On Centroids, Medians and Lines of Symmetry

David E. Dobbs

To make room for the innovations incurred by calculus reform, some traditional topics have been de-emphasized in recent curricula. One such topic concerns centres of mass, more specifically, centroids and moments, although this material remains important in areas such as engineering and statistics. The gradual de-emphasis of this material is apparent if one compares some leading calculus texts of different eras. In this regard, consider the result that justifies the "centroid" terminology: the centroid of any triangular region in the plane is the classical centroid (that is, the intersection of the medians) of the triangle bounding the region. This result was proved in the leading text, *Calculus with Analytic Geometry* (Johnson and Kiokemeister 1969, 404, example 2) 30 years ago, relegated to the exercises in another leading text, *Calculus* (Stewart 1991, 506, exercise 23) a decade ago, but omitted from the reform version *Calculus (Concepts and Contexts, Single Variable)* (Stewart 1998) of *Calculus* (Stewart 1991). One purpose of the present article is to sketch a self-contained proof of this theorem (see following section). The main purpose is to investigate a possible explanation for this result on triangles in terms of principles applicable to more general planar regions. In particular, consider the Symmetry Principle (which was stated in Stewart 1991 and 1998 and was often proved in the more rigorous texts of yesteryear [Leithold 1972, theorem 7.8.1]: if L is a line of symmetry of a planar region R, then the centroid of R lies on L. In view of the result in the next section, it is natural to ask whether the medians of a triangle Δ are lines of symmetry of the planar region bounded by Δ. The answer is in the negative (and so the result in the next section is still needed), for we show in the subsequent section that the medians of a triangle Δ are lines of symmetry for the region bounded by Δ if and only if Δ is equilateral. The material in the subsequent section can be used to enrich the precalculus unit on symmetry, as well as the precalculus topics of equations of lines and solution of linear inequalities and systems of linear equations. It would also reinforce the geometric/graphical approach in any first-year reform calculus sequence. The second section can be read independently of the third section. The material in the second section can be used to enrich the calculus unit on applications of the definite integral; this material also reinforces the solution of linear systems.

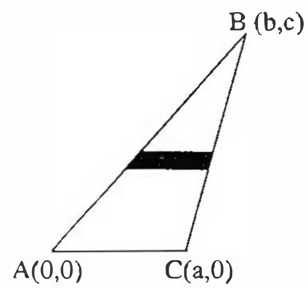
Centroids Are Centroids

The centroid $(\bar{x}, \bar{y})$ of a planar region R is given by $\bar{x} = M_y / A$ and $\bar{y} = M_x / A$, where M_y (resp., M_x) is the moment of R with respect to the $y-$ (resp., $x-$) axis and A is the area of R. If the boundary of R consists of a triangle Δ, Theorem 1 establishes that $(\bar{x}, \bar{y})$ is the intersection of the medians of Δ. A relatively short proof of Theorem 1 is available by using facts about similar triangles [(Johnson and Kiokemeister 1969, 404, example 2, 404), but the proof sketched below uses only equations of lines and calculus.

Theorem 1. *If a triangle Δ is the boundary of a planar region R, then the centroid of R is the intersection of the medians of Δ.*

Proof. For simplicity, we suppose that $\Delta = \Delta ABC$ has a vertex at the origin and a horizontal side, as in Figure 1.

Figure 1



If $D(a/2,0)$ denotes the midpoint of AC, then the median of Δ which passes through B is the line BD, whose point-slope equation is

$$y - c = \left(\frac{c-0}{b-\frac{a}{2}}\right)(x-b).$$

Similarly, the median of Δ which passes through A has equation $y = (c/(b+a))x$. Solving these equations simultaneously, we obtain that the classical centroid of Δ has coordinates $((a+b)/3,\ c/3)$. For a class needing assurance that the medians of Δ are concurrent, note that the given coordinates also satisfy $y = [(-c/2)/(a-b/2)](x-a)$, an equation for the median of Δ which passes through C. On the other hand, for a class that knows that the medians of Δ meet at the point $^2/_3$ of the way from B to D, one need only observe that

$$\left(\frac{1}{3}b + \frac{2}{3}\frac{a}{2}, \frac{1}{3}c + \frac{2}{3}\cdot 0\right) = \left(\frac{a+b}{3}, \frac{c}{3}\right).$$

The area of R (or Δ) is $A = ac/2$. Since $(\bar{x}, \bar{y}) = (M_y/A,\ M_x/A)$, a proof that $(\bar{x}, \bar{y}) = ((a+b)/3, c/3)$ reduces to showing that

$$M_y = \frac{ac(a+b)}{6} \text{ and } M_x = \frac{ac^2}{6}.$$

Consider a horizontal strip, as in Figure 1. The area of the strip is $dA = (x_2 - x_1)\,dy$, where x_2 (resp., x_1) is the expression for x in terms of y obtained by solving for x in a point-slope equation of BC (resp., BA). One readily finds that

$$x_2 = \left(\frac{b-a}{c}\right)\left(y + \frac{ca}{b-a}\right) \text{ and } x_1 = \frac{by}{c}.$$

Now, M_y is the definite integral of $x'\,dA$, where $x' = (x_2 + x_1)/2$ is the $x-$ coordinate of the geometric centre of the horizontal strip. Thus,

$$M_y = \int x'\,dA = \int \left(\frac{x_2 + x_1}{2}\right)(x_2 - x_1)\,dy$$

$$= \int_0^c \left(\frac{\frac{(b-a)}{c}\left(y+\frac{ca}{b-a}\right)+\frac{by}{c}}{c}\right)\left(\left(\frac{b-a}{c}\right)\left(y+\frac{c\alpha}{b-\alpha}\right)-\frac{by}{c}\right)dy.$$

After the integrand is algebraically simplified as a polynomial in y, a routine integration via the Fundamental Theorem of Calculus yields that $M_y = (abc+a^2c)/6 = ac(a+b)/6$, as required.

Finally, $M_x = \int y'\,dA$ where y', the second coordinate of the geometric centre of the horizontal strip, is essentially $y' = y'$. Thus,

$$M_x = \int_0^c y\left(\left(\frac{b-a}{c}\right)\left(y+\frac{c\alpha}{b-\alpha}\right)-\frac{by}{c}\right)dy = ac^2/6$$

where the final equality follows by another routine application of the Fundamental Theorem of Calculus. The proof is complete.

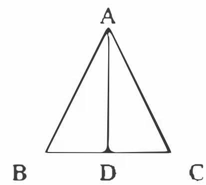
When Medians Are Lines Of Symmetry

Unfortunately, Theorem 1 is not a consequence of the Symmetry Principle since the medians of a triangle Δ are, in general, not lines of symmetry of the region bounded by Δ. As Theorem 4 and Corollary 5 document, requiring such symmetry restricts the nature of Δ severely. First, we isolate two pieces of the argument.

Lemma 2. *Let Δ = Δ ABC be an isosceles triangle, with AB=AC. Let D be the midpoint of BC, and let R be the planar region bounded by Δ. Then the median AD is perpendicular to BC, and AD is a line of symmetry of R.*

Proof. The data are summarized in Figure 2.

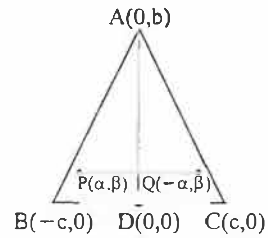
Figure 2



Since $AB=AC$ by hypothesis and $BD=CD$ by the definition of midpoint, it follows from the SSS (Side-Side-Side) congruence criterion that $\Delta ABD \cong ACD$. Therefore, $\angle ADB \cong \angle ADC$, and so $AD \perp BC$.

It remains to show that R is symmetric about AD. For convenience, locate the coordinate axes so that D is the origin and C is on the positive $x-$ axis. (Then AD falls along the $y-$ axis.) Consider an arbitrary point $P(\alpha,\beta)$ in R. Then $Q(-\alpha,\beta)$ is the point symmetric to P with respect to AD. Our task is to show that Q is in R. The data are summarized in Figure 3.

Figure 3



We show that if P is inside ΔABD, then Q is inside ΔACD. While this may seem clear "pictorially," an analytic proof depends on the meaning of "inside" and the fact that the solution sets of linear inequalities are half-planes. Observe that an equation for AB (resp., AC) is $x/(-c)+y/b = 1$ (resp., $x/c + y/b = 1$). The hypothesis that P is in the interior of ΔABD means that P is in the appropriate three half-planes determined by the sides of ΔABD, as follows:

$\alpha/(-c) + \beta/b < 1$, $\beta > 0$, and $\alpha < 0$.

The assertion that Q is in the interior of ΔACD means that $(-\alpha,\beta)$ satisfies the inequalities describing the appropriate three half-planes determined by the sides of ΔACD, as follows:

$(-\alpha)/c + \beta/b < 1$, $\beta > 0$, and $-\alpha > 0$.

These conditions are implied by (in fact, equivalent to) the inequalities imposed by the hypothesis on P, and so the proof is complete.

Lemma 3. Let $\Delta = \Delta ABC$ be a triangle, let D be the midpoint of BC, and let R be the planar region bounded by Δ. Suppose that the median AD is perpendicular to BC. Then Δ is isosceles, with $AB=AC$, and AD is a line of symmetry of R.

Proof. The data are summarized in Figure 2. Since $AD \perp BC$, the angles $\angle ADB$ and $\angle ADC$ are right angles, and hence are congruent to one another. Moreover, $DB=DC$ by the definition of midpoint. It now follows from the SAS (Side-Angle-Side) congruence criterion that $\Delta ADB \cong \Delta ADC$, and so $AB=AC$. An application of Lemma 2 completes the proof.

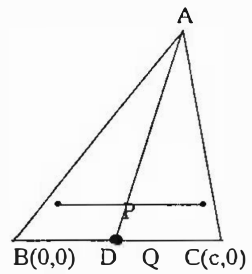
We next present our main result.

Theorem 4. *Let $\Delta = \Delta ABC$ be a triangle, D the midpoint of BC, and R the planar region bounded by Δ. Then the following three statements are equivalent:*

1. *The median AD is a line of symmetry of R;*
2. *The median AD is perpendicular to BC;*
3. *Δ is an isosceles triangle, with $AB=AC$.*

Proof. (2)$\Rightarrow$(3) by Lemma 3, while (3)$\Rightarrow$(1) by Lemma 2. It remains only to prove that (1)$\Rightarrow$(2). We shall prove the contrapositive. Assume, then, that AD has well-defined slope. We shall show that (1) fails by producing a point P in the interior of ΔABD (and hence in R) such that Q, the point symmetric to P with respect to AD, is not in R. There is no harm in locating the coordinate axes so that B is the origin and $C(c,0)$ is on the positive $x-$axis. We may also suppose that $L=AD$ has positive slope. (The proof in the case of negative slope is similar; alternatively, turn the page over!) Finally, let S denote the point of intersection of L and PQ. The data are summarized in Figure 4.

Figure 4



It will be enough to take $S=D$. First, let M denote the ray emanating from D such that M is perpendicular to L and M enters the half-plane determined by L and B. We shall find a suitable P on M. Indeed, since $\tan(\angle ADC)=\text{slope}(L) > 0$, $\angle ADC$ is an acute angle and so its supplement, $\angle ADB$, must be an obtuse angle. Hence, M enters the interior of $\angle ADB$. In particular, M enters the interior of $\angle ABD$. Choose P to be any of the infinitely many points of M which lie in the interior of $\angle ADB$. Then Q, the point symmetric to P with respect to L, has negative $y-$ coordinate, and so Q is not in R. This completes the proof of the contrapositive of (1)$\Rightarrow$(2).

Next, we state a result which was announced in the introduction. The proof of Corollary 5 is immediate from Theorem 4.

Corollary 5. *Let Δ be a triangle and let R be the planar region bounded by Δ. Then the following three statements are equivalent:*

1. At least two of the medians of Δ are lines of symmetry of R;
2. All three medians of Δ are lines of symmetry of R;
3. Δ is an equilateral triangle.

In closing, we indicate an exercise that goes beyond what was established in the proof of Theorem 4. By filling in the details of what is sketched in Remark 6, one would further reinforce the topic of graphs of linear inequalities, as well as basic facts about limits.

Remark 6. Let us return to the context addressed in Figure 4, namely, the proof of the contrapositive of (1)$\Rightarrow$(2) in Theorem 4. Here, we expand upon the assertion that infinitely many points $P(\alpha,\beta)$ in the interior of ΔABD are such their corresponding symmetric points $Q(\gamma,\delta)$ are not in R. Let A have coordinates (a,b). One can prove that there exists ε_1, $0<\varepsilon_1< a-c/2$ with the following properties. If $0<\varepsilon\leq\varepsilon_1$ then $S(c/2+\varepsilon, k)$ is such that P is in the interior of ΔABD (and hence in R) and $\delta < 0$ (so that Q is not in R).

The reasonableness of the preceding assertion can be indicated heuristically by applying a paper-folding experiment to Figure 4. A proof can be fashioned by considering the function of ε, for $0 < \varepsilon < a-c/2$, given by

$$\mu = \frac{\frac{bc}{2a} - \frac{2b\varepsilon}{2a-c} - \frac{(2a-c)\varepsilon}{2b}}{\frac{(2a-c)\varepsilon}{2b} + \frac{b\varepsilon}{a}}.$$

Put $\alpha = c/2 - \lambda\varepsilon$, where λ is a positive real number to be further specified later; for the moment, we require that $0 < \lambda < c/(2\varepsilon)$. Since $k = 2b\varepsilon/(2a-c)$ and $PS \perp L$, the "negative reciprocal" result leads to

$$\beta = \frac{(\lambda+1)\varepsilon(2a-c)}{2b} + \frac{2b\varepsilon}{2a-c}.$$

By considering appropriate half-planes, one shows that P is in the interior of ΔABD if and only if

$$0 < \beta < \frac{b}{a}\left(\frac{c}{2} - \lambda\varepsilon\right);$$

equivalently, if and only if $\lambda < \mu$. Observe via standard limit theorems that

$$\lim_{\varepsilon\to 0^+} \mu = \infty \text{ and } \lim_{\varepsilon\to 0^+} \frac{\mu}{\left(\frac{c}{\varepsilon}\right)} = \frac{\frac{b}{2a}}{\frac{2a-c}{2b} + \frac{b}{c}} < \frac{1}{2}.$$

Thus, putting $\lambda = \mu/2$, we can choose ε (and implicitly ε_1) so that

$$\max\left(\frac{8b^2}{(2a-c)^2} - 1, 0\right) < \lambda < \min\left(\frac{c}{2\varepsilon}, \mu\right).$$

One then verifies the assertions concerning P and Q.

References

Johnson, R. E., and F. L. Kiokemeister. *Calculus with Analytic Geometry*. 4th ed. Boston: Allyn & Bacon, 1969.

Leithold, L. *The Calculus with Analytic Geometry*. Part I, 2d ed. New York: Harper & Row, 1972.

Stewart, J. *Calculus*. 2d ed. Pacific Grove, Calif.: Brooks/Cole, 1991.

——. *Calculus (Concepts and Contexts, Single Variable)*. Pacific Grove, Calif.: Brooks/Cole, 1998.

Area Between Two Circles

What is the area between two circles, one of which circumscribes a regular heptagon and the other of which is inscribed in a regular heptagon? The length of each side of the heptagon is 1 cm.