discourse or application. There are many ways in which people can think about or do mathematics. Therefore, we need space to help students find their mathematical voices and texture their ways of thinking about and doing mathematics.

6. Orchestrate the material. Increase the level of complexity judiciously, make sure that foundational material is covered and the student is psychologically prepared for what is to follow. We need to analyze in much more detail what students are asked to do; this is where members of the university community can be particularly helpful. Too often problems are thrown at pupils with little appreciation of what needs to be in place to begin to solve them. A sound curriculum needs exercises and problems of many different intensities.

This is one area in which the *Principles and Standards* seems to be particularly weak. There are a number of examples thrown in that, upon closer analysis, seem to involve a great deal than first meets the eye. For example, on page 92 is a set of instructions for constructing a golden rectangle. This seems as though it might be pretty heavy sledding for a typical student, and any teacher who embarks on this without careful thought is wading into a treacherous swamp. Why should pupils be interested in a golden section or in ways of constructing such a thing? Will most pupils muster the necessary level of concentration to negotiate the 10-stage set of instructions and understand why it works? This example is meant to illustrate the connectedness of mathematics, but it seems to me to require such a level of maturity and mastery of some basic algebra that it could easily spin out of control in the hands of any but the most adept teacher.

7. Meet different needs. Whatever reason we can give for teaching anything at all applies to mathematics. Some mathematics is taught so that students will be able to accept the privileges and responsibilities of citizenship, some to situate them in the rich culture they will inherit, some to provide recreation and additional options toward a full and rewarding life, and some for professional preparation. These needs can sometimes be met by the same piece of mathematics, but all should inform the curriculum that we set.

8. Foster sound practice and mental attitudes. How well students perform in mathematics seems to depend on their worldview. In designing a curriculum, we should encourage an attitude that involves the following:

- Awareness of structure; appreciation and exploitation of symmetry
- Flow of ideas, analysis and reasoning
- Corroborative quality of mathematics; inner consistency
- Shifting of perspective
- Checking and monitoring one's work
- Organizing and polishing one's work
- Mathematical register; expressing ideas appropriately; the place of heuristic and formalism
- Sense of context
- Attainment of power through understanding, reasoning and technical facility
- Visualization; appropriate imaging
- Appreciation of symbolic representation: numeration, algebra, diagrams
- Making distinctions: classification, equivalence, isomorphism, congruence
- Grasping the interplay between concrete and abstract; progression to higher order structures that in turn become concrete

9. Finally, we come to the subject matter itself. The centrality of arithmetic and algebra must be affirmed. No student can succeed at the secondary level without a good grasp of arithmetic, or succeed at the tertiary level without a good algebraic grounding. But these are not all. Geometry and combinatorics are also indispensable. While students should see how mathematics can be applied in different areas, it is not clear to me that this should necessarily occur in math class. Science, shop, civic, geography and music are areas in which important mathematical concepts can be conveyed in an appropriate and powerful setting. And we should not forget how many important mathematical concepts and processes underly some recreations and puzzles. In fact, if the elementary and secondary curricula are well designed, one could devote the middle school years largely to recreational and cultural issues as a way of consolidating what should have been learned at the elementary level and preparing the ground for a more sophisticated high school program in which symbolism, algorithm and reasoning have important roles.

I would like to close with two courses that might be given at the Grades 9 and 10 levels. The first is designed to give a systematic introduction to algebra and the second to geometry. It is assumed that students have gone through an initiation stage in both areas, that they have some familiarity with formulae and the use of letters to represent numerical quantities and that they have had the opportunity to play around with geometric objects either through tactile or computer models. While technology is not explicitly mentioned in either syllabus, it is understood that it can play a large and appropriate role. These syllabi would be accompanied by resources for the teacher that lay out alternative approaches as well as useful exercises, problems and investigations and that make explicit the pedagogical and mathematical goals that are brought to light through the material.

Course 1: Linear and Quadratic Functions

1. The linear equation $ax = b$. Problems that lead to an equation of this form.
2. The equation of the straight line; slope; various forms.
3. The solution of a system of two linear equations in two unknowns using numerical, algebraic and graphical techniques. Consistency of two linear equations.
4. Factoring difference of squares.
5. The quadratic: factoring over reals and integers, completion of the square, quadratic formula, discriminant and its significance; complex numbers (real quadratic having complex conjugate roots); relation of sum and product of roots to coefficients; first and second order differences; the factor and remainder theorem (for quadratics).
6. Graphing of quadratic functions; comparing the graphs of the functions $f(x)$, $f(x-c)$, $f(ax)$, $f(x)+c$; similarity of all graphs of quadratic functions (role of completion of square).

This is the entire prescription for a full-year course. In some classes, it may be necessary for the teacher to cover only this material and nothing else, spending the time and introducing whatever additional materials are necessary to ensure that students become fluent in the essential components. Normally, one would hope that the teacher would extend the material in some way, either through the introduction of applications, extended investigations and additional topics. It may be that the more able students in the class are given modules to work on independently, possibly with the support of additional pamphlets or computer software. I do not see much point in introducing algebra tiles to the class at large—they amount to an additional code interpolated between the student and the standard code—but they may be useful for individual students who have trouble getting an initial grasp of algebra.

Other topics that might be introduced are motion of projectiles, history of solving equations, linear and quadratic diophantine equations, Pell's equation, quadratic residues, first and second order recursions (characteristic equation), analysis of the dynamical system $x \to \lambda x(1-x)$, polynomials of degree exceeding 2; complex numbers (conjugates, modulus, geometric interpretation of sum, product and inverse, roots of unity up to the fourth); calculus of finite differences applied to polynomials interpolation and extrapolation; linear inequalities.

Course 2: Geometry of Triangles and Circles

1. Triangles: congruence theorems; ambiguous case; Pythagorean theorem; similarity.
2. Trigonometry: six standard ratios and basic identities ($\cos^2 + \sin^2 = 1$; $\sec^2 = 1 + \tan^2$); sine, cosine and tangents of angle sums and differences; double angle formulae; conversion formulae between sums and products; law of sines; law of cosines
3. Circles: subtended angles; cyclic quadrilaterals; tangent theorems
4. Coordinate geometry: circles; area of triangles; family of lines passing through pair of intersecting lines; circles passing through intersection of a pair of circles; radical axis; simple loci problems.

Again, the teacher would have the discretion of reinforcing these core topics for students encountering difficulty, amplifying the material through applications and investigations or moving to additional topics. These could include the following: complex numbers (use in deriving geometric and trigonometric results); vector geometry; statics; loci (conjectures, verification and proof); solid geometry; advanced Euclidean geometry; transformations and their uses (isometries and similarities; composition of isometries; isometry determined by three points; isometries generated by reflections).

The modern school, even at the secondary level, has to serve students all across the spectrum of intelligence, ability, motivation and interest, often in the same class. The only way the system can cope with this situation is that students are trained early to become autonomous in choosing their goals and more self-propelled in accessing the resources needed to learn what they need to know and in seeking validation for their knowledge. Any system of education predicated on the teacher as the sole source of knowledge and evaluation for the student is bound to fail; the more able and motivated will fall short of their potential and others will not be able to really engage with the material. Students must first think about what they value and how far they are willing and able to go in achieving their goals. A successful school regime depends on certain beliefs and characteristics of students as well as the knowledge and experience of teachers; unless we recognize the need to start with this, our principles and standards will become an exercise in merely trying to stave off the most trenchant critics from all sides and leave us seeing "men as trees walking."

Reference

National Council of Teachers of Mathematics. *Principles and Standards for School Mathematics: Discussion Draft.* Reston, Va.: Author, 1998.

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