A Collection of Connections for Junior High Western Canadian Protocol Mathematics

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We have put together "A Collection of Connections" that consists of 12 uses of junior high school mathematics. These activities support the communication and connections strands of the Western Canadian Protocol mathematics curriculum. In using them, teachers may adapt them extensively. They can serve as a basis of one to three mathematics periods. We have also found that teachers need to plan if they intend to incorporate them into their teaching units. They can be used as end-of-unit activities or as focal activities in the development of a unit. We would encourage teachers to use the activities as a means of bringing mathematical skills to life. These contexts provide an opportunity to enhance a student's view of mathematics. As one Grade 9 girl said at the conclusion of one activity: "That just proves that mathematics is everywhere."

The following are samples from the shape and space and the patterns and relations strands.

Shape and Space (Measurement)
The Volume of a Sphere
The Volume of a Sphere Student Activities
Patterns and Relations (Patterns)
Mail Carrier Routes
Mail Carrier Routes Student Activities

The Volume of a Sphere

Intent of the Lesson

The student gets experience in looking at the relationship between the volume of a cube, a cylinder and a sphere. The mathematics involved is practical measurements, volume formulas, data collection and straight line graphs.

General Question

The textbook tells us that the formula for the volume of a sphere is $^4/_3\,\pi r^3$. Why is this a reasonable formula? What meaning does this formula have for us? We know what the formula for the volume of a sphere is but do we understand it? Today we will take a procedure from science and use it to help us understand some mathematics. Usually we use mathematics to help us understand science but today we have the reverse. Why is the formula for the volume of a sphere $^4/_3\,\pi r^3$?

Discussion Questions

- What is the formula for the volume of a cube of side s? (s^3)
- What is the formula for the volume of a cylinder of radius, r, and height, h? (pr^2h)
- Describe what r^3 means physically. (The volume of a cube with each side of length r.)
- What is the area of a circle? (pr^2)
- Do you have a way of remembering the area of a circle?

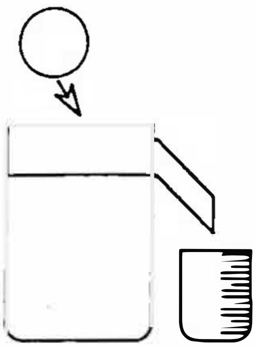
Preliminary Activities

Surface Area of a Sphere

Earlier in this collection, we found out how many stars we could see. We used the formula for the surface area of a dome (half of a sphere). How do you think that was calculated? The formula for the surface area of a half sphere is $2\pi r^2$. Let's compare it to something you already know. Imagine you had an elastic circular disk with radius r and it could be changed into half of a sphere by blowing air on it. The surface area of that disk when it is flat is πr^2. The surface of the curved dome should be about double the area of the flat disk upon which it rests. Although this is not a proof of the surface area of a hemisphere formula, it makes the formula reasonable. Using the same thinking, the surface area of a sphere is four times the area of the circle through its centre, $4\pi r^2$.

The area of a parallelogram can be compared to a rectangle on the same base and the same height. In geometry, it is often useful to compare one figure to another. In this lesson we will compare the volume of a sphere to the volume of the cube and the cylinder into which the sphere could be placed exactly.

A Lesson from Science



We can determine the volume of the sphere by water displacement. The science laboratory may provide us with a displacement jar. We fill the jar so that water flows out of the jar. This means that it is exactly full (as in the picture).

When the sphere is lowered into the water, some water will spill out of the spout. The amount that spills out the spout is exactly equal to the volume of the sphere. Measuring the volume of water that collects in the catch beaker is a way of determining the volume of the sphere. If 20 mL (millilitres) of water is collected, the volume of the sphere is 20 cc (cubic centimetres).

Discussion Questions

- What shape does the object have to be for this to work? (Shape is irrelevant. The method is most often used for irregularly shaped objects.)
- What if the object floats? (A method is needed to submerge the object without displacing extra water.)
- What if the object sinks to the bottom? (The experiment is easier to perform if the objects sink.)
- If our first object is in the displacement jar, can we put another object in? (Yes, as long as you empty the catch beaker before adding the second object, or if you are using a graduated cylinder, just record the volume before the second object is put in.)
- What condition must be met for the displacement test to work? (The object must displace water.)

To understand our lesson, students must be convinced that no matter how far down the sphere is in the displacement jar, the volume of water displaced will be equal to the volume of the sphere.

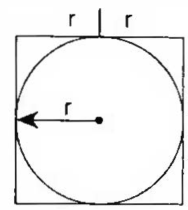
Verifying Volume by Water Displacement

To help students see that the displacement method of finding volume actually works, the teacher can demonstrate the method using an object of known volume. Connecting, for example, eight one-cubic centimetre blocks together would create an object of known volume and could be used for the demonstration. To show that the shape of the object is irrelevant, connect the blocks together to form different shapes and repeat the demonstration.

Answering the General Question

If we simply wanted to find the volume of the sphere, we could displace water and have our answer. However, we want to find the *formula* for volume. We do this by comparing this volume to the volume formula for something we already know. We have two such objects, a cube and a cylinder.

A Cube

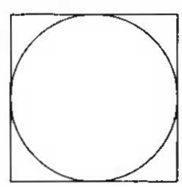


Imagine a cube just large enough so that our sphere fits into it exactly. The dimensions of the cube will be $2r \times 2r \times 2r$. The diagram of a circle in a square illustrates the point.

The diameter of the circle is $2r$. Therefore, each side of the square is $2r$. The volume of our cube is $8r^3$ and we can easily see that the volume of our sphere is less than this. A reasonable guess is that it is about half this amount, or $4r^3$. That is, the volume of the cube outside the sphere is about equal to the volume that is inside the sphere.

A Cylinder

Now let us imagine our sphere fitting exactly into a cylinder which is the same height as the sphere. When we look at this from the top of the cylinder, we see that the sphere touches at all points of the cylinder. It is a perfect fit. However, when we look at it from the side (if we could see through the cylinder) the picture would look like this.



The sphere would not be touching the cylinder in any of its corners. We need to recall that the formula for the volume of a cylinder is $V = \pi r^2 h$. In this case $h = 2r$ (the diameter of the sphere) so the formula becomes $2\pi r^3$. So now we can see that the formula for the volume of a sphere is less than $2\pi r^3$ because the sphere does not entirely fill the cylinder. A good guess might be one-half of $2\pi r^3$ or πr^3.

Looking back to our earlier guess of $4r^3$, based on the volume of a cube, we should notice that the estimate of πr^3 is about the same since this is about $3r^3$ (because $\pi = 3.1$). We might also reasonably guess that πr^3 is in the formula. The constant π does not appear in the cube formula since the cube is not in any way in the shape of a circle while the cylinder is.

These comparisons have yielded two insights:

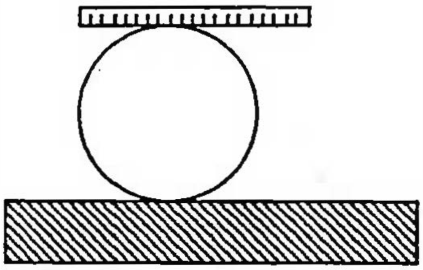
The formula for the volume of a sphere is close to $4r^3$ and πr^3.

The formula probably contains πr^3.

A Measurement Procedure

We now know that the formula for the volume of the sphere is some number K times r^3. We are also reasonably sure that π is part of the number K. We are going to use our science measurement to make a reasonable guess at that number. We know that it must be fairly close to 3 or 4.

Each group is given a different ball. All, however, need to fit in the displacement jar. If they sink on their own accord so much the better. Otherwise we will have to find ways to submerge them. However, before students find the displacement they should figure out the volume of the cube and cylinder that are related to their ball.



Exact measurements are a must, certainly to the nearest millimetre. One way of measuring the diameter of the sphere is to wrap adding machine tape around it and simply measure across the edges of the tape. Placing a ruler on top of the ball parallel to the table top is another way.

The students have four tasks:

1. Find the radius of the sphere.
2. Find the volume of the related cube.
3. Find the volume of the related cylinder.
4. Take the actual sphere and see how many cubic centimetres it displaces.

The Displacement Activity

As each group submerges its sphere in the displacement jar, the teacher will need to supervise the activity. The displacement jar must be full and the catch beaker must be empty. The tricky part is to fully submerge the sphere. If the sphere is heavier than water, this is easy. If it is lighter and floats, a device will need to be used to push the sphere down so that it just submerges. If the pushing device goes into the water, it will displace water giving inaccurate measurements.

The larger the ball is, the less percent error there will be. Small balls give large errors. The line of best fit method (given below) will make it clear when one of the measurements is erroneous. This measurement should be excluded.

Using a Table

In the chart we will include the radius, the volume of a cube, the volume of a cylinder and displacement. The last three numbers will give us a basis for comparison. Since the volume of the sphere is probably going to be some multiple of r^3, we will also put that number in a sixth column. This is a calculation based on the radius.

Once all the groups have contributed their numbers to this chart, the class can proceed to make a graph of r^3 on the x-axis and the displacement on the y-axis.

1	2	3	4	5	6	7 optional column
Name of group	radius (least to greatest)	volume of cube	volume of cylinder	displacement	value of r^3	displacement divided by r^3

Teaching Suggestion

The teacher should supervise the filling in of the chart. Once this is done, students can be assigned the task of drawing the graph of the displacement against r^3. Using the line of best fit method, they will try to determine the line on the graph. Those points that fall far from the line may be due to error and may be eliminated.

A discussion of error is a chance to reflect on the displacement activity and to focus on the physical nature of the measurements. This, of course, contrasts with the error-free use of mathematics that occurs in typical mathematics classes. The discussion of error brings out the real world aspects of this activity.

Determining the Equation

Points on the line can be used to determine the equation for the relation:

D(isplacement) = $K \times r^3$

The value of K should be close to 4. The actual value is $^4/_3\pi$ which is about 4.2. The discussion of the constant K should centre on whether $4r^3$ or $^4/_3\pi r^3$ is closer.

The formula of the volume of a sphere is $^4/_3\pi r^3$. Students can then be asked to find out what fraction the sphere is of the cube and what fraction is of the cylinder. It is ½ of the volume of a cube and $^2/_3$ the volume of the cylinder. A discussion of these results should help to confirm that the formula for the volume of a sphere is reasonable.

An Alternative Method of Data Analysis

A simpler, but less accurate way, of finding K is to create another column for "the displacement divided by r^3." The numbers in this column should all be close to 4.2, that is $^4/_3\pi$.

Discussion Questions

This activity is not a proof of the formula, but the activity shows how the volume of a sphere is related to the volume of a cube, the volume of a cylinder and πr^3.

Materials

Displacement jar, water, spheres of various sizes and grid paper.

Modifications

If a displacement jar is not available, a beaker or a large graduated cylinder may be used. In this case, the volume of the sphere is the difference between the volume reading while the sphere is submerged and the volume reading before the sphere is submerged in the water.

It is essential that the beaker or graduated cylinder be large enough to catch and measure all the water displaced by the ball.

The problem in this activity is accuracy. However, the main idea is to have students thinking of the cube, cylinder and the sphere in comparison terms and also identifying r^3 as the key variable in the change of volume with π being a unique and important constant.

The whole activity might be done as a teacher demonstration. In this case the students miss the multiple calculations and the discussion of error and how it arises. The teacher may point out that error in using a formula arises when the measurements for a variable used in the formula contain error.

Volume of a Sphere Student Activities

General Question

The textbook tells us that the formula for the volume of a sphere with diameter $2r$ is $^4/_3\pi r^3$. How does the volume of this sphere compare to the volume of a cube with side of length $2r$? How does the volume of a cylinder of diameter $2r$ and height $2r$ compare to the sphere? Is there a relationship between the volumes of these three shapes?

Activities

1. a) The surface area of a disk is πr^2. What is the surface area of a dome (hemisphere) placed on that disk? Why is this reasonable?
   b) Based on question 1a, what is the surface area of a sphere?
   c) What is the area of a disk with radius π cm? (Do not do the actual calculations.) What is the surface area of a hemisphere on that disk? Of a sphere formed by adding the other half of that hemisphere?
2. a) We can find the area of a parallelogram by relating it to a rectangle on the same base. What can a trapezoid be related to in order to find the area? (In the trapezoid we know the length of the two parallel lines and the height of it.)

   b) There are at least three options for finding the area of a trapezoid: break it into triangles, double its area and make it into a parallelogram, or make a rectangle on its longer base and one on its shorter base. (Hint: the width of the rectangles is half the height of the trapezoid.) Show one of these ways (one that you did not do in question 2a).

3. a) A cube measures 8 cm on a side. It has a lid. A cylinder and a sphere fit perfectly into it. Assuming the cylinder also has a lid, find the surface areas of all three of these shapes. What is the ratio of the surface area of the cylinder compared to the surface area of the cube and the sphere compared to the cube?
   b) Find the volume of the three shapes in Question 3a. What is the ratio of the volume of the cylinder compared to the volume of the cube and the sphere compared to the cube?
4. a) In measuring the diameter of a sphere, Jerry measured 4.5 when the actual measurement was 4.2. Jerry said a 0.3 error was not a problem. What percentage was Jerry's error of the actual measurement? Would you agree with Jerry's conclusion?
   b) Jerry used this measurement to determine the volume of the sphere. What percentage error occurs compared to the actual volume? Do you think Jerry would agree with his conclusion now?
   c) Why is there such a big difference in the percentage error in question 4a compared with 4b?
5. a) A displacement jar (which is a cylinder) is full of water and contains 100 cc of water. A sphere is submerged which fits perfectly into the jar so that the top of the sphere just touches the top of the water when submerged. What percent of the water in the displacement jar remains?
   b) When the sphere is lifted out, how many cc of water remain in the displacement jar?

Mail Carrier Routes

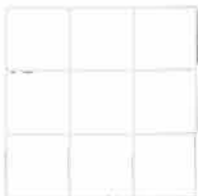
Intent of the Lesson

This lesson relates to operations research, which is an area of mathematics used to solve problems in engineering and city planning. It shows how mathematical reasoning can be applied to questions of logistics and efficiency. It relates to drawing networks of geometric figures and uses line segments and intersections.

The General Question

The general question asked here is, What is the most efficient way for a mail carrier to cover a route? Can we make a route that covers every street without going on the same street twice? This is not related to any part of the mathematics curriculum but can be engaged in as an exercise in a systematic approach to an abstract problem. It comes from an important application of mathematics called operations research. Operations research examines issues such as industrial efficiencies, transportation routings and pipeline oil flow to and from refineries.

Mail carriers want to cover every street in a certain town (or subdivision) but they want the most efficient route. For example, suppose they needed to deliver mail in a four-by-four town.



What is the most efficient way for a mail carrier to go over every street of the town? Several attempts will convince students that however they proceed, it will be necessary to backtrack in order to get to every street.

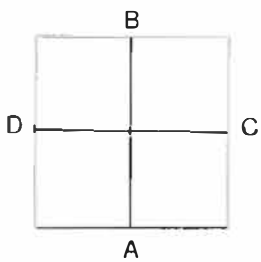
1	2	3	4	5	6	7 optional column
Name of group	radius (least to greatest)	volume of cube	volume of cylinder	displacement	value of r^3	displacement divided by r^3

Mail carriers, puzzled by the problem, have invited you, a mathematician, to help them solve the problem. You want to solve the problem for a four-by-four town but you would also like to find a general solution that would apply to any type of town or district. A general solution would be of help not only to mail carriers but also to paper carriers and snowplow drivers all across the country. Your few trials should convince you that the four-by-four town cannot be covered without going over some streets twice.

Discussion Questions

- Where do problems occur? (At intersections with an odd number of paths)
- Besides paths, what other feature is obvious on this town map? (Intersections)
- All paths are straight lines but are all intersections the same? (No, the number of paths leading to and from them may differ.)
- Which do you think we are going to be more interested in—paths or intersections?

Preliminary Activity



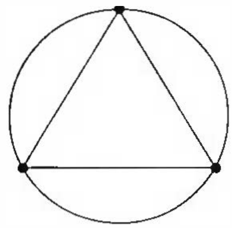
Let us take a very simple town, say three by three.

Since this town has a certain symmetry, we can only start at three mathematically "different" places: a corner, at point A or at the centre. Intersections A, B, C and D are similar in one respect. They all have three paths connecting them to other points. No matter where we start, we have to come to one of those points, let's say C.

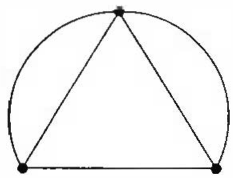
Having arrived at C, our journey will have to end at C because one of the remaining roads (from C) will take us away and the other will bring us back. Once back, we will be unable to leave again because all three roads will have been covered. Because A, B and D also have three paths, we can argue that the journey will have to end at each of the points, which is impossible since a journey can only end in one place. So, regardless of where we start our journey—a corner, at point A or the centre---once we arrive at any of the points B, C or D, our journey has to end there.

Discussion Questions

- Our small three-by-three town has how many intersections? (Nine)
- Does each intersection have the same number of paths? (No, two paths [corners], three paths [points A, B, C and D] and four paths [the centre)].
- What is the main problem in making a route?
- How many intersections does the town below have? (Three)



- How many paths does each intersection have? (Four)
- Can we make a route easily? (Yes)
- How many intersections does the town below have? (Three)
- How many paths are there to each intersection? (Two have three, one has four)



- There is only one way for making a path in this town. (We must start at one odd number intersection and end at the other.)
- What happens if a town has more than two intersections each with three paths such as our three-by-three town? (You cannot make a route without doubling back on some path.)

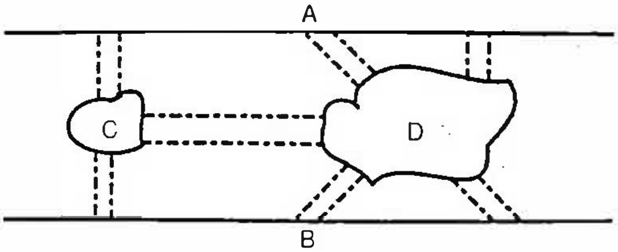
A Mathematician's Advice

Mathematics people speak of paths and nodes. Nodes are important. Furthermore, the important characteristic of nodes is whether they are even or odd nodes. A node that has an even number of paths is even, and a node that has an odd number of paths is odd. Based on this type of analysis, they have come up with four guidelines:

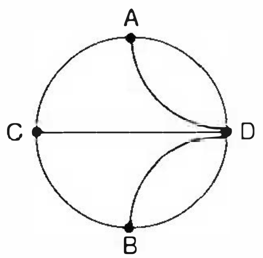
1. A town with *more than two* odd nodes cannot be covered without going over some roads twice.
2. Odd nodes always occur *in pairs*. You cannot make a town that has an odd number of odd nodes. We might try to make a town with three nodes that each have three paths.
3. If a town has *exactly two odd nodes*, the only route that can be made is one that begins at one odd node and ends at the other.
4. If a town has more than two odd nodes, we can make a route by making a path that *connects the pairs of two nodes*, thereby making them even nodes. Once we have a town with only even nodes, the route can begin at any point.

A Famous Problem—The Bridges of Konigsberg

The bridges of Konigsberg look like this. There are two islands in a river.



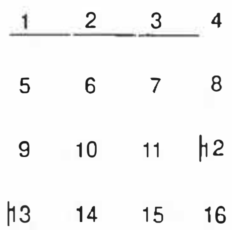
The Sunday afternoon strollers of Konigsberg wondered if they could make the walk by crossing over every bridge only once. With our knowledge we can help them, but first let's make the problem look like paths and nodes. We label the point A (one shore), B (the other shore), C (the small island) and D (the large island). Each of these four nodes is odd.



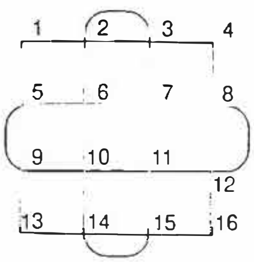
The route is impossible. The Konigsberg strollers did not have a serious practical problem, only a curious mind. However, our problem of the shortest route for mail carriers is practical.

Answering the General Question

Looking back at the four-by-four town, we can number the intersections which we now call nodes. There are 16 in all.



Nodes 2, 3, 5, 8, 9, 12, 14 and 15 are odd nodes. Each has three paths, so our rule says that a simple route is not possible. However, we can make it possible by making an extra path between pairs of odd nodes. "Making an extra path" simply means walking back over the same path.



Actually, if we can start at an odd node and arrange to end at its pair (another odd node), we will not have to make an extra path there. However, this solution is not acceptable if we want to end at the same point that we started. Usually mail carriers want to end where they started because they may have left their car there or they may be able to catch the bus home.

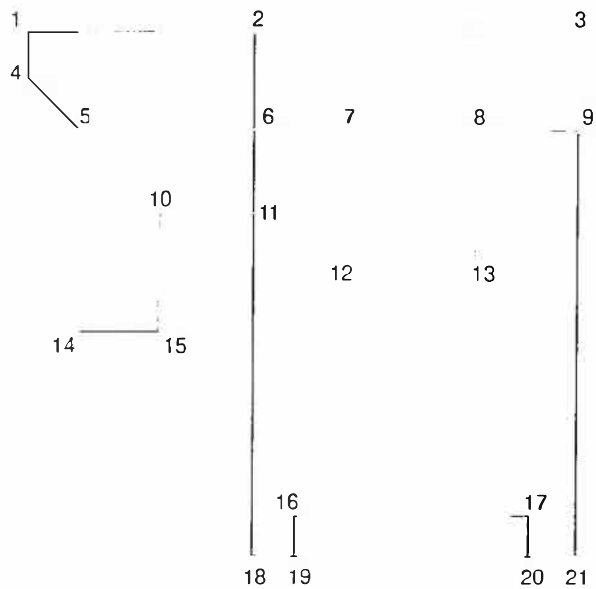
An Edmonton Problem Where Street and Avenues Are Not the Same Length

When making a route by joining two odd nodes to make extra paths, some choice is possible, particularly, in a city like Edmonton, where in certain districts blocks along an avenue are much longer than blocks along a street. Consider the following neighborhood.

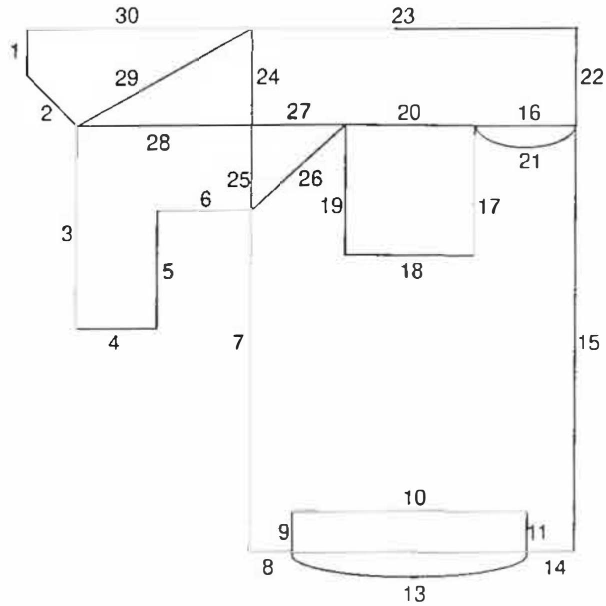
We have four odd nodes. Joining 3 to 4 and 5 to 6 is more efficient than joining 3 to 5 and 4 to 6. This is an extreme case to show that some choice of node joining is necessary.

Your Own Neighborhood

Students can be asked to find a city map and decide upon a paper route that they might have in a six-by-six block neighborhood. They should begin the journey at their house and make the shortest path possible. Remind the students that *all nodes should be numbered* and identified as odd or even nodes. They should then proceed to join nodes in an efficient manner. One such path is given below for the teacher to assign if desired.



One of many solutions is provided below. Additional paths were added between nodes 2 and 5, 7 and 11, 8 and 9, and 19 and 20. The numbers here represent the order in which the paths were followed to make the route.



An alternative manner of identifying the route is to list the numbered nodes in order. In this case it would be 1, 4, 14, 15, 10, 11, 18, 19, 16, 17, 20, 19, 20, 21, 9, 8, 13, 12, 7, 8, 9, 3, 2, 6, 11, 7, 6, 5, 2, 1.

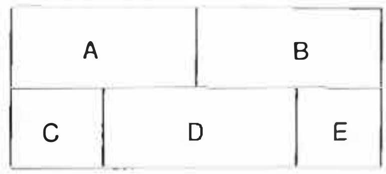
Materials

No special materials other than grid paper are needed for this lesson.

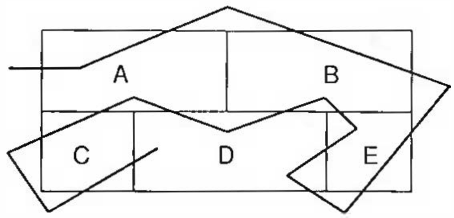
Modifications

The teacher can easily make up easy or difficult problems depending on the capability and interest of the class. The discussion opens up a large mathematics area of operations research that deals with efficiencies of systems often without numbers. Although these are logistical problems, mathematicians often solve them by attaching numbers and making the problem as concrete as possible. For example, we could assign fractional kilometre distances to the Edmonton street problem above. We could rephrase the question as, "What is the shortest distance a mail carrier needs to travel to cover every street?"

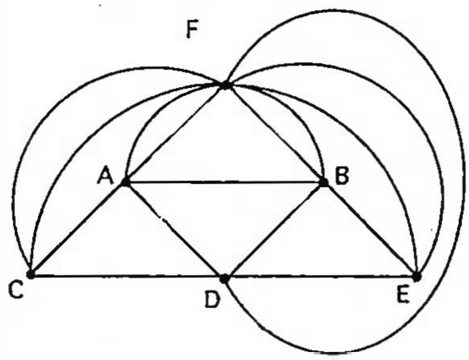
Another approach to these problems is to make a *path node diagram* out of problems that don't look that way, such as the Bridges of Konigsberg. Here is another such problem. The idea here is to draw a single line that cuts the side of each rectangle only once.



Here is one attempt. Which segment is not cut?



The node diagram looks like this: (F is the region outside the square.)

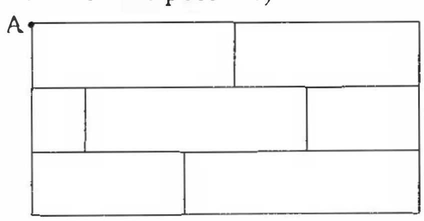


The drawing of the node diagram is fairly difficult but it is a nice example of taking the fun out of a puzzle by analyzing it mathematically.

Mail Carrier Routes Student Activities

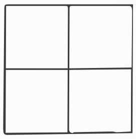
General Question

Imagine a small town (shown here). Can you travel the streets of this town, starting at A, covering each street only once? You will soon see that this is impossible. In other words, if you want to walk on every street of the town, you will have to use some of the streets twice. The question is, Which streets should you cover twice to be the most efficient (cover the shortest distance possible)?



Activities

1. a) For the three-by-three town shown below, number the intersections from 1 to 9 and show three paths starting at each of three points: a corner, a side intersection and the middle intersection. Describe these paths by writing down the sequence of intersections that they go through.



   b) In each of the three paths above (Questions 1a), at which of the three intersections (a corner, a side intersection or the middle intersection) do your paths end? To answer this question, complete these three sentences:
   i) When I start at a corner, I end at a ______ or______.
   ii) When I start at a side intersection, I end at a ______.
   iii) When I start at the middle intersection, I end at a ______ or ______.

   c) Explain your observations in question 1b? Why is a simple three-by-three town so complicated?

2. a) Make a ***three-intersection*** town in which each intersection has three paths connecting it to other intersections. Mathematicians say this cannot be done. Explain why towns work this way.

   b) Here's a task that is possible: make a three-intersection town with two intersections that have three paths leading to other intersections.

   c) In the three-intersection town that you made in question 2b, begin at the intersection that does *not* have three paths and make a route that doesn't involve going down one street twice. Why is this impossible?

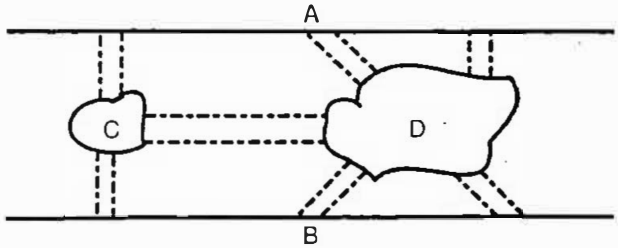
   d) Where would a possible path start? Where would it end?

3. a) Below is a town. Label all the nodes. Begin at point A and try to make a route without thinking about where the nodes are odd or even. Which paths did you end up covering twice?

   b) Now use the rule of connecting odd nodes by noting which nodes are odd. Which odd nodes would you connect to make the *shortest* complete route?

   c) Was your route in 3a different than in 3b? Does this example illustrate why mail carriers can benefit from studying mathematics? Elaborate.

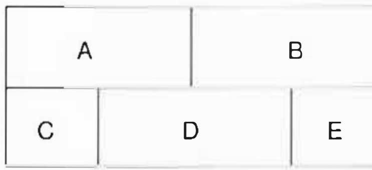
4. a) Here is a diagram of the Bridges of Konigsberg. The question is whether a person can make a path that crosses each bridge only once. A and B are the banks of the river and C and D are islands.



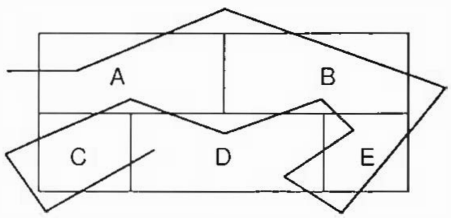
   Draw a node diagram of the Bridges of Konigsberg. Discuss your answer to the question based on the node diagram.

   b) How many bridges would need to be covered twice? Which bridges should these be to involve the least walking?

5. a) Here is a common puzzle which most of your parents will have seen. Draw a continuous path which intersects every segment of the diagram only once.



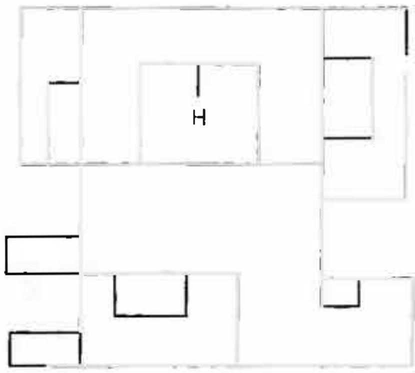
One attempt is shown below. Which segment remains uncrossed?



   b) Draw a node diagram for this problem. (This is not easy.) Label the regions A, B, C, D, E, F (exterior). Now the *paths* are the connections between regions. How many odd nodes do we have?

   c) We can now *change the problem* to make it possible. Add a box (internally) to this diagram to make it possible. Now it is possible but only if you start inside one box and end up inside another.

6. Here is a neighborhood with your home indicated. Number the nodes. Determine which of these are odd and indicate by number which nodes to connect so that you can deliver papers most efficiently.



7. From a city map (or from your own knowledge), draw a six-by-six block neighborhood around your house. Label the intersections and describe an efficient route indicating which streets would have to be covered twice.

Authors' Note: Readers interested in the entire volume of "A Collection of Connections" may contact Sol E. Sigurdson, Faculty of Education, Department of Secondary Education, University of Alberta, Edmonton T6G 2G5; phone 780-492-0753.

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Two Bridge Tables

At a family party held at Green's place, bridge was being played at two tables. The players were Mr. Green, Mr. Pink, Mr. Black and Mr. White and their respective wives. The partner of White was his daughter. Pink played against his mother. Black's partner was his sister. Mrs. Green played against her mother. Pink and his partner have the same mother. Green's partner was his mother-in-law. No player had an uncle participating. Who plays with whom at what table?