An Enigmatic E-Mail Riddle with an Explanation Using Modular Arithmetic

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The following riddle was circulated by e-mail at Augustana University College by Tom Bateman of the political studies department. Perhaps you have already encountered it.

This is pretty cool, but you must follow directions!

Scroll down slowly!

Do not skip ahead. Read this message *one line at a time* and just do what it says. You will be glad you did. If not, you will be sorry and you wished you had listened.

1. Pick a number from 1 to 9
2. Subtract 5
3. Multiply by 3
4. Square the number (multiply by the same number, not square root)
5. Add the digits until you get only one digit (that is, $64 = 6 + 4 = 10 \rightarrow 1 + 0 = 1$)
6. If the number is less than 5, add 5. Otherwise, subtract 4
7. Multiply by 2
8. Subtract 6
9. Map the digit to a letter in the alphabet 1 = A, 2 = B, 3 = C and so on
10. Pick a name of a country that begins with that letter
11. Take the second letter in the country name and think of a mammal that begins with that letter
12. Think of the color of that mammal

(Keep scrolling.)

DO NOT SCROLL DOWN UNTIL YOU HAVE DONE ALL OF THE ABOVE!

Here it comes, NO CHEATING or you will be sorry. Now scroll down.

You have a grey elephant from Denmark. Right?

The Enigma

Doesn't it seem rather mysterious that you should get the same answer as someone who started with a different number?

The Explanation

The following is an explanation based on responses sent to Tom Bateman by me and by Bill Hackborn, a colleague of mine in Augustana's math department.

After picking a number from 1 to 9 and subtracting 5 in steps 1 and 2, one has a number from −4 to 4. Then, after multiplying by 3 and squaring, one obtains one of 0, 9, 36, 81 or 144. The sum of the digit 0 is obviously 0. The remarkable thing about the other four numbers, and the fact that allows the riddle to work, is that in each case the sum of their digits is 9. The result of summing the digits in step 5 is thus either a 0 or a 9. After adding a 5 (or subtracting a 4) in step 6, everyone will then have the same answer, 5. From there on, everyone participating is on common ground, achieving the number 4 at step 8, and the corresponding letter D at step 9. For those with a modest knowledge of geography and mammalogy, the D suggests Denmark and the E an elephant. For most, the associated color is grey.

The Mathematical Basis

It turns out that the restriction on the size of the number in step 1 is not necessary. The riddle will work for any number. Steps 1 and 2 are used only to keep the computations simple.[1] The square of a multiple of 3 is always a multiple of 9. Further, if the digits of any multiple of 9, regardless of its size, are summed recursively, the sum will be a multiple of 9. For example, $3{,}888 = 9 \times 432$, a multiple of 9. Summing its digits recursively, one obtains $3 + 8 + 8 + 8 = 27 \rightarrow 2 + 7 = 9$.

Proof (for a 4-digit multiple of 9): the following is a proof of this property for any 4-digit multiple of 9.

Suppose that $N = 9m$, m an integer, is a multiple of 9. Let the decimal representation of N be $T\,h\,t\,u$, where T is the thousands digit, h is the hundreds digit, t the tens digit and u the unit digit.

Then $9m = 1000T + 100h + 10t + u$

$\Rightarrow \quad 9m = (999 + 1)T + (99 + 1)h + (9 + 1)t + u$

$\Rightarrow \quad 9m = 999T + T + 99h + h + 9t + t + u$

$\Rightarrow \quad T + h + t + u = 9m - 999T - 99h - 9t$

$\Rightarrow \quad T + h + t + u = 9(m - 111T - 11h - t)$

$\Rightarrow \quad T + h + t + u$ is a multiple of 9.

Thus, the sum of the digits of any 4-digit multiple of 9 is itself a multiple of 9. Summing the digits recursively will eventually result in one of the single-digit multiples of 9, namely, 0 or 9.

It is easy to see how this proof could be adapted to a multiple of 9 with any number of digits. There is a more general theorem that extends these results even to numbers in other bases, but it involves concepts of modular arithmetic that are too extensive to include here.

Note

1. In my response to Tom Bateman, I suggested that restricting the size of the number in step 1 and subtracting 5 from it in step 2 were useful in keeping the computations within the single digit capabilities of today's generation of students and professors.

References

Lauber, M. R. "Casting Out Nines—An Explanation and Extensions." *The Mathematics Teacher* 83, no. 8: 661–65.

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Meserve, B. E., et al. *Contemporary Mathematics*. 4th ed. Englewood Cliffs, N.J.: Prentice Hall, 1987, 160–67.

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Fireworks

My grandfather really likes fireworks. This year he bought as many rockets as his age in years to really celebrate his birthday. However, half the rockets became wet, the grandchildren borrowed one third of the rockets for their bush party and 21 rockets were without the ignition device. "It does not matter," said my grandfather, "that means that instead of one rocket for each year, each rocket has to count 10 years." This would really be a poor show. We will have a better fireworks show when my grandfather turns 100. When will that be?