Four-Color Map Problem

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The renowned conjecture in topology that arose circa 1852 asserts that four colors are both sufficient and necessary for coloring all maps drawn on a plane or sphere so that no two regions that touch (that is, share a segment of a boundary) are the same color. While this conjecture has always been an accepted fact for cartographers, for mathematicians it remained an unproved supposition. What has been known since 1890 is that five colors suffice and that there are maps for which three colors are insufficient. The chromatic number of a surface is the least number of colors that suffice to color any map on that surface. Except for the Klein bottle, the chromatic number of a surface is equal to the greatest integer not greater than, $\frac{1}{2}(7+\sqrt{49-24\chi})$, where χ is the Euler characteristic of the surface. Since $\chi = 0$ for the cylinder, Möbius strip and torus (doughnut), their chromatic numbers are 7. But $\chi = 0$ for the Klein bottle and its chromatic number is 6. For the projective plane, $\chi = 1$ and the chromatic number is 6. For the plane or sphere, $\chi = 2$ and the chromatic number is 4.

All of this has driven topologists to endless exasperation. They have been able to prove that only six colors are needed on a Möbius strip and seven on a torus, but no one has been able to prove what mapmakers have known for ages—that four colors are enough for any flat map or sphere. Topologists since Möbius' time have tried to draw a flat map on which five colors are needed; no one had done it, but neither had anyone proved that it cannot be done.

It remained a great-unsolved problem of topology until 1976 when two American mathematicians, Wolfgang Haken and Kenneth Appel, from the University of Illinois, presented a proof for this famous conjecture. Their method of proving this conjecture made unprecedented use of computers. Although this proof is an extraordinary achievement, many mathematicians found it to be deeply unsatisfying. This feeling was largely driven by their suspicion that a counter-example could still be found and that a proof in the rigorous mathematical sense does not exist. What mathematicians have seen, they claim, is a program for attacking the problem by computer along with the results of an "experiment" performed on a computer. Furthermore, the Haken-Appel proof of the four-color theorem is unsatisfying to mathematicians because it is not simple, beautiful or elegant. Haken and Appel both think that it is unlikely that a proof can be found that does not require an equally intensive application of computers, but there is no way of knowing this for sure. So one may ask, have Haken and Appel really proved the four-color-map conjecture? Many mathematicians who have examined the proof say yes, but the trial period is still not over. One thing is certain though: the four-color theorem has introduced a new era in mathematics—proof by using computers—and no one knows where it will lead.

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