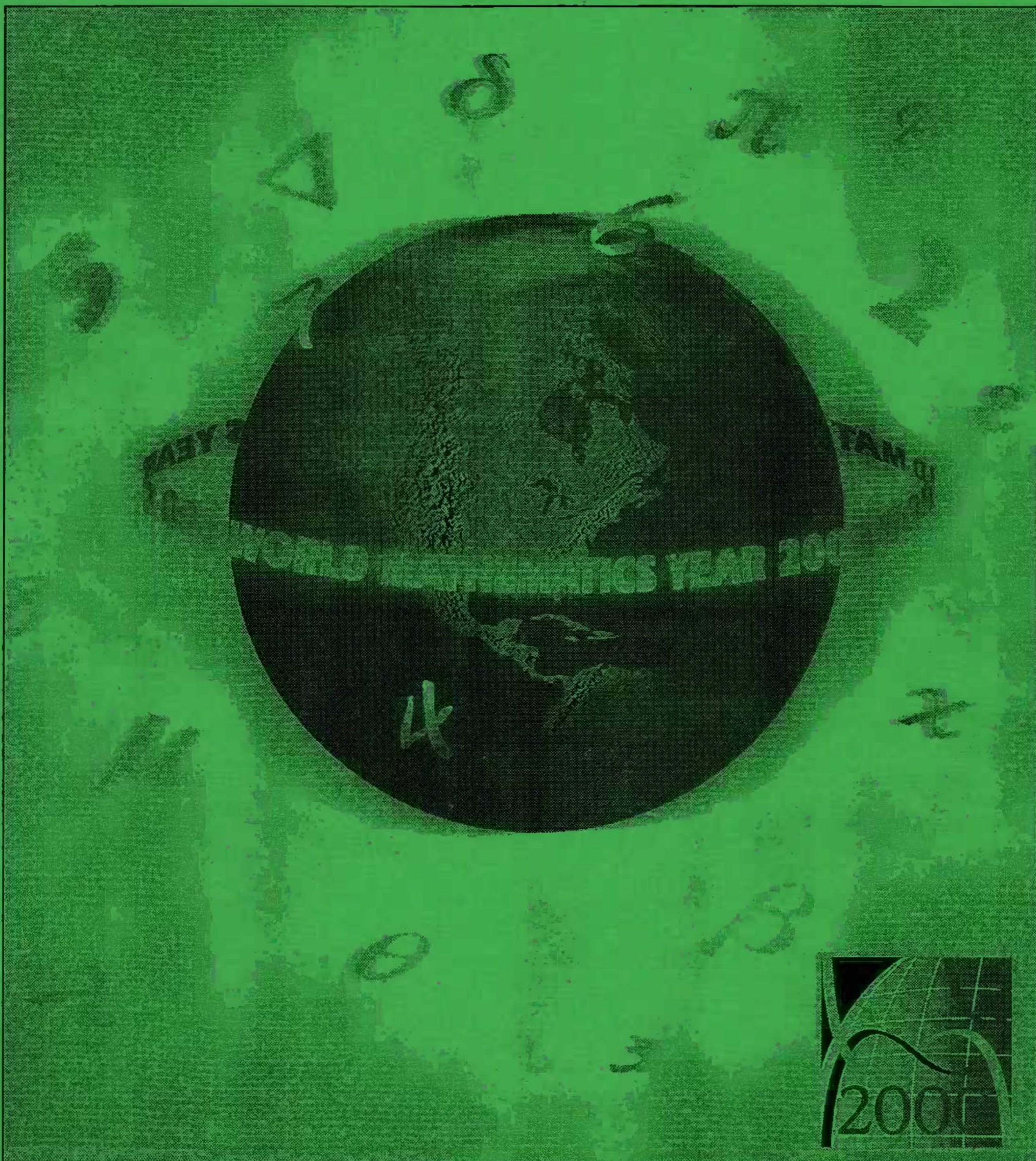
delta-k

JOURNAL OF THE
MATHEMATICS COUNCIL
OF THE ALBERTA
TEACHERS' ASSOCIATION

Volume 37, Number 2 June 2000



GUIDELINES FOR MANUSCRIPTS

delta-K is a professional journal for mathematics teachers in Alberta. It is published to

- promote the professional development of mathematics educators, and
- stimulate thinking, explore new ideas and offer various viewpoints.

Submissions are requested that have a classroom as well as a scholarly focus. They may include

- personal explorations of significant classroom experiences;
- descriptions of innovative classroom and school practices;
- reviews or evaluations of instructional and curricular methods, programs or materials;
- discussions of trends, issues or policies;
- a specific focus on technology in the classroom; and
- a focus on the curriculum, professional and assessment standards of the NCTM.

Manuscript Guidelines

1. All manuscripts should be typewritten, double-spaced and properly referenced.
2. Preference will be given to manuscripts submitted on 3.5-inch disks using WordPerfect 5.1 or 6.0 or a generic ASCII file. Microsoft Word and AmiPro are also acceptable formats.
3. Pictures or illustrations should be clearly labeled and placed where you want them to appear in the article. A caption and photo credit should accompany each photograph.
4. If any student sample work is included, please provide a release letter from the student's parent allowing publication in the journal.
5. Limit your manuscripts to no more than eight pages double-spaced.
6. A 250–350-word abstract should accompany your manuscript for inclusion on the Mathematics Council's Web page.
7. Letters to the editor or reviews of curriculum materials are welcome.
8. *delta-K* is not refereed. Contributions are reviewed by the editor(s) who reserve the right to edit for clarity and space. **The editor shall have the final decision to publish any article.** Send manuscripts to Klaus Puhlmann, Editor, PO Box 6482, Edson, Alberta T7E 1T9; fax 723-2414, e-mail klaupuhl@gyrd.ab.ca.

Submission Deadlines

delta-K is published twice a year. Submissions must be received by August 31 for the fall issue and December 15 for the spring issue.

MCATA Mission Statement

Providing leadership to encourage the continuing enhancement of teaching, learning and understanding mathematics.

delta-K

Volume 37, Number 2 June 2000

CONTENTS

 delta-K is published by the ATA for the Mathematics Council (MCATA). EDITOR: Klaus Puhlmann, PO Box 6482, Edson, Alberta T7E 1T9. EDITORIAL AND PRODUCTION SERVICES: Central Word Services staff, ATA. Opinions expressed herein are not necessarily those of MCATA or of the ATA. Address correspondence regarding this publication to the editor. *delta-K* is indexed in the Canadian Education Index. ISSN 0319-8367

FEATURE ARTICLES

MATHEMATICAL PROBLEM SOLVING FOR THINKERS

HISTORICAL CORNER

COMMENTS ON CONTRIBUTORS

Ed Barbeau is a professor of mathematics at the University of Toronto, Toronto, Ontario, and he runs the Olympiad correspondence program of the Canadian Mathematical Society. He is also on the board of the Ontario Association for Mathematics Education, the professional organization for mathematics teachers in Ontario.

Daryl M. J. Chichak is the Grade 6 mathematics assessment specialist with Alberta Learning and the Alberta Learning representative and membership director on the MCATA executive, Edmonton, Alberta.

David E. Dobbs is a professor of mathematics at the University of Tennessee, Knoxville, Tennessee.

David R. Duncan is a professor of mathematics at the University of Northern Iowa, Cedar Falls, Iowa.

Richard T. Edgerton is a science resource teacher with Seattle Public Schools, Seattle, Washington.

Brenda Healing teaches at Bentley High School, Bentley, Alberta.

Victoria R. Jacobs teaches in the Center for Research in Mathematics and Science Education at San Diego State University, San Diego, California.

Art Jorgensen is a retired junior high school principal from Edson, Alberta, and a longtime MCATA member.

Thomas E. Kieren is a professor of education (emeritus) in the Faculty of Education, Department of Secondary Education, University of Alberta, Edmonton, Alberta.

Murray Lauber is a professor in the Division of Mathematical Sciences at Augustana University College in Camrose, Alberta.

Terri-Lynn McLeod teaches at Bonnyville Centralized High School in Bonnyville, Alberta.

Bonnie H. Litwiller is a professor of mathematics at the University of Northern Iowa, Cedar Falls, Iowa.

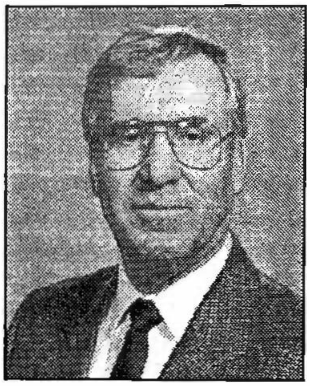
Kay McClain teaches methods courses for preservice teachers at Vanderbilt University, Nashville, Tennessee. She also teaches middle school mathematics in conjunction with her research on students' statistical understandings.

Sandra M. Pulver is a professor of mathematics at Pace University in New York.

Sol E. Sigurdson is a professor of education (emeritus) in the Faculty of Education, Department of Secondary Education, University of Alberta, Edmonton, Alberta.

Sandra L. Takis teaches advanced placement statistics at Yorktown High School in Arlington, Virginia.

EDITORIAL



As you read this issue of *delta-K*, you are well into the year 2000 and near the end of another school year. For many of you, this will have been your second year teaching the Applied and Pure Mathematics streams that were developed as part of the Western Canadian Protocol. I am sure that some of you are still struggling, while others are beginning to reach an appropriate comfort level within these new streams. Any call for significant curricular change poses new challenges for teachers. The new curriculum is no exception. It not only identifies the strands and general and specific outcomes for students to learn but also lists seven mathematical processes, which are seen as critical components that students must encounter in their mathematics learning.

These seven mathematical processes—communication, connection, estimation and mental mathematics, problem solving, reasoning, technology and visualization—permeate the teaching and learning of mathematics from Kindergarten to Grade 12. While incorporating all these mathematical processes in the lessons requires thoughtful planning by teachers, problem solving continues to challenge teachers in a significant way. This is partly due to the fact that as teachers employ problem solving throughout the strands, they are required to develop new techniques for evaluating what students have learned and the effectiveness of instruction. Problem solving also draws in other mathematical procedures, such as opportunities for students to be active in constructing mathematical meaning, communicating mathematical ideas, reasoning and the use of technology. In addition, problem-solving activities should provide the students with the opportunity to work cooperatively.

Teachers are generally very supportive of the importance of problem solving as the focus of mathematics at all grade levels, but they question current evaluation/assessment methods. While this is a critical issue, it does not mean that problem solving can be ignored. Some suggestions that you might find useful in implementing problem-solving processes in your classroom follow.

What am I trying to evaluate? Problem solving involves many subskills, knowledge and attitudes, and your assessment can focus on them individually. It is important to recognize that successful problem solving not only involves the mastery of these subskills but also their coordination.

What are some evaluation techniques that I can use in my classroom? Your selection of an evaluation method is probably guided by a number of factors, such as the type of problem, class size, the time available for evaluation, your experience and availability of resources. Depending on the outcomes to be evaluated—student performance or student attitudes/beliefs—several techniques can be employed, including observation, checklists and questioning of students, using assessment data from students, holistic scoring methods or multiple-choice and completion tests.

How do I organize and manage my evaluation program? This important question requires you to reflect on your own beliefs about evaluation and, in this case, problem solving. Is your evaluation plan part of your instructional plan, its content and activities? As you develop your plan, be mindful of the following guidelines:

- Evaluate your students' work on a regular and systematic basis.
- Evaluate their thinking processes as well as their answers.
- Match your evaluation plan to your instructional goal.
- Assess attitudes and beliefs about problem solving as well as performance.
- Interview students individually whenever possible.
- Observe students' small-group efforts and their written work as an important part of your evaluation plan.
- Do not feel compelled to evaluate all students at the same time or to record their performance on every problem-solving experience.
- Advise students of your evaluation plan and how it works.

How do I use the evaluation results? Clearly, the major reason for developing and using an evaluation plan for problem solving should be to gain information that enables you to make instructional decisions based on students' identified strengths and weaknesses. More specifically, one might ask, Are the problems appropriate with respect to the level of difficulty? Is problem solving part of my instructional program or is it an extra?

Are the experiences properly sequenced to develop students' skills? Do I incorporate appropriate content and teaching methods? or Do I use my results for evaluating student progress in problem solving or for grading?

The assessment/evaluation of the mathematical process of problem solving is indeed complex and challenging, but with the appropriate attention to its purpose, techniques, organization/management and the use of the results, problem solving can be a very positive experience for students and teacher.

As George Polya (1949), the great teacher of problem solving, said:

> No one can give away what he has not got. No teacher can impart to his students the experience of discovery if he has not got it himself.

References

Charles, R., F. Lester and P. O'Daffer. *How to Evaluate Progress in Problem Solving.* Reston, Va.: National Council of Teachers of Mathematics, 1987.

Polya, G. *California Mathematics Council Bulletin* 7, no. 2 (1949).

Klaus Puhlmann

From the President's Pen

We are now several months into the new century but as we march forward into Y2K and beyond, it is important to remember where we have been. Here are a few recollections from the past and a few hopes for the future.

Once upon a time in a land very much like this one, there lived a group of people who wanted to educate their children and another group of people who were willing (for a price) to do so.

They gathered the children together, separated them by age and ability, and put them all together in institutions that were called schools. Schools were modeled after prisons, hospitals and factories—the only other large buildings in existence at the time. Laws were passed which required children (up to a certain age) to attend these schools on a daily basis (usually from 9 to 5) modeled after the workday. Day visits home were allowed on weekends and the timing of extended paroles varied according to the location (farm, city, climate zone and so on). Days at school had to be organized efficiently and orderly as literally hundreds of students would be attending them and "doing their time" each day. Bells rang, blocks of time called periods were devised and a curriculum was instituted that would emphasize the prevailing culture, thus ensuring the easy integration into society and uniform conformity for all. Our discipline of mathematics more than adequately reflects this time from long ago. Over time, however, beliefs have changed. In 1989, the National Research Council in the United States published a document called *Everybody Counts* which is a report to the nation on the future of mathematics education. In the margins of this document can be found numerous "myths" regarding mathematics education. Although over a decade old, many of these myths are still espoused today. Here is a sample:

1. As computers become more powerful, the need for mathematics will decline.
2. What is good enough for me is good enough for my child.
3. Early use of calculators will prevent children from learning the basic facts of arithmetic.
4. Learning mathematics means mastering an immutable set of basic skills.
5. Students learn by remembering what they are taught.
6. The way to improve students' mathematical performance is to stress the basics.
7. Only objective tests yield reliable results.

After reading this list of myths, I am reminded of my youngest son's definition of myth—a female moth—and wonder why these myths were relegated to the margins of the piece. Clearly they should be front and centre and represent a potpourri of the traditional customs, tales and sayings of the common people with respect to teaching, learning and even research. Certainly they represent the life and the spirit of the mathematics education that we were given and a good chunk of the mathematics education that we have delivered!

All of the myths listed above can easily be debunked by a heavy dose of reality. Arguments could be put forward that are clear, concise and logical for each one of them. In fact, some of the myths seem so trivial that it is a wonder that anyone still holds them! Life marches on and we can choose to move along with the flow or simply be run down by the stampede. However, we need to understand our past in order to make better sense of the future. We must recognize these myths for what they are—female moths ready to eat away our significance and *not* Moses' stone tablets of truth. The point is that they were believed and adhered to from generation to generation partly because they were useful for the time and because nobody ever questioned them. It is time to cast these myths from our belief system and create a new reality about teaching and learning of mathematics that befits the 21st century. Watch MCATA as together we march into the new millennium.

Cynthia Ballheim

The Right Angle

Daryl M. J. Chichak

Student Evaluation News

The Grades 3, 6 and 9 Mathematics Information Bulletins for the 1999–2000 school year are available at Student Evaluation's website: http://ednet.edc.gov.ab.ca/studenteval/. This website should be checked regularly for new information.

Direct questions or comments regarding the bulletins to Terry Gamble, Grade 3 math assessment specialist, at tgamble@edc.gov.ab.ca or Daryl M. J. Chichak, Grades 6 and 9 math assessment specialist, at dchichak@edc.gov.ab.ca.

Implementation of Pure and Applied Math 30 Examinations

Exams in Pure and Applied Math 30 will be offered beginning in the 2000–2001 school year. This is a mandatory implementation year for Pure Math 30 and an optional implementation year for Applied Math 30. Because of the implementation schedules, the exams in Pure and Applied Math 30 for the 2000–2001 school year will be weighted differently than other exams. FOR THESE EXAMS ONLY, the school-awarded mark will count for 80 percent and the diploma exam mark will count for 20 percent of the final blended mark. This weighting will continue in the 2001–2002 school year for APPLIED MATH 30 ONLY. Exams in all other subjects, including Math 30 and Math 33, will continue to be weighted at 50 percent for the school-awarded mark and 50 percent for the diploma-exam mark.

The new calculator policy goes into effect in September 2000 for ALL math and science exams.

There will be no November 2000 exam in Math 30 or Math 30 Pure. November exams in Math 30 Pure will commence in November 2001.

Exams for the old Math 30 program will be available in the 2000–2001 and 2001–2002 school years for students repeating the course or for students completing a course sequence.

Exams for the Math 33 program will be available to all students in the 2000–2001 and 2001–2002 school years because these are optional implementation years for Applied Math 30. In the 2002–2003 school year, Math 33 exams will be available only to students repeating the course or completing course sequences.

For further information on mathematics diploma examinations, see the implementation schedule on page 9.

The standards for Math 30 Pure and Applied have been reviewed by many teachers throughout the province and are now in final draft form. The bulletins for these courses should be available soon.

Student Evaluation Branch staff have been giving presentations through the regional consortia across the province. These presentations have focused on the format and standards for the new diploma exams in Pure and Applied Math 30.

Keystrokes Required for Clearing Approved Calculators

On September 1, 2000, the new Alberta Learning Calculator Policy will come into effect. The following information is provided to help familiarize students, teachers and presiding examiners with the procedures involved in clearing all calculator memories.

It is not required that calculators be cleared according to these procedures for the January, April, June or August 2000 diploma examinations.

Using the methods on page 10 to clear calculators for these examinations will result in the erasure of programs that are allowed under the Calculator Policy for the 1999–2000 school year, that is, the "Conics" program.

Procedures to Follow Prior to Writing a Diploma Examination

1. At the beginning of any mathematics or science diploma examination course, teachers must advise students of the types of calculators approved by Alberta Learning for use when writing diploma examinations.
2. Students must clear all programmable calculators, both graphing and scientific, that are brought into diploma examinations. All information that is stored in the programmable or parametric memory must be cleared.
3. Presiding examiners are responsible for ensuring that
   - all calculators operate in silent mode,
   - students do not share calculators or information contained within them,
   - calculator cases are stored on the floor throughout the examination and
   - all examination rules are followed.

Note 1: If you have problems with any of the clearing techniques, please contact the Mathematics/Science Diploma Examination Unit of Student Evaluation at 780-427-0010 (toll-free 310-0000), fax 780-422-4200 or e-mail cmccabe@edc.gov.ab.ca.

Note 2: Resetting calculators may result in altering the calculator mode settings. Please remember to check the mode settings before proceeding with the diploma examination.

Note 3: Programs downloaded from the Web are not allowed on the calculators used during diploma examinations and will be erased by these procedures.

Note 4: The memory values given on the next pages refer to memory expected to be available as a factory setting. The values available in student calculators should match these values when the calculator has been reset. If the values in the student calculators do not match these values, then the calculators should be reset a second time. If this fails to change the values, then the calculator should not be used on the examination.

Language Services Branch News

There has been early implementation of Applied Math 10 and Pure Math 20 during the second semester of the 1999–2000 school year. If you require further information on the French mathematics program, contact François Lizaire at 780-427-2940.

Curriculum Standards Branch News

Standards Documents

The Standards Documents for Applied Mathematics 10 and 20 and Pure Mathematics 10 and 20 are available at Alberta Learning's website: http://ednet.edc.gov.ab.ca.

Distance Learning Information

The following junior and senior high school distance learning materials are available from the Learning Resources Distributing Centre (LRDC) for distance learning and regular classroom use:

- Mathematics 7 (LRDC Product #311069): This package has seven modules with accompanying assignment booklets. No textbook is required to complete the course, but students need a scientific calculator and two videos (LRDC Product #313495 and 313502). Note: The Mathematics 7 Learning Facilitator's Manual (LRDC Product #311035) includes the answers for marking the assignments for this course.
- Mathematics 8 (LRDC Product #349812): This package has six modules with accompanying assignment booklets. No textbook is required to complete the course, but students need a scientific calculator and two videos (LRDC Product #356883 and 356891). Note: The Mathematics 8 Learning Facilitator's Manual (LRDC Product #349838) includes the answers for marking the assignments for this course.
- Mathematics 9 (LRDC Product #348103): This package has six modules with accompanying assignment booklets. No textbook is required to complete the course, but students need a scientific calculator and one video (LRDC Product #356908). Note: The Mathematics 9 Learning Facilitator's Manual (LRDC Product #348111) includes the answers for marking the assignments for this course.
- Pure Mathematics 10 (LRDC Product #381468): This package has five modules with accompanying assignment booklets. The *Mathpower 10* textbook and a graphing calculator are required to complete the course. Note: The Pure Mathematics 10 Learning Facilitator's Manual (LRDC Product #383472) includes the answers for marking the assignments for this course.
- Pure Mathematics 20 (LRDC Product #398265): This package has six modules with accompanying assignment booklets. The *Mathpower 11* textbook and a graphing calculator are required to complete the course. The Pure Mathematics 20 Learning Facilitator's Manual (LRDC #389257) includes the answers for marking the assignments for this course.
- Pure Mathematics 10b (LRDC Product #407644) Interim Bridging Course (3 credits): This package includes facilitator's cover package, student cover package with parts of Module 1 of Pure Math 10, Modules 3 and 4 of Pure Math 10 with accompanying assignment booklets. The *Mathpower 10* textbook and a graphing calculator are required to complete this course. Note: The Pure Mathematics 10 Learning Facilitator's Manual LRDC #383472 includes the answers for marking this material.
- Pure Mathematics 20b (LRDC Product #407652) Interim Bridging Course (5 credits): This package includes facilitator's cover package; student cover package with parts of Module 1 of Pure Math 10; and parts of Module 2 of Pure Math 20; Modules 3 and 4 of Pure Math 10 with accompanying assignment booklets; and Modules 3, 4, 5, and 6 of Pure Math 20 with accompanying assignment booklets. The *Mathpower 10* and *Mathpower 11*

textbooks and a graphing calculator are required to complete this course. Note: The Pure Mathematics 10 Learning Facilitator's Manual (LRDC #383472) and Pure Mathematics 20 Learning Facilitator's Manual (LRDC #389257) may be used to mark this material.

The following courses will be available for the fall semester (the first two courses will be available in a split shipment. Package A will be available by August 15, 2000; package B will be available by October 15):

- Pure Mathematics 30: This package will have seven modules with accompanying assignment booklets. Students will require the *Mathpower 12* textbook and a graphing calculator.
- Applied Mathematics 10: This package will have seven modules with accompanying assignment booklets. Students will require the *Addison Wesley Applied Mathematics 10 Source Book*, a graphing calculator, a micrometer (LRDC Product #394643) and a vernier caliper (LRDC Product #394635).
- Mathematics Preparation 10: This package will have five modules with accompanying assignment booklets. No textbook is required, but students will need a scientific calculator. Note: This five-module course can be offered for 5 credits; alternatively, individual modules may be offered for 1 credit each.

For further information, call Linda Chase at 780-674-5350, ext. 138.

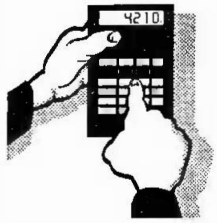
Daryl M. J. Chichak

Implementation Schedule for Mathematics Diploma Examinations
1999–2003

School Year	Math 30 (Old)	Math 30 Pure	Math 33 (Old)	Math 30 Applied
1999–2000	No Field Tests		Field Tests	
Nov–99	Dip (Eng/Fr)—S			
Jan–00	Dip (Eng/Fr)—R		Dip (Eng/Fr)—R	
Jun–00	Dip (Eng/Fr)—R		Dip (Eng/Fr)—R	
Aug–00	Dip (Eng/Fr)—S		Dip (Eng/Fr)—S	
2000–2001	No Field Tests	Field Tests	No Field Tests	Field Tests
Nov–00	No Diploma Exam	No Diploma Exam		
Jan–01	Dip (Eng/Fr)—S	Pilot Dip (Eng) 20%—R	Dip (Eng/Fr)—R	Pilot Dip (Eng) 20%—R
Jun–01	Dip (Eng/Fr)—S	Pilot Dip (Eng/Fr) 20%—R	Dip (Eng/Fr)—R	Pilot Dip (Eng) 20%—R
Aug–01	Dip (Eng/Fr)—S	Pilot Dip (Eng/Fr) 20%—S	Dip (Eng/Fr)—S	Pilot Dip (Eng) 20%—S
2001–2002	No Field Tests	Field Tests	No Field Tests	Field Tests
Nov–01	No Diploma Exam	Dip (Eng/Fr)—S		
Jan–02	Dip (Eng/Fr)—S	Dip (Eng/Fr)—R	Dip (Eng/Fr)—S	Pilot Dip (Eng/Fr) 20%—R
Jun–02	Dip (Eng/Fr)—S	Dip (Eng/Fr)—R	Dip (Eng/Fr)—S	Pilot Dip (Eng/Fr) 20%—R
Aug–02	Dip (Eng/Fr)—S	Dip (Eng/Fr)—S	Dip (Eng/Fr)—S	Pilot Dip (Eng/Fr) 20%—S
2002–2003		Field Tests	No Field Tests	Field Tests
Nov–02		Dip (Eng/Fr)—S		
Jan–03		Dip (Eng/Fr)—R	Dip (Eng/Fr)—S	Dip (Eng/Fr)—R
Jun–03		Dip (Eng/Fr)—R	Dip (Eng/Fr)—S	Dip (Eng/Fr)—R
Aug–03		Dip (Eng/Fr)—S	Dip (Eng/Fr)—S	Dip (Eng/Fr)—S

Key
R = Examination released to educators, students and the general public after the administration is complete
S = Examination secured; not released to educators, students or the general public after the administration is complete
Eng = Examination available only in English
Eng/Fr = Examination available in both English and French
Pilot = Pilot examination worth 20 percent and school-awarded mark worth 80 percent of the final mark

Calculator Clearing Techniques



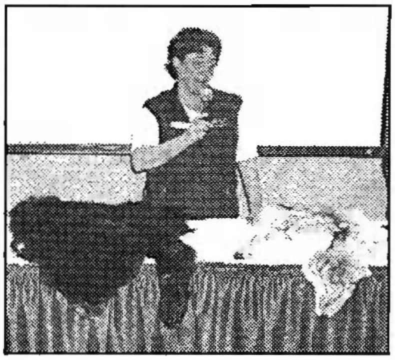
Casio	Memory Remaining
Casio 9700 Go to Menu Cursor to Reset EXE (All memory) F1 (Yes–Reset All)	Menu 24 000 Bytes available EXE Clear Shift MDISP(CAPA)
Casio 9800 Go to Menu Options EXE (Memory) (Reset) EXE (Reset) F1 (Yes–Reset All)	Menu 24 000 Bytes available EXE Clear Shift MDISP(CAPA)
Casio 9850 Go to Menu ALPHA E (Memory) ⇒ (Reset) EXE	Menu 30677 Alpha E EXE Check usage
Casio CFX-9850G Go to Menu ALPHA E (Memory) ⇓ (Reset) EXE F1 (Yes–Reset All)	Menu 30677 Alpha E EXE Check usage

Sharp	Memory Remaining
Sharp EL 9600 and 9600C 2nd XθTN(Option) log (Reset) 2 (All memory) CL (Clear all data) Note:There is also a reset switch on the back. (Use round tip of pen, press, then CL)	2nd XθTN 18562 ⇓

Texas Instruments	Memory Remaining
TI 82 2nd+ (Mem) 3 (Reset) 2 (Reset) Note: If, on clearing, the screen is blank, the contrast needs to be reset. To do this, use 2nd ⇑ both repeatedly.	2nd + MEM FREE 28734 1
TI 83 2nd+ (Mem) 5 (Reset) 1 (All memory) 1 (Reset) Note: If, on clearing, the screen is blank, the contrast needs to be reset. To do this, use 2nd ⇑ both repeatedly.	2nd + RAM 27118 1
TI 83 Plus 2nd+ (Mem) 7 (Reset) 1 (All RAM) Enter 2 (Reset) Note: If, on clearing, the screen is blank, the contrast needs to be reset. To do this, use 2nd ⇑ both repeatedly.	2nd + RAM 24317 2 ARC 163840
TI 86 2nd3 (Mem menu) F3 (Reset) F1 (All) F4 (Yes) Note: If, on clearing, the screen is blank, the contrast needs to be reset. To do this, use 2nd ⇑ both repeatedly.	2nd 3 MEM FREE 98226 F1
TI 89 2nd6 (Mem) F1 (ALL) 1 (Reset) Enter	2nd 6 RAM 199154 ARC 393204
TI 92 2nd6 (Mem) F1 (Reset) 1 (All) Enter Note: If, on clearing, the screen is blank, the contrast needs to be reset. To do this, use ◊ (green) and + or – repeatedly.	2nd 6 System 61064 Memory Free 70008

MCATA 1999 Annual Conference

"Mathematics in Harmony with the New Millennium," was the theme of MCATA's 1999 annual conference, held October 21–23 at Jasper Park Lodge in Jasper. More than 600 registrants attended this conference. The following is a potpourri of images from the conference.



Cleone Todgham, naturalist, giving a presentation about Jasper National Park.

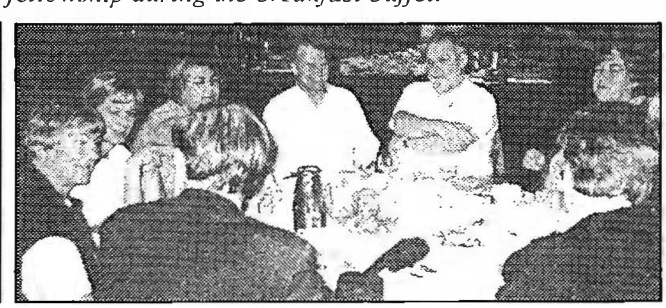
Carol Greenes giving her keynote opening address.

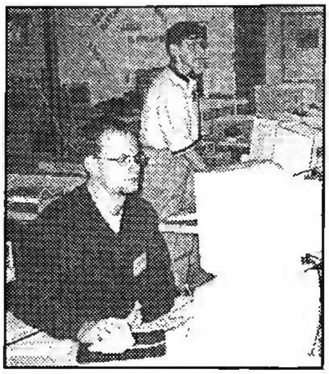
Conference presenters John Percevault and Art Jorgensen.

Some "Friends of MCATA." (l–r) Marian Florence, Hugh Sanders, Bob Hart, Craig Loewen, Cynthia Ballheim and Paula Bruner.

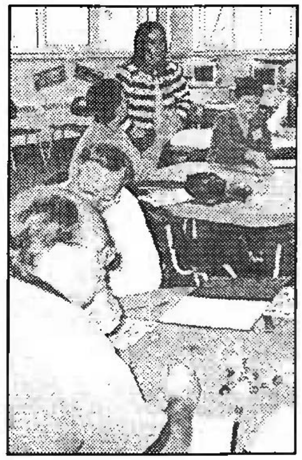
Math Educator of the Year Award recipients (non-classroom teacher category) Betty Morris (l) and (classroom teacher category) Gail Poshtar (r), pictured with Cynthia Ballheim, MCATA president.

Conference participants enjoying the fellowship during the breakfast buffet.



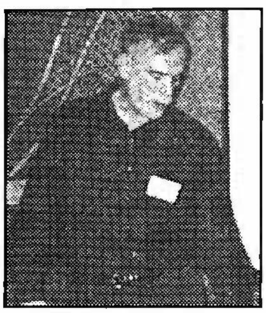


Edna Dach (standing) talking about "Technology Outcomes and Mathematics."



Joanne Currah (standing) giving her presentation, "Fun 'Die' Mentals."

Len Arden presenting "Texas Instrument Graph Links and Software."



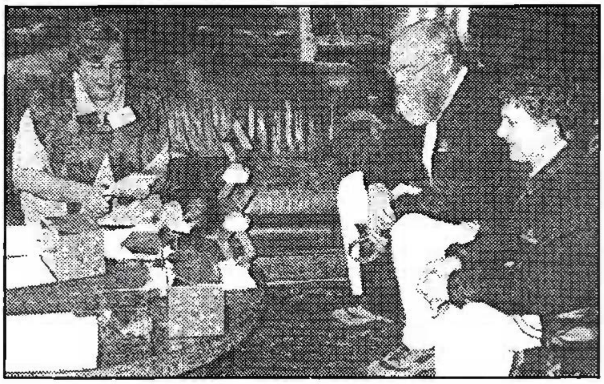
Doug Knight reviewed the "Information and Communication Program of Studies."

Craig Loewen working with teachers on "Literature and Problem Solving."

Hugh Sanders enlightened teachers on "The Future of Mathematics in Alberta Schools."

Daryl Chichak talks about "Statistics and Probability Made Easy."

Doug MacLachlan involving teachers in "Pure Math 10 Teaching and Learning Online."

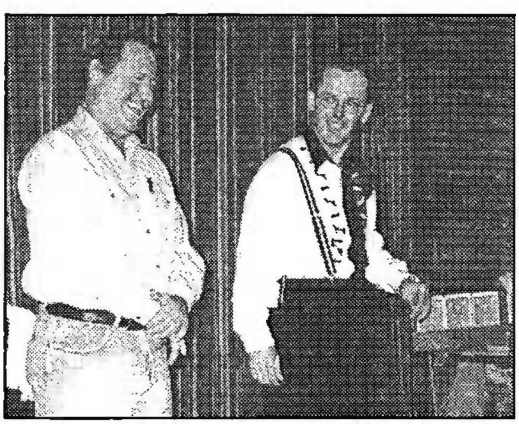


Marian Oberg (l) conducted a Games Emporium in which participants had a chance to learn some new mathematical games.

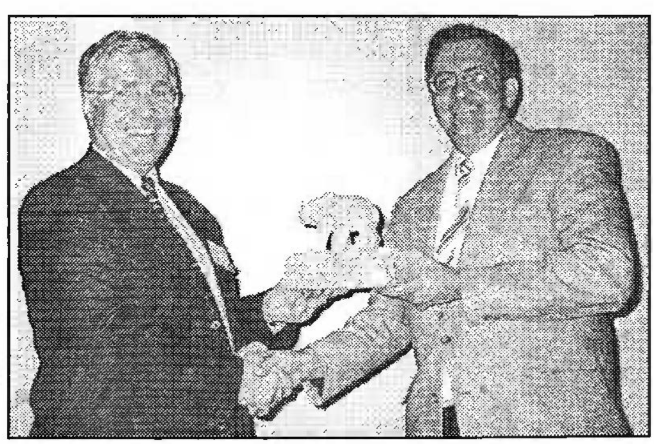
Marianne Nisson conducted a workshop on "Protocol Perspectives on Problem Solving."

Doug Weisbeck introduced the lunch session.

(l–r) Len Bonifacio, Kathy McCabe and Shauna Boyce reviewing the "Math 30 Pure and Applied Diploma Exam Blueprint."



Presenter Phil Radomsky (r) involving a teacher in his presentation.



Klaus Puhlmann presenting a wood carving to Dale Burnett.

READER REFLECTIONS

In this section, we will share your points of view on teaching mathematics and your responses to anything contained in this journal. We appreciate your interest and value the views of those who write. In the following article, "Men as Trees Walking," Ed Barbeau from the University of Toronto offers his opinion about the Principles and Standards of School Mathematics.

Men as Trees Walking

Ed Barbeau

This article is based on a plenary talk given May 29, 1999, at the summer meeting of the Canadian Mathematical Society in St. John's, Newfoundland.

"Men as trees walking"—this was the phrase that came into my mind as I examined the draft *Principles and Standards of School Mathematics* (NCTM 1998) recently issued by the National Council of Teachers of Mathematics. The phrase is from the Bible (Mark 8:22–26), and describes a curious miracle—a blind man touched by Christ could at first see only "men as trees walking" and required a second touch in order to see clearly.

One has the same feeling about the *Principles and Standards*; while it is generally praiseworthy, it is hard to bring the document into a clear focus. It contains a mixture of recommendations about topics, processes, teaching styles and general philosophy, but, as a lecturer of first-year university students, I did not get a clear sense of what I would be able to count on from the students in front of me. It seems that there is such a plethora of ideas put forward that perhaps a second healing touch is needed to tease out the main threads.

The central issue became clear to me one evening as I watched Rob Buckman on TV Ontario present a documentary on alternative medicine. Many people have become alienated from traditional medicine because they find it too reductionist and narrowly focused. Whatever the specifics, alternative medical regimes are attractive because they are holistic, proceed from a broader worldview and, significantly, involve the patient in the diagnosis and choice of treatment. Cases that might seem identical to traditional doctors might receive quite different treatments in different environments. Surely something like this is behind the pressure for reform in education. Teachers and students are reacting against a curriculum that seems to be reduced to a list of topics and processes, against an imposed canon robbing students and teachers of their autonomy. Whatever the details might be, we want students to be intimately involved in an educational process that cares about who they are and what characteristics they bring to the mathematics class.

But the charge that traditional education (as traditional medicine) has consistently lacked the human touch is far too stereotypical; it has resulted in a number of ghosts that have haunted recent educational reform. Let us raise a few of them.

Ghost 1: failure, streaming, elitism. There is no doubt that school was a brutal experience for many students in the past, but the fact remains that success in mathematics depends on a certain level of ability and application. To deny students the opportunity to fail is also to deny them the opportunity to enjoy success. This is not a call to ignore students who are floundering. Instead, we must create conditions that do not neglect the imperatives of learning mathematics and the possibility for achievement and that provide adequate support for students to move on in confidence. Many students now are uncertain about what they can do or should know; even good students are denied the chance to demonstrate their capabilities. While there is much that can be done to make mathematics more generally accessible, it needs to be recognized that the subject is often difficult and beyond the ability or interest of some segment of the population.

Ghost 2: the syllabus, list of prescribed topics, facts and procedures. The charge here is that having a list is too confining and leads to an emaciated mathematics of disembodied set of facts and routines.

It is not clear why this should necessarily be the case. In fact, the criticism seems to be misplaced; it should be directed at matters of design. In the hands of a teacher familiar with mathematics, the syllabus can serve as a set of markers and goals that frame what will happen in the class. An uncertain or ignorant teacher will hold to the list without any regard to connections and deeper understanding.

Ghost 3: drill, rote learning and memorization. Memorization was an important part of traditional education, and many cultures put a great deal of emphasis on what children should remember. Children seem to have good memories, and it seems foolish not to take advantage of this. But they need to be taught to evaluate what is worth memorizing, how mastery can be reached and how they can exploit the coherence of mathematics to leverage their knowledge of a few facts into fluency in a larger domain. The most able children can do this naturally; others need to have the issue explicitly addressed, so there is an underlying equity issue for students without natural talent.

Ghost 4: arithmetic. This is a word that seems to have fallen upon hard times in the curriculum stakes. But it is through arithmetic that most ordinary citizens see the connection between mathematics and the world, and lack of numeracy can present a severe handicap. Arithmetic has come to symbolize mindless memorization and manipulation. We need to detach it from this calumny and exalt it to the level of mathematical richness that it deserves.

Ghost 5: paper-and-pencil algorithms. This is also sometimes seen as drudgery, and the advent of modern technology has provided detractors of traditional calculation with a pretext for summarily discarding it. The issue is really how traditional arithmetic should be handled in a modern curriculum. Perhaps we need to see it more as additional means of accustoming students to working with figures or as examples of algorithms. Long division seems to be particularly suitable for showing how one can move from the idea of tallying a continued subtraction to a mechanical algorithm that is fast and accurate; one can point out that it was a human invention and replaced a method that was decidedly inferior (the "scratch method"). What has changed is that the importance of paper-and-pencil methods as practical techniques is reduced and we now have alternatives for reaching children encountering difficulties.

Ghost 6: word problems. Traditional word problems are criticized as being artificial, but one can argue that this is precisely the point of most word problems. They provide an imaginary situation in which certain points about interpretation and formulation can be made. The question again is not whether we should retain word problems, but how appropriate they are to the situation and how we plan to move beyond them.

Ghost 7: authoritarianism. This ghost has two manifestations, depending on whether you are referring to the teacher, who, we are told, should be the "guide on the side" rather than the "sage on the stage," or to the subject itself in which pupils are oppressed by the tyranny of "one right answer." Without disputing the advantages of the more open and friendly classrooms that modern students enjoy, it remains the case that a teacher's effectiveness depends on what she knows and that sometimes students need to submerge their egos and pay attention to what she has to say. In the same way, it seems mischievous to deny the power of mathematical certainty, especially given the diverse ways in which one might think about concepts and approach problems. (One might say that the work of Gödel and Lakatos, however lauded by serious mathematicians, have had a particularly pernicious effect on some mathematics educators seduced by the vamps of relativism.) There are many ways to encourage the individuality of pupils without permitting them to believe that black is white.

All of these ghosts are the traces of essential components of mathematical education in the past which must be part of the future as well. Children are going to either succeed or fail at any worthwhile task, and the question is how humanely the failure is handled and whether pupils are held back or advanced for frivolous reasons. Mathematics is a hierarchical subject, and we need to spell out what students need to master at each stage; the issue of the syllabus is one of design and focus. We cannot deny the need for practice; the question is whether the student has the strategies and perspective to learn and memorize efficiently and effectively. Arithmetic and the standard algorisms are as important as ever; we need to be sure that they are put in the proper context and conceptual framework.

These ghosts are accompanied by a number of sirens who drive a lot of educational reform. Like the ghosts, the sirens also speak to important aspects of the mathematics education. Here are some of them.

Siren 1: problem solving and investigation. This siren calls us away from the ghost of drill and of dry and unilluminating exercises. There is nothing wrong with wanting our children to solve problems and explore mathematical situations, but we can run into serious distortions if we do not take the trouble to ground children mathematically and psychologically.

Any problem we present to children should be carefully analyzed for its mathematical content and appropriateness. Strategic decisions need to be made as to what mathematics has to be presented beforehand as background and what can be brought out in the analysis of the problem. Let me give an example that I have used with students and prospective teachers.

Let ABCD be a unit square and let E and F be the respective midpoints of the sides BC and CD. The three line segments AE, AF and BF partition the square into five regions. It is required to determine the area of each region.

There is actually quite a lot in this. At what point should this example be introduced---before or after the pupils see the area formula (half-base-times-height) for triangles? This will govern how they might approach the problem. If the students are to try a structural rather than a formulaic approach, they might need some understanding of isometries and might need to understand that areas of nonoverlapping sets add, that areas are invariant under rigid transformations and that they vary as the square of the factor of a dilatation. Should some of this be discussed ahead of time, or can it emerge as the problem is covered? If the students exploit the similarity of the two subtriangles of ABE, they may need to know that AE is perpendicular to BF; what tools can they be expected to deploy? Will students who assign letters to the five regions have the necessary algebraic understanding to proceed, or is this a nice vehicle to introduce them to this approach? Finally, will they be able to negotiate the fractions? Are the fractions appropriately expressed in vulgar or in decimal form? Why? When we take all this into consideration, this example could take quite a bit of time to do properly---has this been anticipated by the teacher?

The problem-solving approach to the curriculum has a great deal to be said for it, but the teacher must be able to envisage what might happen and, importantly, what can possibly go wrong. It is risky, and we should be sure that teachers are equipped to accept the risks.

Siren 2: relevance; real life. Mathematics has been seen as alienating because of the artificiality of what we ask students to do. To counter this, we should make reference to students' daily lives and concern ourselves with what they will need to succeed in their later careers. But what passes for relevance often raises the question, "For whom?" Often children are subject to tedious arithmetic problems dressed up with clowns or are brought into the world of adult concerns in the name of relevance. Play is a part of the world of a child, and mathematics gives lots of opportunity for this. I have never been aware of any nice number, geometric, topological or combinational novelties being beyond the pale for the children I have taught.

Siren 3: patterns. No mathematician can gainsay the importance of a sense of structure in doing mathematics, and the ability to recognize pattern is an important part of this. What has been forgotten in a lot of modern educational reform is that the power of patterning is seen in its use in analyzing and generalizing mathematical situations. Much of what passes for patterning is a kind of teacher "guess-my-rule" game and rather ad hoc examples. There is no excuse for this. It is hard to progress through the curriculum without finding natural opportunities to exploit patterns, and teachers need to be alert to this and make them explicit to their pupils at the right time. One opportunity that seems to be neglected is giving students the mathematical voice to describe and analyze patterns.

Siren 4: data analysis. We do not have to look very far to realize how much of our daily lives is governed by a flood of data of one sort or another. So there is certainly a duty to help pupils become number-wise and to interpret what they read astutely. But the danger is that we do not just cover a lot of canned techniques and introduce jargon, which may in some cases stand in the way of good discussion and analysis.

Siren 5: technology. First, let me say that I agree that through modern technology we are undergoing changes that are at least as profound as those introduced by the invention of the printing press and the Industrial Revolution. But any revolution, no matter how pervasive, is not completely disengaged from the past. Many of the issues raised by technology are old ones, and the charge of misuse and mindlessness can be and has been leveled against Arabic numeration, algebraic symbolism, logarithms, slide rules, mechanical adding machines and numerical algorithms of all types whether executed on paper or on a bench with pebbles. Certainly, the modern computer has greatly expanded both the range of the mathematics we *can* do as well as the mathematics we *want to* do: our curriculum should reflect this. But we do not need idolatry. Technology is a part of our environment, as are books and pens, and a general purpose of education is to produce students who can understand and work within their environment. There are core mathematical issues that are independent of technology, but can be greatly informed by our use of technology—it is in this spirit that we should embrace the use of calculators and computers in our classrooms.

All of the sirens speak to the important aspects of the modern mathematics curriculum, but they all involve subtle issues and risk of being trivialized. The big question is whether we will have teachers in front of our children who will handle the issues in a sensitive, moderate and intelligent way.

In designing a curriculum, we need to keep both the ghosts and sirens in mind, for each of them not only speaks to important goals but also carries a warning of distortion and counterproductivity. There are a number of factors that we need to be cognizant of.

1. For a body of mathematics, I believe that there are three stages a learner must pass through: *initiation, formalization* and *consolidation*. In the stage of initiation, the learner encounters the ideas in a somewhat haphazard way, feeling her way about, exploring; there has to be some motivation, some reason that the person is interested in the material at hand. At the formalization stage, the ideas are drawn together and organized; at this point, the pupil should learn proper concepts, processes and conventions. There may indeed be a lot of artificiality; the payoff should be a growth in the knowledge and mathematical power of the learner. In the consolidation stage the learner reflects upon what has gone before, contextualizing it, detecting relationships and connecting it to other material. It may be only here that the learner truly appreciates the reasons for what she has been taught before. Traditional education has emphasized the second of these stages while modern practice seems to focus on the first and third without the buttress of the second stage. A good curriculum allows for all three stages to occur. The first signifies to the pupil that the subject matter *could* belong to them, the second provides the tools to *allow* it to belong to them and the third ensures that it *does* belong to them.

2. Have a strong focus and a slender core for each course in the curriculum. Make sure that context is established, the purpose of the material becomes clear, there is enough depth to support its assimilation and allow students to develop the necessary skills and understanding to proceed.

3. Have one or two attainable goals for each year; plan to achieve them and *move on*. This means that the sterile spiraling that now occurs in schooling should cease, but it does not preclude returning to previous material to inform and consolidate it.

4. Have enforceable entry requirements for secondary and tertiary courses. No teacher should be asked to deal with a student who does not have reasonable prerequisites for the material to be studied. There is an important pedagogical purpose in requiring students to review and mentally organize the work of several months or a year; it is this process that makes purposeful curriculum progress possible.

5. Change pace. The diversity of the mathematical enterprise should be reflected in our curriculum, whether it be with respect to subject matter, level of discourse or application. There are many ways in which people can think about or do mathematics. Therefore, we need space to help students find their mathematical voices and texture their ways of thinking about and doing mathematics.

6. Orchestrate the material. Increase the level of complexity judiciously, make sure that foundational material is covered and the student is psychologically prepared for what is to follow. We need to analyze in much more detail what students are asked to do; this is where members of the university community can be particularly helpful. Too often problems are thrown at pupils with little appreciation of what needs to be in place to begin to solve them. A sound curriculum needs exercises and problems of many different intensities.

This is one area in which the *Principles and Standards* seems to be particularly weak. There are a number of examples thrown in that, upon closer analysis, seem to involve a great deal than first meets the eye. For example, on page 92 is a set of instructions for constructing a golden rectangle. This seems as though it might be pretty heavy sledding for a typical student, and any teacher who embarks on this without careful thought is wading into a treacherous swamp. Why should pupils be interested in a golden section or in ways of constructing such a thing? Will most pupils muster the necessary level of concentration to negotiate the 10-stage set of instructions and understand why it works? This example is meant to illustrate the connectedness of mathematics, but it seems to me to require such a level of maturity and mastery of some basic algebra that it could easily spin out of control in the hands of any but the most adept teacher.

7. Meet different needs. Whatever reason we can give for teaching anything at all applies to mathematics. Some mathematics is taught so that students will be able to accept the privileges and responsibilities of citizenship, some to situate them in the rich culture they will inherit, some to provide recreation and additional options toward a full and rewarding life, and some for professional preparation. These needs can sometimes be met by the same piece of mathematics, but all should inform the curriculum that we set.

8. Foster sound practice and mental attitudes. How well students perform in mathematics seems to depend on their worldview. In designing a curriculum, we should encourage an attitude that involves the following:

- Awareness of structure; appreciation and exploitation of symmetry
- Flow of ideas, analysis and reasoning
- Corroborative quality of mathematics; inner consistency
- Shifting of perspective
- Checking and monitoring one's work
- Organizing and polishing one's work
- Mathematical register; expressing ideas appropriately; the place of heuristic and formalism
- Sense of context
- Attainment of power through understanding, reasoning and technical facility
- Visualization; appropriate imaging
- Appreciation of symbolic representation: numeration, algebra, diagrams
- Making distinctions: classification, equivalence, isomorphism, congruence
- Grasping the interplay between concrete and abstract; progression to higher order structures that in turn become concrete

9. Finally, we come to the subject matter itself. The centrality of arithmetic and algebra must be affirmed. No student can succeed at the secondary level without a good grasp of arithmetic, or succeed at the tertiary level without a good algebraic grounding. But these are not all. Geometry and combinatorics are also indispensable. While students should see how mathematics can be applied in different areas, it is not clear to me that this should necessarily occur in math class. Science, shop, civic, geography and music are areas in which important mathematical concepts can be conveyed in an appropriate and powerful setting. And we should not forget how many important mathematical concepts and processes underly some recreations and puzzles. In fact, if the elementary and secondary curricula are well designed, one could devote the middle school years largely to recreational and cultural issues as a way of consolidating what should have been learned at the elementary level and preparing the ground for a more sophisticated high school program in which symbolism, algorithm and reasoning have important roles.

I would like to close with two courses that might be given at the Grades 9 and 10 levels. The first is designed to give a systematic introduction to algebra and the second to geometry. It is assumed that students have gone through an initiation stage in both areas, that they have some familiarity with formulae and the use of letters to represent numerical quantities and that they have had the opportunity to play around with geometric objects either through tactile or computer models. While technology is not explicitly mentioned in either syllabus, it is understood that it can play a large and appropriate role. These syllabi would be accompanied by resources for the teacher that lay out alternative approaches as well as useful exercises, problems and investigations and that make explicit the pedagogical and mathematical goals that are brought to light through the material.

Course 1: Linear and Quadratic Functions

1. The linear equation $ax = b$. Problems that lead to an equation of this form.
2. The equation of the straight line; slope; various forms.
3. The solution of a system of two linear equations in two unknowns using numerical, algebraic and graphical techniques. Consistency of two linear equations.
4. Factoring difference of squares.
5. The quadratic: factoring over reals and integers, completion of the square, quadratic formula, discriminant and its significance; complex numbers (real quadratic having complex conjugate roots); relation of sum and product of roots to coefficients; first and second order differences; the factor and remainder theorem (for quadratics).
6. Graphing of quadratic functions; comparing the graphs of the functions $f(x)$, $f(x-c)$, $f(ax)$, $f(x)+c$; similarity of all graphs of quadratic functions (role of completion of square).

This is the entire prescription for a full-year course. In some classes, it may be necessary for the teacher to cover only this material and nothing else, spending the time and introducing whatever additional materials are necessary to ensure that students become fluent in the essential components. Normally, one would hope that the teacher would extend the material in some way, either through the introduction of applications, extended investigations and additional topics. It may be that the more able students in the class are given modules to work on independently, possibly with the support of additional pamphlets or computer software. I do not see much point in introducing algebra tiles to the class at large—they amount to an additional code interpolated between the student and the standard code—but they may be useful for individual students who have trouble getting an initial grasp of algebra.

Other topics that might be introduced are motion of projectiles, history of solving equations, linear and quadratic diophantine equations, Pell's equation, quadratic residues, first and second order recursions (characteristic equation), analysis of the dynamical system $x \to \lambda x(1-x)$, polynomials of degree exceeding 2; complex numbers (conjugates, modulus, geometric interpretation of sum, product and inverse, roots of unity up to the fourth); calculus of finite differences applied to polynomials interpolation and extrapolation; linear inequalities.

Course 2: Geometry of Triangles and Circles

1. Triangles: congruence theorems; ambiguous case; Pythagorean theorem; similarity.
2. Trigonometry: six standard ratios and basic identities ($\cos^2 + \sin^2 = 1$; $\sec^2 = 1 + \tan^2$); sine, cosine and tangents of angle sums and differences; double angle formulae; conversion formulae between sums and products; law of sines; law of cosines
3. Circles: subtended angles; cyclic quadrilaterals; tangent theorems
4. Coordinate geometry: circles; area of triangles; family of lines passing through pair of intersecting lines; circles passing through intersection of a pair of circles; radical axis; simple loci problems.

Again, the teacher would have the discretion of reinforcing these core topics for students encountering difficulty, amplifying the material through applications and investigations or moving to additional topics. These could include the following: complex numbers (use in deriving geometric and trigonometric results); vector geometry; statics; loci (conjectures, verification and proof); solid geometry; advanced Euclidean geometry; transformations and their uses (isometries and similarities; composition of isometries; isometry determined by three points; isometries generated by reflections).

The modern school, even at the secondary level, has to serve students all across the spectrum of intelligence, ability, motivation and interest, often in the same class. The only way the system can cope with this situation is that students are trained early to become autonomous in choosing their goals and more self-propelled in accessing the resources needed to learn what they need to know and in seeking validation for their knowledge. Any system of education predicated on the teacher as the sole source of knowledge and evaluation for the student is bound to fail; the more able and motivated will fall short of their potential and others will not be able to really engage with the material. Students must first think about what they value and how far they are willing and able to go in achieving their goals. A successful school regime depends on certain beliefs and characteristics of students as well as the knowledge and experience of teachers; unless we recognize the need to start with this, our principles and standards will become an exercise in merely trying to stave off the most trenchant critics from all sides and leave us seeing "men as trees walking."

Reference

National Council of Teachers of Mathematics. *Principles and Standards for School Mathematics: Discussion Draft.* Reston, Va.: Author, 1998.

Reprinted with permission from Canadian Mathematics Society Notes de la SMC, *Volume 31, Number 6, October 1999, pages 3–8, a CMS publication. Minor changes have been made to spelling and punctuation to fit ATA style.*

STUDENT CORNER

Mathematics as communication is an important curriculum standard, hence the mathematics curriculum emphasizes the continued development of language and symbolism to communicate mathematical ideas. Communication includes regular opportunities to discuss mathematical ideas and explain strategies and solutions using words, mathematical symbols, diagrams and graphs. While all students need extensive experience to express mathematical ideas orally and in writing, some students may have the desire—or should be encouraged by teachers—to publish their work in journals.

delta-K *invites students to share their work with others beyond their classroom. Such submissions could include, for example, papers on a particular mathematical topic, an elegant solution to a mathematical problem, posing interesting problems, an interesting discovery, a mathematical proof, a mathematical challenge, an alternative solution to a familiar problem, poetry about mathematics or anything that is deemed to be of mathematical interest.*

Teachers are encouraged to review students' work prior to submission. Please attach a dated statement that permission is granted to the Mathematics Council of the Alberta Teachers' Association to publish [insert title] in one of its issues of delta-K. *The student author must sign this statement (or the parents in the case of the student being under 18 years of age), indicate the student's grade level, and provide an address and telephone number.*

No submissions were received for this issue. We look forward to receiving your submissions for the next issue.

The Missing Digit

When the multiplication for 35! = 1 2 3 ... 34 35 is carried out, the result is a number with 41 digits. If the middle digit is replaced with a question mark, is it possible to determine what the digit is without carrying out the multiplication of the 35 factors?

35! = 10 333 147 966 386 144 929 ?66 651 337 523 200 000 000

NCTM Standards in Action

Klaus Puhlmann

The Standards recognize the importance of having all students develop an awareness of the concepts and processes of data analysis, statistics and probability. These concepts are more than just reading and interpreting graphs. Statistics and probability are important links to other content areas, such as social studies and science. They also can reinforce communication skills as the students discuss and write about their activities and their conclusions. Also within mathematics, these topics often involve the uses of number, measurement, estimation and problem solving.

The Standards for K–12 suggest that mathematics instructional programs should include attention to data analysis, statistics and probability so that all students

- pose questions and collect, organize, and represent data to answer these questions;
- interpret data using methods of exploratory data analysis;
- develop and evaluate inferences, predictions, and arguments that are based on data;
- understand and apply basic notions of chance and probability. (NCTM 1998)

Students in the early grades should be exposed to real objects that embody the characteristics to be studied. Activities such as comparing, sorting and counting are essential in order to develop the students' understanding of data and data analysis. Although the depth and variety of activities in relation to the above-mentioned standards differ greatly, the spirit of investigation and exploration is to permeate statistics instruction at all levels. Students' questions about the world around them can often be answered by collecting and analyzing data. As the students develop questions, they also decide what information to collect that will answer these questions. This process also involves evaluating the data collected, interpreting them and drawing conclusions from them. The focus areas for the primary grades include

- gathering data about themselves and their surroundings;
- sorting and classifying objects and organizing data according to attributes;
- representing data to convey results using concrete objects, pictures and numbers;
- describing parts of the data and the data as a whole;
- identifying parts of the data with special characteristics; and
- understanding notions such as *certain, impossible, more likely, less likely*.

Students need to understand and develop an appreciation that the overall purpose of collecting, organizing and representing data is to answer questions that are otherwise difficult to answer. Our teaching needs to build on the informal experiences that the primary students bring to this task. The students must eventually see that when the data are organized and represented, they often convey powerful information that either leads to a conclusion or more questions. This is also an excellent opportunity to engage students in discussions, allowing them to communicate their thoughts and understanding.

Making inferences and predictions is generally deferred to the upper elementary grades, as both require probabilistic thinking. However, answering questions about notions like *certain, impossible, more likely* and *less likely* begins at this level. Again, the treatment of these notions must be built on the intuitive understanding that these primary students bring to these discussions. Therefore, the theoretical treatment of the notion of probability is not appropriate at this grade level.

At the upper-elementary level, students continue to focus on the same standards, but the focus areas increase in variety and depth. Students at this level should

- formulate questions they want to investigate;
- design data investigations to address a question;
- collect data using observations, measurement, surveys, or experiments;
- organize data using tables and graphs (e.g., bar graph, line plot, stem-and-leaf plot, circle graph, and line graph);

- use graphs to analyze data and to present information to an audience;
- compare data representations to determine which aspects of the data they highlight or obscure;
- describe the shape and important features of a set of numerical data, including its range, where the data are concentrated or sparse, and whether there are outliers;
- describe the center of sets of numerical data, first informally, then using the median;
- classify and describe categorical data (e.g., ways we travel to school) in different ways; analyze and compare the information highlighted by different classifications;
- compare related data sets, with emphasis on the range, center, and how the data are distributed;
- propose and justify conclusions based on data;
- formulate questions or hypotheses based on initial data collection, and design further studies to explore them;
- describe how data collection methods can impact the nature of the data set;
- discuss the concept of representativeness of a sample within the context of a particular example (e.g., is the class representative of other fifth-grader classes in our town? In Alberta? In Canada? Why or why not?);
- compare the data from one sample to other samples and consider why there is variability;
- in simple experiments, infer the structure of the population through drawing repeated samples (with replacement);
- discuss events as likely or unlikely and give descriptions of the degree of likelihood in informal terms (e.g., unlikely, very unlikely, certain, impossible);
- estimate, describe, and test probabilities of outcomes by associating the degree of certainty with a value ranging from 0 to 1 (e.g., in simple experiments involving spinners with different fractions shaded). (NCTM 1998)

Students at this grade level must be engaged in activities that involve responding to questions about a variety of situations that interest them. They should also become familiar with a variety of representations and their appropriateness for different data and purposes. The focus at this level must also include looking at data in more than one way, comparing data and developing the idea of "typical" or average value. Furthermore, their experiences must also include comparing several, related data sets and the use of evidence as validation.

Toward the end of this grade level, students should also develop an understanding that data sets are samples of larger populations. At this point, too, they are beginning to experiment with probability in situations with a few outcomes, and they develop the language to discuss their informal notions of probability.

Students at the middle grade level continue to build on their previous experiences, but they are now engaging in the full process of data investigation: posing questions, collecting data, organizing data, analyzing data, interpreting data and answering questions. The students now draw on their knowledge of ratios, fractions, decimals, percent, graphs and measurement as they engage in data analysis.

Furthermore, they develop and extend their understanding of ideas that are central to the study of statistics, such as data distribution, central tendency and variance. New representational forms are being added as well (for example, box-and-whisker plots) to allow more complex considerations of data distributions. Students are also engaged in the analysis of scatter plots for related variables, allowing them for the first time to develop linear approximations (line of best fit) for the plots. With respect to the following focus areas the students should

- design experiments and surveys, and consider potential sources of bias in design and data collection;
- recognize types of data (for example, categorical, count, continuous or measurement) and organize collections of data;
- choose, create and use various graphical representations of data (line plots, bar graphs, stem-and-leaf plots, histograms, scatter plots, circle graphs and box-and-whisker plots) appropriately and effectively;
- find, describe and interpret mean, median and mode as measure of the centre of a data set; know which measure is best to use in particular situations; and understand how each does and does not represent the data;
- describe and interpret the spread of a set of data using tools such as range, interquarterly range and box-and-whiskers graphs;
- interpret graphical representations of data, including description and discussion of the meaning of the shape and features of the graph, such as symmetry, skewness and outliers;
- analyze associations between variables by comparing the centres, spreads and graphical representations of related data sets;
- examine and interpret relationships between two variables using tools such as scatter plots and approximate lines of best fit;
- develop conclusions about a characteristic in the population from a well-constructed sample;
- through simulations, develop an understanding about when differences in data may indicate an

actual difference in the populations from which the data were collected and when the differences may result from natural variation in samples;

- use data to answer the questions that were posed, understand the limitations of those answers and pose new questions that arise from the data;
- make judgments about the likelihood of uncertain events and be able to connect those judgments to percents of proportions;
- understand what it means for events to be equally likely and for a game or process to be fair;
- compute simple probabilities using appropriate methods, such as lists, tree diagrams or area models;
- identify complementary, mutually exclusive, independent and dependent events and understand how these relationships affect the determination of probabilities.

During the final four years of schooling, the complexity and difficulty levels are further enhanced. While the standards continue to be the same, the focus areas are further increasing in depth and breadth. More specifically, students at this level should

- design and carry out appropriate methods for gathering univariate data, both to study the distribution of a variable in one population and to compare the distributions of the same variable in two different populations;
- design appropriate methods for collecting, recording and organizing data to obtain bivariate data to study the association between two variables;
- select appropriate graphical representations and numerical summaries of data;
- understand how a change in a representation (for example, scales on a scatterplot, categories in a two-way table and bin size of a histogram) affects the information it conveys;
- use calculators and computer applications (for example, spreadsheets, simulation software and statistical software) appropriately to assist in data collection, organization and representation;
- compute, identify and interpret measures of centre and spread (for example, range, variance and standard deviation, and interquartile range);
- describe shapes of one- and two-dimensional data sets;
- look for symmetry and skewness, clusters and gaps, and possible outliers in data and consider their effects on the interpretation of the data;
- recognize how sample size or transformations of data affect shape, centre and spread;
- use a variety of representations of data, including scatterplots, frequency distributions and two-way tables;
- be able to recognize trends in bivariate data, visually and numerically, and use technology to determine how well different models (for example, linear. exponential, and quadratic) fit data, while understanding that a perfect fit is unlikely for empirical data;
- understand the elements involved in finding good models for phenomena;
- apply well-fitting models to predict unobserved outcomes;
- evaluate conclusions based on data;
- use data from samples to estimate population statistics;
- use and interpret the normal and binomial distributions appropriately;
- understand and compute probabilities of independent, disjoint and conditional events;
- understand that some phenomena are random and apply and the law of large numbers to predict long-term behavior;
- use probability distributions to compute probabilities of events.

The emphasis at this level has shifted from univariate data to binomial data. Students will also deal with the concepts of randomness and chance in greater depth. Their increased mathematical knowledge will also allow them to formalize their understanding of the relationships among the graph of a distribution, measures of shape, centre and spread. Linear approximations are now including finding regression curves, and technology is used more extensively to study data, illustrate concepts, perform calculations, create representations and provide data from situations.

Students at this level will have studied probability distributions, binomial and normal distributions, designing surveys and experiments, the influence of sample size on the closeness of the estimate of a population statistic and drawing inferences about a population.

The three articles that follow provide two examples of the level of student understanding relative to certain statistical concepts and one example of how a real-life situation can be used to capture students' interest in statistics.

Bibliography

Alberta Education. *The Common Curriculum Framework for K–12 Mathematics*. Edmonton: Author, 1985.

National Council of Teachers of Mathematics (NCTM). *Curriculum and Evaluation Standards for Teaching School Mathematics*. Reston, Va.: NCTM, 1989.

——. *Principles and Standards for School Mathematics: Discussion Draft. NCTM Standards 2000*. Reston, Va.: Author, 1998.

How Do Students Think About Statistical Sampling Before Instruction?

Victoria R. Jacobs

How would your students respond to the Raffle Scenario in Figure 1? What information about your students' knowledge would help you plan instruction for statistical issues related to the Raffle Scenario? This article highlights students' thinking and instructional implications from two studies that examined upper-elementary students' written and oral explanations as they responded to survey situations. Specifically, students were asked to reason about statistical issues related to how the participants were selected—the *sampling method*—for the following:

- *Random* sampling methods that gave each member of the population the same chance of being selected (for example, survey 1 in the Raffle Scenario) or each member within subgroups of the population having the same chance of being selected (for example, survey 2).
- *Restricted* sampling methods that asked particular groups of people who might be more likely to select a certain response and, consequently, bias the results in a particular direction, for example, surveys 3 and 4 in the Raffle Scenario.
- *Self-selected* sampling methods that had the participants select themselves and, consequently, might bias the results because individuals who choose to participate in a survey often have opinions different from the opinions of people who choose not to participate, for example, surveys 5 and 6 in the Raffle Scenario.

None of the students had formally studied sampling, but my goal was not to instruct the students on sampling issues. Rather, I wanted to find out what they already knew from their experiences outside of school—their informal knowledge.

Interviewer: What is a sample?

Melanie: A piece of something whole—it's like a peek.

Georgia: A piece of food or carpet [that] gives you an idea of what the real thing is.

Figure 1. Raffle Scenario (adapted from Schwartz et al. [1994]) for Examining Sampling Issues with Surveys

The Raffle Scenario

A Grade 5 class wants to raise money for a field trip to an amusement park. Students are considering several options, including selling raffle tickets for a videogame system. As part of their efforts to determine how to raise the most money, students conducted different surveys to estimate how many students in the whole school would buy a raffle ticket to win this prize. For each survey, 60 students were asked their opinion. The school consisted of 600 students in Grades 1–6 with 100 students in each grade. The surveys and their results follow:

1. Shannon got the names of all 600 kids in the school, put them in a hat and pulled out 60 of them. (Thirty-five percent said that they would buy tickets.)
2. Kyle put the names of all the first-grade boys in one hat and all the first-grade girls in another hat. He pulled out [the names of] five boys and five girls from each hat. He did the same thing for each grade until he had [the names of] five boys and five girls from each grade. (Thirty percent said that they would buy tickets.)
3. Raffi asked 60 of his friends. (Seventy-five percent said that they would buy tickets.)
4. Jake asked 60 kids at an after-school meeting of the games club. (Ninety percent said they would buy tickets.)
5. Claire set up a booth outside the lunchroom. Anyone who wanted to could stop by and fill out her survey. She stopped collecting surveys when she got 60 completed. (Ninety-five percent said that they would buy tickets.)
6. Abby sent out a questionnaire to every kid in the school and then used the first 60 that were returned to her. (Eighty-five percent said that they would buy tickets.)

What do you think about each of these survey methods? Good? Bad? Why?

What do you think is the best estimate of the percentage of kids in the whole school who will buy a raffle ticket?

Despite a lack of formal instruction, these students included some of the elements of statistical samples in their definitions: (a) the sample is part of the whole and (b) the smaller part gives an idea of the whole. This partial understanding could be a good starting point for instruction; teachers can build instruction on the basis of students' informal knowledge. This approach has proved successful in many areas of mathematics (Hiebert and Carpenter 1992). For instance, teachers have used students' experiences with fair sharing in out-of-school contexts, such as sharing four cookies with a sibling, as the basis for instruction on division with children as young as Kindergartners (Carpenter et al. 1993). Similarly, middle school teachers can benefit from understanding how upper-elementary students like Melanie and Georgia think about sampling issues before they receive formal instruction.

This article identifies students' informal knowledge of sampling by describing the major categories of their responses in two studies that asked them to evaluate individual surveys and then to draw conclusions from multiple surveys with conflicting results. In the first study, I interviewed 17 students to begin categorizing their conceptions of sampling. On the basis of a written assessment about sampling in surveys. I selected these students from 31 fourth graders and 32 fifth-graders in three multiage, fourth- and fifth-grade classrooms to include a range of understanding. In the second study, I developed a series of written activities from the categories of sampling conceptions identified in the first study. See Figures 2 and 3 for examples of these written activities. I asked 110 students to complete these written activities to confirm the categories identified in the first study and to determine the prevalence of the categories in a larger sample. These students came from eight fifth-grade classrooms in three elementary schools that were chosen to reflect the minority population—31.6 percent—and income distribution—29.2 percent of students receiving free and reduced-cost lunch—of this medium-sized city in Wisconsin.

Figure 2. Written Activity for Assessing Students' Abilities to Draw Conclusions from Multiple Surveys

	Will Buy Raffle Tickets	Will Not Buy Raffle Tickets
Shannon pulled 60 names out of a hat.	35%	65%
Claire set up a booth to collect 60 surveys.	95%	5%
Jake asked 60 kids in the games club.	90%	10%

What percentage of students in the whole school will buy a raffle ticket?

Here are some ideas that other students had. Circle any of the ideas that you agree with. Put a star next to the idea that you agree with most. If you do not agree with any of the ideas, circle the last choice and explain what you think.

A. I thought that Shannon's survey was the only one that was done well, so I ignored the other two surveys and used Shannon's results. She found that 35 percent said that they would buy a raffle ticket.

B. I thought that Claire's survey was the only one that was done well, so I ignored that other two surveys and used Claire's results. She found that 95 percent said that they would buy a raffle ticket.

C. I thought that Jake's survey was the only one that was done well, so I ignored the other two surveys and used Jake's results. He found that 90 percent said that they would buy a raffle ticket.

D. I just knew that most kids like video games and would buy a raffle ticket, so I picked a high percent. ____ percent of kids will buy a raffle ticket.

E. I just knew that most kids would not buy a raffle ticket, so I picked a low percent. ____ percent of kids will buy a raffle ticket.

F. I took the average of the three surveys. The average of the kids who said that they would buy a raffle ticket is 73 percent.

G. I don't know because they got different results.

H. I think ____ percent of kids in the whole school are willing to buy a raffle ticket because ____.

Throughout this article, I illustrate categories with students' responses to the Raffle Scenario from interviews in the first study. I also indicate the prevalence of each response category by reporting percentages of the 110 students in the second study who used each response category in their evaluations of nine surveys. Percentages are not reported for the first study because they would not be meaningful. Each interview was individualized to investigate fully each student's thinking; therefore, different students had different opportunities to give particular responses.

Evaluating Individual Surveys

Each student evaluated the quality of individual surveys in scenarios like the Raffle Scenario, adapted from Schwartz and colleagues (1994). Other scenarios included statewide reporting on recycling programs, choosing classroom pets and identifying favorite lunchroom items. Some students showed sound reasoning by basing their evaluations on the potential of the sampling method for producing a biased sample—a group of people who would be likely to produce results that were not reflective of the population. Other students showed more problematic reasoning by basing their evaluations on other issues. On average across nine surveys, 50 percent of the students used sound reasoning; 47 percent of them used problematic reasoning; and 3 percent used reasoning that was missing, unique or unclassifiable. All students used multiple types of rationales when evaluating different surveys instead of sticking with a favorite rationale.

Sound Reasoning

Some students evaluated sampling methods by focusing on just what we would want them to consider: the quality of the sample and the potential for bias. This sound reasoning based on the potential for bias led students to make both accurate and inaccurate evaluations of surveys.

Accurate Evaluations

On average, 34 percent of the students used sound reasoning that led to accurate evaluations of survey quality. It is important to note that this sophisticated way of evaluating sampling methods was not restricted to a small group of "smart" students. Ninety-four percent of the students used this approach at least once when evaluating the nine surveys. With this approach, students positively evaluated random sampling on the basis of their tendency to produce unbiased samples, and they negatively evaluated restricted and self-selected sampling methods on the basis of their tendency to produce biased samples. For example, students negatively evaluated restricted sampling methods if they recognized that these methods were problematic because they were likely to produce samples in which everyone has the same opinion. In the Raffle Scenario, one student negatively evaluated the restricted sampling method of selecting only friends because—

> friends a lot of times are friends because they have the same opinions ... so a lot of his friends are going to like one thing or the other. And it seems to me they mostly like getting raffle tickets.

Similarly, students negatively evaluated self-selected sampling methods when they recognized that these methods were problematic because the individuals who choose to complete surveys are likely to have opinions different from those people who do not choose to participate. In contrast, students positively evaluated random sampling methods if they were able to recognize the potential for producing an unbiased sample. For instance, one student positively evaluated the stratified random sampling method of selecting five boys and five girls from each grade because—

> that way he has a mixture of boys and girls and who are different ages ... because sometimes girls and boys can have different opinions on things and also one age might really like something, but an older age might think that was a terrible idea.

Inaccurate Evaluations

On average, 16 percent of the students used basically sound reasoning, measured against the potential for bias, but this reasoning was applied inappropriately and consequently led to inaccurate evaluations of survey quality. For example, some students positively evaluated self-selected sampling methods. They incorrectly assumed that these methods would produce a good mixture of respondents because no sample restrictions were specified, such as selecting only girls or friends. The following student positively evaluated the self-selected sampling method of sending a questionnaire to every student in the school and then using the first 60 returned:

> I think Abby's was a good idea because ... she got a variety of people ... because you weren't just giving them to a couple people and then giving it back to you. You were asking every kid, and whoever wanted to return it, could.

Striving for a mixture is sound reasoning, but a mixture does not usually result from a self-selected sampling method.

Similarly, some students negatively evaluated random sampling methods when the mixture was not clearly specified, for example, putting names in a hat. They did not like the uncertainty as to who would be selected. One student suggested that with simple random sampling,

> you could get like all your friends, or all girls, or all boys, like all in the first grade or something and everybody else has different opinions.

Avoiding restrictions, such as all girls, is sound reasoning, but the likelihood of random sampling's producing a sample with all girls is very low. However these students seemed to focus on the *possibility* of extreme outcomes without realizing that the *probability* of their occurrence was low.

Problematic Reasoning

In contrast to the foregoing responses, some students in their evaluations showed more problematic reasoning by focusing on issues other than the potential for bias. These students based their evaluations on fairness issues, practical issues, results or all three.

Fairness Rationales

On average, 23 percent of the students inappropriately evaluated sampling methods on the basis of whether they were fair. However these students were not thinking of fair in the statistical sense of whether the sample would be fair, that is, whether everyone has an equal chance of being selected so that the sample is not biased. Rather, they were concerned about how people *feel* when they are selected or not selected to participate in the survey. Students using fairness rationales assumed that everyone wanted to participate in the surveys. To be fair, in an equitable sense, they believed that everyone should have the chance to participate.

Sometimes fairness rationales, and their problematic reasoning, led to accurate evaluations. For example, some students appropriately evaluated a restricted sampling method negatively, but their reasoning was based on the idea that the people who were left out would feel bad, not that the responses would be restricted and potentially biased by the people they selected. The following student negatively evaluated the restricted sampling method of selecting only friends because—

> that still wouldn't be fair. Because some people don't know him ... and they would say, "Hey, but this person told me that you picked them and not me. How come?"

In other situations, fairness rationales and their problematic reasoning led to inaccurate evaluations, especially in self-selected sampling methods in which everyone initially has a chance to participate. For example, one student positively evaluated the self-selected sampling method of setting up a booth for volunteers to complete the survey because—

> the people will choose if they want to. ... Like if they wanted to do the survey, they will, but if they would not want to, they don't have to—so they're not pressuring anybody.

For these students, the fact that everyone had a chance to participate—the students' idea of fairness—was more important than the fact that, using the self-selected method, people with particular opinions were more likely to participate than others.

Practical Rationales

On average, 12 percent of the students inappropriately evaluated sampling methods on the basis of whether actually conducting the survey would be practical. For example, was the sampling method efficient, easy to implement, confusing or even possible? This reasoning is not sufficient for evaluating the quality of sampling methods. In addition, students were not always accurate in their evaluations of which sampling methods were, in fact, practical. They would even sometimes suggest asking *everyone* instead of taking a sample, because they drastically underestimated the difficulties of asking everyone in large surveys, such as surveys of entire states.

Results-Based Rationales

On average, 12 percent of the students inappropriately evaluated sampling methods on the basis of whether the results were decisive, the results of the survey corresponded with their expectations or both.

First, some students based their evaluations on the decisiveness of the results. They believed that a survey with a completely decisive result—such as 100 percent of the students will buy raffle tickets—was more useful than a survey with indecisive results—such as 50 percent will buy tickets. These students concluded that a sampling method producing a 100 percent result was of a higher quality than a sampling method producing a 50-50 split because, as one student suggested, "50-50's not going to decide for you."

Second, some students based their evaluations on whether the results of the survey corresponded with their expectations of what would actually occur in the real world. When the results corresponded with their expectations, these students evaluated the sampling method positively. Conversely, when the results did not correspond, they concluded that the sampling method was inappropriate. This finding is consistent with adult research that has found that we are more critical of ideas that are not consistent with our own (Lord, Ross and Lepper 1979). It is important to note that some students were able to separate their own evaluations of sampling methods from their own opinions *and* to articulate that separation. For example, one student commented on the stratified random sampling method of selecting five boys and five girls from each grade:

> I think that the way they picked the same number of boys and girls in each grade was a good way.... I don't know about those results, though.... They don't seem to really match what most of the kids I know would think.

Drawing Conclusions from Multiple Surveys

After evaluating the quality of individual surveys in scenarios like the Raffle Scenario, each student was asked to draw conclusions from these multiple surveys with conflicting results. See Figure 2 for an example of this type of activity. On average, 64 percent of the students drew their conclusions on the basis of survey results, whereas 31 percent did not. Five percent of the students' responses were missing, unique or unclassifiable.

Conclusions Based on Survey Results

On average, 20 percent of the students used survey results in a way that we would advocate. They initially evaluated the quality of the surveys and then used information only from the survey or surveys that they thought were done well (options A–C in Figure 2). For example, if students believed that Claire's and Jake's surveys were done poorly, they ignored those surveys and drew their conclusions from Shannon's survey (option A). Some students—on average, 44 percent—aggregated *all* the available information regardless of the quality of the surveys. In Figure 2, these students looked at all three surveys and computed a total, or average, percent of students who would buy a ticket (option F). This approach was particularly disappointing when students showed that they were capable of accurately evaluating individual sampling methods but then did not use this information when drawing conclusions from multiple surveys. Rather they aggregated all the information—often immediately after they had identified potential problems with some of the individual sampling methods.

Conclusions Not Based on Survey Results

On average, 31 percent of the students did not use the survey results to draw conclusions. Instead, some of them based their conclusions on other information, such as personal experiences or opinions (options D and E in Figure 2). Others essentially refused to draw conclusions from either their own experiences or survey results (option G in Figure 2), in particular, if the answer was not clear cut—for example, if one survey's results suggested that more students would buy a raffle ticket, whereas another survey's results suggested that more students would not buy a raffle ticket. As one student stated, "I can't think of what I would do if there were two different answers." Sometimes these students suggested conducting additional surveys to add support to one conclusion or another. At other times, they delegated the decision making to other individuals, such as authority figures like principals.

Assessing Students' Thinking

Teachers can most accurately assess their students' knowledge through oral questioning, such as interviewing, but this approach may be excessively time-consuming. Alternatively, teachers may want to use written activities similar to those used for the second study. Figure 2 is an example of a written activity to assess students' abilities to draw conclusions from multiple surveys. Figure 3 offers an example of a written activity to assess students' abilities to evaluate individual sampling methods. I developed both activities to correspond with the categories of students' thinking described in this article.

In Figure 3, option A reflects a problematic evaluation based on practical issues; option B, a well-reasoned evaluation based on the potential for bias; option C, a problematic evaluation based on results; and option D, a problematic evaluation based on fairness issues. Option E provides the opportunity for students to express an opinion in their own words. Given the time-consuming nature of interviews and the tendency to respond minimally to open-ended questions, this type of format may be a practical compromise to help teachers gain an initial picture of their students' understanding or identify which students' thinking they might need to explore in more depth. Whatever methods teachers choose to assess their students' thinking, they must probe for students' reasoning behind their answers.

Building Instruction on Students' Thinking

I found that even without formal instruction, upper-elementary students have substantial informal knowledge about sampling. Although this knowledge is not always complete, it can be a starting point for instruction. For example, instruction could begin with the basic premise of the fairness rationale: everyone should have an equal opportunity to participate. Instruction would then lead students to the realization that the importance of equal opportunity is to minimize the potential for bias in the resulting sample rather than to minimize the negative feelings of the nonparticipants.

Similarly, some students positively evaluated self-selected sampling methods because they inappropriately based their decisions on the idea that without stated restrictions, sampling methods are of high quality because they result in a mixture of people. Instruction should help these students retain this basically sound reasoning but recognize situations that require special consideration. For example, with self-selected sampling methods, everyone does not begin with the same probability of participating; therefore, even without stated restrictions, the resulting samples often do not include a mixture of people.

In addition to identifying students' specific conceptions, these studies suggest two considerations for designing instructional activities. First, teachers should give students opportunities to make decisions from the surveys. Students need practice in reasoning through decision making from surveys. It was not enough for students to evaluate individual surveys effectively, because many ignored their evaluations when drawing conclusions and making decisions. They often chose to aggregate the information from all surveys regardless of survey quality—even when they had already identified problems in some sampling methods.

Second, teachers should use surveys based in multiple situations. Students need experiences with situations both in and out of school. School situations are effective because they are familiar and interesting to students. However, out-of-school situations present more of a reason to sample because they have larger populations, which underscores the difficulty of asking everyone. It is important to note that a few students consistently evaluated sampling methods poorly because the students preferred to ask *everyone* rather than take a sample. However, students were less likely to insist on asking everyone if the survey referred to a situation that took place outside a school context. Experience with multiple situations should also help students base their evaluations on survey results rather than rely solely on personal experiences.

Figure 3. Written Activity for Assessing Students' Abilities to Evaluate Individual Sampling Methods

Raffi asked 60 of his friends, and he found that 75 percent said that they would buy raffle tickets and 25 percent said that they would not buy raffle tickets.

What do you think of Raffi's survey?

(Circle one) good bad I'm not sure

Here are some ideas that other kids had. Circle any of the ideas that you agree with. Put a star next to the idea that you agree with most. If you don't agree with any of the ideas, circle the last choice and explain what you think.

A. I made my decision because it was easy to do. He just had to ask people he already knew.
B. I made my decision because his friends probably agree with him. So the survey doesn't tell you how the people who are not friends with Raffi think.
C. I made my decision because most of the kids said they would buy raffle tickets.
D. I made my decision because it's not nice to the people who are not his friends. They want to answer the survey too, but they aren't allowed.
E. I made my decision because ______________________________

____

______________________________.

Conclusions

These studies suggest that students may be entering middle school with surprisingly rich informal knowledge about sampling that should be useful as a starting point for instruction. Therefore, although statistics instruction has traditionally been delayed until postsecondary education, my research convinced me that this delay is unnecessary. These results support the NCTM's (1989) recommendations that Grades 5–8 are appropriate times to introduce students to inferential statistics, of which sampling is a central component. This article identifies students' substantial, although incomplete, informal understanding of sampling in the transitional, upper-elementary grades, thereby underscoring the potential for instruction in middle school.

References

Carpenter, T., et al. "Models of Problem Solving: A Study of Kindergarten Children's Problem-Solving Processes." *Journal for Research in Mathematics Education* 24 (November 1993): 428–41.

Hiebert, J., and T. P. Carpenter. "Learning and Teaching with Understanding." *Handbook of Research on Mathematics Teaching and Learning*, edited by D. A. Grouws, 65–97. New York: Macmillan, 1992.

Lord, C. G., L. Ross and M. R. Lepper. "Biased Assimilation and Attitude Polarization: The Effects of Prior Theories on Subsequently Considered Evidence." *Journal of Personality and Social Psychology* 37 (1979): 2,098–109.

National Council of Teachers of Mathematics (NCTM). *Curriculum and Evaluation Standards for School Mathematics.* Reston, Va.: NCTM, 1989.

Schwartz, D., et al. "Adolescent Understandings of Sampling in the Context of a Survey." Paper presented at the annual meeting of the American Educational Research Association, New Orleans, La., April 1994.

Reprinted with permission from Mathematics Teaching in the Middle School. *Volume 5, Number 4 (December 1999), pages 240–46, 263, an NCTM publication. Minor changes have been made to spelling and punctuation to fit ATA style.*

Multiplication Against Time

How is it possible to carry out the following multiplication in less than a minute?

81 624 324 048 566 472 808 896 × 12.5 =

Reflecting on Students' Understanding of Data

Kay McClain

Ongoing discussions about students' experiences with data analysis include debates about the role of graphical representations in supporting students' understandings (Shaughnessy 1992; Lehrer and Romberg, in press; Hancock, Kaput and Goldsmith 1992). More specifically, do students first need to know how to construct various types of graphs before they can engage in an analysis of data, or can they learn to construct various types of graphs by engaging in data analysis? Further, can they engage in data analysis before they have acquired a conceptual understanding of the multiple forms of data representation? These discussions also include a debate over the role of representative values, such as the mean, mode and median (Mokros and Russell 1995). What are the students' understandings of representative values? How should representative values be introduced? What role does students' understanding of representative values play in their ability to analyze sets of data?

This article is intended to address these issues by discussing what happened as I observed several groups of Grade 7 students working on performance tasks designed to assess their understanding of both graphical representations and representative values. During my interactions with these students, their activity made me reflect on my then-current beliefs about what it means to know and do "data analysis" in the middle grades. In particular, as I observed students attempting to find a way to "represent" a data set by debating the advantages and disadvantages of the graph as a visual, single-glance impression of the entire data set versus the mean as a single-number numerical summary, I began to rethink what might be involved in an instructional sequence that addresses these concepts. By taking an in-depth look at the way that students reasoned about representing a set of data and conveying my reflections on their processes, this article highlights the importance of teachers' supporting students' development of conceptual understandings of multiple forms of data representation and representative values in the context of ongoing data analysis.

Classroom Episode

A shift in the way I thought about students' experiences with data occurred as I was working with Grade 7 students on several performance-assessment tasks. The tasks were intended to yield information about how the students reasoned about a set of data and were designed to be accomplished in group settings. Specifically, I wanted to try to understand how the students might go about organizing and subsequently representing a set of data to generate a summary of the information. This preassessment information would, in turn, be used to guide the research team's decisions about instructional tasks that would be introduced in a unit on exploratory analysis. As a result of the need to clearly understand students' interpretation of and reasoning about the task, two of the group discussions and the subsequent whole class discussion were videotaped. Viewing the tape gave me the opportunity to analyze the students' ways of reasoning beyond what I observed as I monitored the different groups.

One task asked students to summarize the results of a hypothetical survey to create a report for the principal and parents. The survey results included the number of hours of television that 30 Grade 7 students watched in one week. The task is shown in Figure 1. In anticipating how the students might begin to reason about organizing the data, it seemed obvious to me that a histogram would be a clear and concise way to represent the information. For me, the histogram would preserve the variation in the data set while simultaneously giving a holistic impression of the trends and patterns within the data set without the need to digest each piece. The use of the mean for this particular set of data would appear to oversimplify the information, eliminating the nature of the variation. However, as this task was not designed for preassessment purposes, I was not sure how ready-to-hand the use of histograms was for the students. Further, my assessment of their performance would not be based solely on whether they made a histogram and made it correctly but would focus more on how they reasoned about organizing and representing the data. My colleagues and I needed this type of information to guide the design of the instructional tasks for the unit on exploratory data analysis.

As students began working in their groups, I walked around the classroom to try to monitor their activity and begin to understand how they were reasoning. Several groups began by finding the mean of the data set. One group asked for the calculators, and it could be argued that their request triggered a request from the other groups. Nonetheless, several groups began to methodically calculate the mean of the 30 responses.

As I continued monitoring the groups, I noticed that of the groups that found the mean, some of them subsequently rejected it as inappropriate. They reasoned that just reporting a mean of 10.56 for this set of data was insufficient, arguing that just one number did not provide enough information for this particular data set because "a bunch of the numbers were way above and a bunch of the numbers were way below" the mean. They then proceeded to discuss how to make a graph that would better represent the data, keeping all the features visible. However, for one group, the use of the mean became an intense topic of discussion. In particular, Amee and Latisha argued that you cannot use the mean, whereas Tony insisted that the assignment was to tell about the 30 students "all together," which was exactly what the mean did. Tony seemed to recognize the need to take the variation into account, and he thought that the mean, as an arithmetic average, did just that.

Amee: You cannot average it out, Tony.
Latisha: You cannot average it.
Tony: If I want to, I can.
Latisha: Listen, Tony . . . .
Tony: [Interrupts] They said all together, *all together* now.

Latisha and Amee then tried to explain to Tony that some of the students in the survey only watched 1.5 hours, so the average is "way off."

Amee: Tony, when you average it out, it is supposed to come somewhere close to [1.5].
Tony: It's not supposed to come out close to 1.5.
Amee: But you cannot do [the average].
Tony: I can. You really can, it just might not be accurate.
Amee: It's way off, it's *very way off.* It's so off, you cannot use the answer.
Tony: Yes, I can.
Amee: Well, you use the answer, but I'm not going to use that answer.

I found it interesting to note that this discussion appeared to focus on whether you could actually use the mean, not whether it would provide a clear representation in this particular task. I interpreted the students' discussion as suggesting that in certain instances you can use the mean and in others it simply does not work. However, their inability to clarify their true understandings was problematic for me. As a result, I intervened in the conversation to help clarify my understanding of their reasoning.

Teacher: Let her give her argument, and then you can give yours.
Latisha: Tony, when I averaged it, it came out to 10. This person here, this is 23 hours.
Tony: So? They said all together, *all together*, not just one person by hisself [*sic*].
Teacher: [Latisha's] agreeing with you that [10.56] is the average. What she is saying is that she doesn't think it is a good way to tell the story of these numbers because there is so much variation here. Because if you say 10, then people might think that everybody watches 10. Some don't even watch any.
Nathan: See, you don't watch 10, you watch about 20.
Tony: So, y'all are just picturing one guy by himself.
Amee: That's what you are doing.
Tony: No, I'm not.

Figure 1. Survey Data Given to Students

How Much Television?

Below are the results of a survey taken of 30 Grade 7 students to find out how many hours of television they watch in a week. The principal has asked you to summarize and represent these data in some form so that parents will be able to understand them quickly when they are posted on the bulletin board. The principal also asks you to write a short report for parents, explaining what the data show.

1.5	21	12.5
0	2.5	15
23	19	4
14	8	16
13.5	16.5	6
4.5	9	18
5	10.5	8.5
6	3	9
11.5	3.5	19.5
13	10	9

At this point, I was still unclear whether Nathan, Latisha and Amee thought that the variability in the data set resulted in the mean's inability to give a complete report of the situation in this particular instance or whether they thought that since some of the data points differed so markedly from the mean that therefore it did not work. I clearly needed to understand those two very different interpretations of the mean to plan how to proceed with further instructional tasks.

As a result of the groups's ability to reconcile their differing interpretations, Tony decided to use the mean and others in the group decided to make a graph. They began by discussing the possibility of grouping the data. Initially they discussed grouping the data from 0 to 10, from 10 to 20, and from 20 to 30. However, Nathan pointed out that the data did not go to 30, so they charged the upper bound to 25. I found this decision intriguing, as the largest data point was not 25 but 23, and changing the upper bound from 30 to 25 would in no way affect the height of the bar. It would, however, create unequal data intervals. The fact that their intervals were not of the same size was not problematic for the students, nor was it represented in the width of the bars in their graph (see Figure 2 for the final version). However, since I was anticipating students' decision that this set of data could best be represented by a histogram, the issue of inconsistent intervals was problematic for me. Further, I was not clear whether the students were making a modified histogram or simply grouping the data points into categories that they named with numeric intervals.

I chose not to intervene in their discussion other than to ask clarifying questions. At this point, I hoped that the issue of inconsistent intervals would arise during whole-class discussions. I judged that it would be much more productive for all the students to be engaged in a discussion about this issue than for me simply to "correct" their mistake. My telling the students would not constitute a basis for their understanding.

As the groups finished their investigations, I asked each group to come to the chalkboard and present the results of their work. The first group to report was Latisha, Nathan and Amee. When they finished, other students in the class appeared to accept their graph as a reasonable way to proceed and a very significant way to represent the data. Interestingly, in creating the graph, they modified their intervals so that they now ranged from 0 to 10, 11 to 20, and 21 to 25. This change occurred as they were placing data into three categories. When I asked where they placed 10.5, they explained that they had rounded each data point to the nearest whole number.

At this juncture, I did not point out the problem with rounding the data and the effect that it has on the representation. Further, I did not see it as my role to ensure that the students made the graphs "correctly" or used the representation that I had envisioned. Instead, I was much more interested in ascertaining what the students did know and thinking about how to use their current understandings as building blocks for later lessons. If these issues did not emerge as problematic for the students, I would be forced to impose conventions, such as how to "correctly" make the graph. Had I simply corrected what I perceived as a mistake, subsequent activities might have caused the students to focus on guessing what I wanted instead of allowing them the opportunity to reason independently. Further, I was still unclear whether the students viewed the data along a continuum of values or had simply categorized the data points. My imposing a continuous scale, as was necessary for the histogram, would be very problematic if the students, at this point, were not reasoning in that manner. For me, a better alternative was to use my knowledge of their current understandings to try to structure subsequent tasks that would make these issues the focus of the investigation.

The second group to share its results drew what I would classify as a modified histogram (see Figure 3).

Figure 2. Group 1's Graph

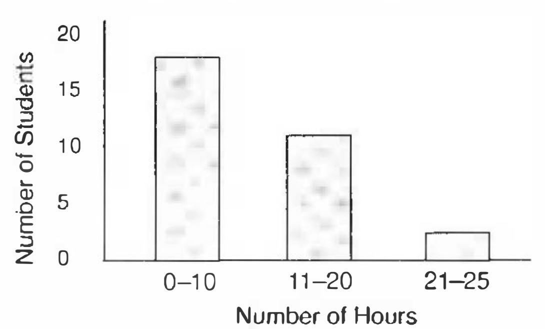
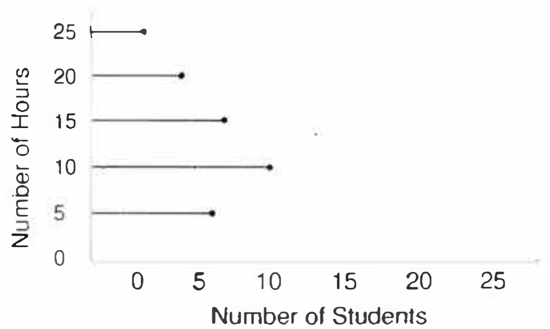


Figure 3. Group 2's graph



The members grouped their data into intervals with a range of five, placing the bar at the upper limit of the range of the interval. In addition, instead of making clearly defined bars, they used line segments. Further, in constructing their graph, they had used a large dot to mark the endpoint of the segment. Therefore, the length of the segment represented the number of data points that fell in the interval (that is, from 0 to 5, from 6 to 10 and so on).

The dot became an issue for other students in the class when I asked, "How are these two graphs alike, and how are they different?"

Andy: That's [points to group 2's graph] a line graph and that's [points to group 1's graph] a bar graph.
Teacher: That's a line graph, and this is a bar graph. Anything else?
Carla: I thought it was supposed to be bars and not like little lines . . . like bars.
Teacher: Why do you think it is bars?
Carla: Because if you do a line, it is supposed to go up at the time when it goes up, and it goes down when . . . it goes up when the rate is high and goes low. . . .
Lynn: I think it is supposed to have bars when it goes vertical [*sic*] like that.
Teacher: What if she is calling these skinny bars? What if she is saying that these are really just skinny bars?
Maggie: With dots on the ends of them?
Jose: With dots on them? I mean, you could do that, but you wouldn't have a line on them, you would just have the dots.
Paul: A line graph is supposed to be connected to other line segments.

For the students, certain conventions were associated with making graphs, and if you used dots, then you must be making a line graph of connected dots. The rules for the use of dots in making graphs seemed very clear to the students; however, they appeared to take great liberties with what I had interpreted as histograms. Their notations of "school mathematics" became intertwined with their goal of making a representation of the data.

The third group to present their graph also drew what I would call a version of a histogram. They appeared very clear on the rule that the intervals all had to be of the same size. However, in their "histogram" the intervals were each composed of 6 of the 30 data points. They said that they decided to use six data points in each interval because you could divide 6 evenly into 30 to get five groups. They had ranked the data from least number of hours of television watched to the greatest, as if finding the median, and then divided the ranked data into five groups of six numbers each (see Figure 4a). They then totaled the six data points in each group, which gave them the height of the respective bar (see Figure 4b). The result obviously gave a series of bars with increasing heights, since the subsequent bars contained the larger data points (see Figure 4c). The group members labeled the horizontal axis according to which 6 of the 30 data points were contained in the interval, and they used the vertical axis as a scale for the total hours for each of the six groups of data points.

After group 3 finished explaining its graph, I again asked how the three graphs were alike and different. The students' discussions tended to focus on the direction of the bars (that is, vertical or horizontal) and the labeling of the axes. For instance, many students noted that group 2 and group 3 both labeled the horizontal axis as "number of students." The fact that the number represented very different reports was of no consequence. They also noted that group 2 was the only group whose graph was "sideways." Their focus was on the superficial features of the graphs instead of the underlying meaning and intent.

The members of the next group to report stated that they had calculated the mean. I then asked why they thought that the mean was a good way to represent the data.

Mark: Well, we averaged it out, and it worked pretty good; well, because that's like saying, well, when you make a 100 on something and 60 on something and they average that out, like on your report card, that's really not right because you made a high grade on the one thing and a low grade on another.

It appeared that in Mark's justification for the use of the mean, he was questioning whether it was "right," since he related it to finding an average grade, he would not want to use the mean if he had a high score and a low score. I found it intriguing that, as he offered his justification to the class, he appeared to be reconceptualizing his understanding of the mean as a way to represent his group's data. Of the remaining three groups, two calculated the mean and the third made a bar graph of each value in the data set.

As the class drew to a close, I realized that I had generated more questions for myself than answers. The students had obviously acquired ways to reason about the data. However, their ability to make judgments tended to be impaired by their notions about "rules," for example, if you use dots, they have to be connected. Also, group 3 was certain that each bar had to represent the same number of data points, but the intent of the graph was never clarified. Initially, I had been concemed not about whether the students could make the graphs correctly but with how they reasoned about the data set. However, as a result of interacting with the students on multiple tasks, I realized that their previous experiences with learning how to make graphs would clearly have a large impact on future instructional goals. Apparently, the set of rules associated with the construction of graphs formed the basis of their initial investigations with graphs. They perceived that I was now asking them to remember all the rules as they made reasonable judgments about the data sets rather than to reason and construct their own ideas. Asking them to create their own representation appeared to be a shift in the type of activity that they normally associated with sets of data. Also, their interpretation of the mean as a representative value was unclear to me. Students clearly knew how to calculate the mean, but their ability to talk about it as a reasonable way to represent data set seemed limited.

Implications for Instructional Tasks

The results of my observations of the students' ways of reasoning on this and other tasks had serious implications for instructional "next steps." To support students' development of ways to reason logically about data, instructional tasks would need to build from the students' current understandings. Simply restating rules and procedures for the proper ways to construct types of graphs would not equip the students with powerful tools for data exploration. Their subsequent activity would be reduced to deciding which graph to use for which set of data according to some predetermined criteria. Further, problematic areas in their current understandings would need to be highlighted in the context of investigations so that those areas could then become the focus of discussion. This outcome would require finding situations in which the data presented the opportunity for students to reason about the implications of their decisions.

In working with my colleagues to develop a sequence of specific tasks that would capitalize on the students' current understandings, we decided to focus on the use of the mean as a representative value. We thought that this understanding could be best achieved by having students work to compare two sets of data to make a decision. In this situation, each choice would result in a consequence. We hoped that students would then come to see the value and limitations of the mean. As we deliberated, we decided to find situations in which the two data sets had very similar means even though the individual data points in one of the sets varied greatly. In these particular situations, it might be important to know more than just the mean; one might also need to know the consistency, or the range, of each set. Students would then need to work to find another way to tell the whole story of the data.

One such investigation involved comparing the hours of use of a sample of 10 separate batteries from each of two different brands of batteries. Students were immediately engaged, since most of them depended on batteries for some type of personal electronic apparatus. As they began to analyze the two sets of data, their investigation led to discussions about what brand had "more bad batteries" and which brand was "more dependable." During the discussion, some students argued that if you used the battery with the higher average, over time that brand would be the better

Figure 4. Group 3's Data Points, Rankings and Graph

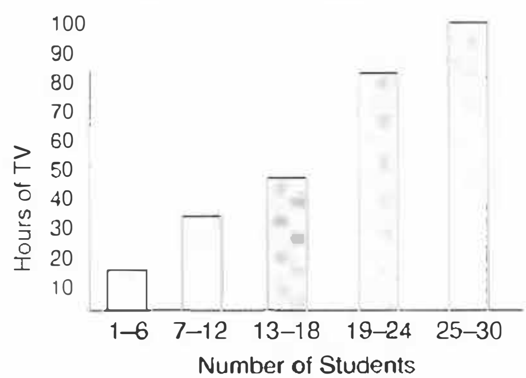
(a)
Ranked data separated into five groups

0 1.5 2.5 3 3.5 4 / 4.5 5 6 6 8 8.5
/ 9 9 9 10 10.5 11.5 / 12.5 13 13.5 14 15 16
/ 16.5 18 19 19.5 21 23

(b)
Totals for the five groups of data

0	1.5	2.5	3	3.5	4	= 14.5
4.5	5	6	6	8	8.5	= 38
9	9	9	10	10.5	11.5	= 59
12.5	13	13.5	14	15	16	= 84
16.5	18	19	19.5	21	23	= 117

(c)
Group 3's "histogram"



choice. The students then worked to find other ways to represent the information to support their arguments. This type of discussion provided opportunities to build from students' investigations to support conceptual understanding of the mean in relation to a set of data and of the mean in relation to a graph. The development of this type of task clearly requires that we find situations that not only appeal to the students but also feature data that fit with our pedagogical agenda. This approach is far more work-intensive than teaching from a textbook. However, I believe it to be both necessary and appropriate.

Conclusion

As a result of my interactions with these students, I have further refined my ideas about how to support students' development of statistical reasoning. I believe that students need to develop ways to reason logically about data—not memorize rules and procedures. For me, this belief involves students in tasks that permit the exploration of multiple forms of representation. Students' experiences with data, graphs and representative values need to reflect the expectation that they reason about situations rather than simply apply rules and algorithms. For instance, students should use their analysis as the basis of arguments. By making reasoning the focus of our instruction, we then change our expectations. We focus on the meanings that graphs and representative values have for the students in light of their investigations.

In the past, the distinction I often made between "graphing" and "finding the average" caused students to infer that these two types of representations are not related. They therefore lacked the understanding that both representations are models of the data set, each highlighting different aspects of the set. By allowing students' initial exposure to data analysis to be in the context of exploration of data sets, we give purpose to the investigations and subsequent representations. The specifics of constructing graphs can emerge from the importance of clarifying the representation. For instance, the graph presented by group 1 offered an opportunity to build on students' contributions while clarifying the purpose of a histogram. In this setting, the problem of rounding data points could be highlighted as students wrestled with the notion of a continuous scale.

As a result of thinking differently about how graphical representations should be incorporated in the middle school curriculum, I see the value in allowing students to engage in analyzing data before they have mastered all the conventional "tools." In this way, the tools emerge from students' investigations, thereby acquiring meaning in the context of their activity.

References

Hancock, C., J. Kaput and L. Goldsmith. "Authentic Inquiry with Data: Critical Barrier to Classroom Implementation." *Educational Psychologist* 27 (1992): 337–64.

Lehrer, R., and T. Romberg. *Children's Constructions of Data, Chance, and Inference in Design Contexts*, in press.

Mokros, J., and S. J. Russell. "Children's Concepts of Average and Representativeness." *Journal for Research in Mathematics Education* 26 (January 1995): 20–39.

Shaughnessy, M. "Research in Probability and Statistics: Reflections and Directions." In *Handbook of Research on Mathematics Teaching and Learning*, edited by D. Grouws, 465–94. New York: Macmillan, 1992.

The research and analysis reported in this article were supported by the Office of Educational Research and Improvement (OERI) under grant number R305A60007 through the National Center for Improving Student Learning and Achievement in Mathematics and Science. The opinions expressed do not necessarily reflect the views of OERI.

The author would like to thank Judi Zawojewski for her comments on an early draft of this article as well as Paul Cobb, Koeno Gravemeijer and other members of the research team for their efforts at analyzing students' work and in designing instructional tasks.

Reprinted with permission from Mathematics Teaching in the Middle School. *Volume 4, Number 6, (March 1999) pages 377–80, an NCTM publication. Minor changes have been made to spelling and punctuation to fit ATA style.*

Titanic: A Statistical Exploration

Sandra L. Takis

The tremendously popular movie *Titanic* has rejuvenated interest in the *Titanic* and its passengers. Students are particularly captivated by the story and by the people involved. Consequently, when I was preparing to explore categorical data and the chi-square distribution with my class, I decided to use the available data about the *Titanic's* passengers to interest students in these topics. This article describes the activities that I incorporated into my statistics class and gives additional resources for collecting information about the *Titanic*.

Analyzing Categorical Data

A topic in the descriptive-statistics strand of the Advanced Placement curriculum is analyzing categorical data using conditional and marginal frequencies. In this analysis, students examined the distributions of specific outcomes for different groupings of the population, comparing proportions rather than counts of the data. To perform this type of analysis using a real-world example, we examined the overall population of *Titanic* passengers and survivors, as shown in Table 1.

Table 1

***Titanic* Passengers and Survivors by Class and Age or Gender**

Passenger Category	Number of Passengers	Number of Survivors
Children, first class	6	6
Children, second class	24	24
Children, third class	79	27
Women, first class	144	140
Women, second class	93	80
Women, third class	165	76
Men, first class	175	57
Men, second class	168	14
Men, third class	462	75

Source: His Majesty's Stationery Office, 1912

The first two questions that the students explored were the following:

- Was a difference in the survival rates related to the class of passenger?
- Was a difference in the survival rates related to the gender or age of the passenger?

Both of these questions are interesting not only from a statistical perspective but also from a historical perspective. The makers of the movie *Titanic* imply that lower-class passengers were treated unfairly. This analysis of categorical data allows students to determine whether that portrayal is accurate.

To examine these issues, my students started by developing relative frequencies of survival and loss for first-class, second-class and third-class passengers and similar rates for men, women and children. Figures 1 and 2 illustrate these data. After a preliminary graphical analysis, students likely will conclude that survival depended on both class and gender or age. Approximately 60 percent of the first-class passengers survived, compared with approximately 40 percent of the second-class passengers and approximately 25 percent of the third-class passengers. Women had the highest survival rate at approximately 75 percent; children were next at approximately 50 percent; and finally men at approximately 20 percent.

Figure 1

Survival Rates by Class

Because two factors appeared to influence survival, we examined survival data further. Specifically, we examined whether the differences in survival rate by class were caused by the very different distributions of men and women in each class. Many more men than women were in second and third class, whereas the numbers of men and women in first class were more similar. To determine the effect that the gender distribution had on the survival rates of the first-class, second-class and third-class passengers, we examined the data in more detail. The histogram in Figure 3 illustrates the disaggregated data on survival rates by both class and gender or age.

This analysis helped students examine the effect of class and gender or age together and to think about some of the shortfalls of examining aggregate data. For example, although the overall rate of survival for second-class passengers was higher than for third-class passengers, the rate was lower for second-class adult males than for third-class adult males.

Although this situation does not represent the complete reversal of the relationship known as *Simpson's paradox*, it did show the students that they need to be cautious about making conclusions based on aggregate data. Simpson's paradox occurs when an association found between two variables that are examined at disaggregate levels is reversed when the variables are examined at an aggregate level. In this situation, Simpson's paradox would occur if the overall survival rates were higher for second-class passengers than for third-class passengers even though the individual survival rates for second-class adult males, adult females and children were lower than those of the third-class adult males, adult females and children, respectively. Such a reversal occurs when differences exist in the distribution of each group—here, adult males, adult females and children—for each aggregate group—here, second- and third-class passengers.

From the analysis, the students more clearly realized that gender played a larger role than class. Several specific questions made the students think about the differences in the overall survival rates. A few examples follow:

- What proportion of the first-class passengers were adult males, adult females or children?
- How did this proportion affect the overall survival rate of the first-class passengers?
- How were the gender distributions different for second- and third-class passengers, and how would these distributions affect the overall survival rate of these groups?

In the future, I plan to ask the students to write a summary of the relationship between passenger class and survival rates from a data-analysis perspective. To follow up this report, students could examine the formal investigation report of the disaster prepared for the British Parliament, as well as other reports prepared at the time.

Applying the Chi-Square Test of Independence to an Entire Population

A second activity in which I used the *Titanic* data was in introducing the chi-square (χ^2) test of independence. This test can be used to examine

Figure 2
Survival Rates by Gender/Age

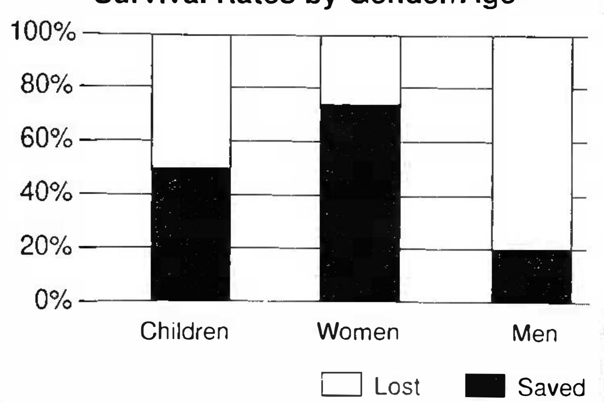
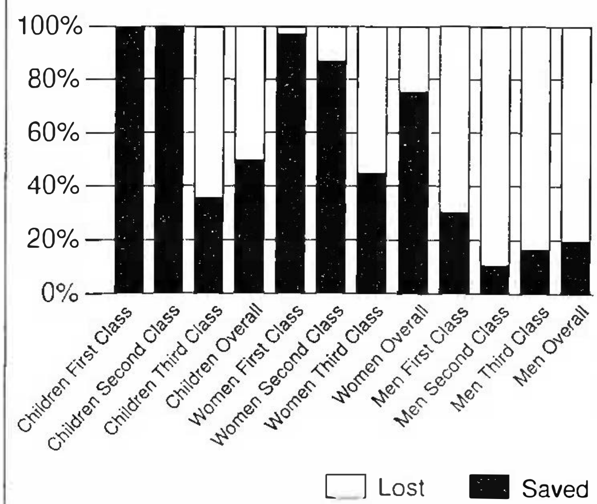


Figure 3
Survival Rates by Gender/Age and Class



whether a relationship exists between two categorical variables. In this example, we applied the test to the entire population of *Titanic* passengers to determine whether a statistically significant relationship existed between passenger class and survival. The χ^2 test can be applied to data from independent samples from several populations where the populations act as one of the two categorical variables or to data from a single sample where subjects are classified by two categorical variables. For further discussion, see Moore (1995, 535–37).

Table 2 summarizes survival and loss by passenger class. The crucial concept in the test of independence is understanding what the data would look like if no relationship existed between the variables, in this example, survival rate and passenger class. First, if survival did not depend on class, the overall survival rate should stay constant regardless of the passenger class. The overall survival rate was 37.9 percent, and the overall loss rate was 62.1 percent. These rates were applied to the total number of passengers for each class to determine the expected number of survivals and losses in each class. Students needed to understand that they were applying the survival and loss rates to the total number of passengers in each category and were consequently assuming that survival and class are independent.

Table 2

Survival by Class: Summary and Percents

Class	Survived	Lost	Total
First	203	122	325
Second	118	167	285
Third	178	528	706
Total	499	817	1,316
Percent	37.9%	62.1%	

After we calculated the expected values for each category, we calculated the chi-square statistic for the sample data. The formula for this calculation is

$$\chi^2 = \sum \left(\frac{(\text{observed} - \text{expected})^2}{\text{expected}}\right).$$

The calculation is similar to that of standard deviation in that we measured how far the observed value was from the expected outcome, just as we measured how far individual data points are from the mean when calculating standard deviation. This difference is squared to eliminate negative values and is divided by the expected outcome to standardize the measure. The χ^2-test statistic is the sum of these measures for each of the categories. Table 3 shows the calculation of the chi-square statistic for the population data.

Table 3

Calculation of Expected Outcome and Chi-Square-Test Statistic

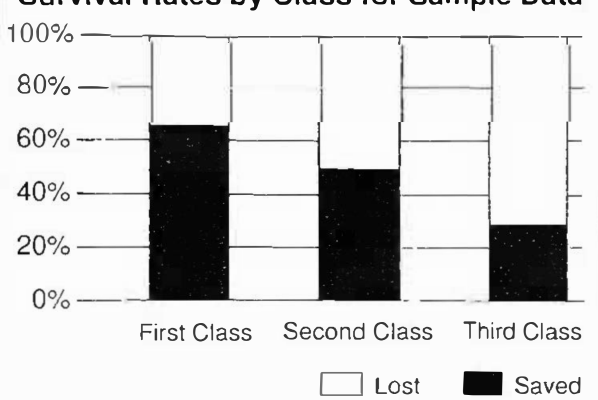
Passenger Category	Observed Data	Expected Outcome	Differences between Observed and Expected	Difference Squared and Divided by the Expected
First class survived	203	$\frac{499}{1316} \cdot 325 = 123.23$	$203 - 123.23 = 79.77$	$\frac{79.77^2}{123.23} = 51.64$
First class lost	122	$\frac{817}{1316} \cdot 325 = 201.77$	$122 - 201.77 = -79.77$	$\frac{(-79.77)^2}{201.77} = 31.54$
Second class survived	118	$\frac{499}{1316} \cdot 285 = 108.07$	$118 - 108.07 = 9.93$	$\frac{9.93^2}{108.07} = 0.91$
Second class lost	167	$\frac{817}{1316} \cdot 285 = 176.93$	$167 - 176.93 = -9.93$	$\frac{(-9.93)^2}{176.93} = 0.56$
Third class survived	178	$\frac{499}{1316} \cdot 706 = 267.70$	$178 - 267.70 = -89.70$	$\frac{(-89.70)^2}{267.70} = 30.06$
Third class lost	528	$\frac{817}{1316} \cdot 706 = 438.30$	$528 - 438.30 = 89.70$	$\frac{89.70^2}{438.30} = 18.36$
Total	1316	1316		$\chi^2 = 133.07$

As in other tests of significance, we compared the calculated measure with a standardized distribution of measures to determine whether the statistic's measured differences were caused by random variation or whether these differences were too large to be caused by chance. We made this determination in two ways. One approach was to compare the calculated measure with a cutoff value in the chi-square distribution. The distribution represented variation in the chi-square value that would occur randomly when no relationship existed between our two variables. Most random variation will result in small chi-square scores. We set a cutoff, or critical, value at a point above which very few randomly generated scores occurred. We set our critical value at 5.99, using a significance level of 5 percent and degrees of freedom of 2. Degrees of freedom for a test of independence equals (number of rows − 1) times (number of columns − 1). In this example, it is $(3 - 1)(2 - 1) = 2$; therefore, at a 5 percent level of significance, the critical chi-square score is 5.99. On the basis of this cutoff value, we saw that the population showed more variation from the expected value than would generally occur by chance. We therefore concluded that a statistically significant relationship existed between survival rate and class.

A second approach was to use a p-value rather than a critical value. The TI-83 calculator will perform the chi-square calculation using matrices and the statistical test functions. It will also supply a p-value from which our conclusion can be drawn. In this case, it calculates a p-value of 1.28×10^{-29}. We interpreted this result as meaning that very little chance, a 1.28×10^{-29} probability, exists that such extreme differences in survival between classes would occur when truly no difference exists in survival by class. The ability to calculate a p-value forges a strong link to the underlying probability concepts that give meaning to statistical tests.

Figure 4

Survival Rates by Class for Sample Data



Comparing Conclusions Drawn from Sample Data with Those Drawn from the Entire Population

A third activity for using the *Titanic* data was to have students answer the previously proposed questions using sample data. Because data on the entire population were available, the sample test that follows was not necessary to draw a conclusion. However, we used sample data from a known population so that students could compare their results with the known results and revisit such concepts as sampling error and variation in samples. The following is a synopsis of the students' process.

To study the questions of differing survival rates by class and gender, students broke up into small groups and developed a method for taking random samples of 100 passengers. Each group took a different approach to its sampling plan. Some groups randomly selected a starting point and systematically chose every 20th passenger. Others used their calculators to randomly generate numbers and counted off the list to find those passengers. When students had their samples, I asked them to create a contingency table by class and a relative-frequency histogram summarizing the survival rates. Table 4 and Figure 4 illustrate an example of sample results.

The students then calculated expected values and chi-square scores for each of their samples. Each group completed a test of independence on its sample data. Of the seven groups that performed the analysis in my class, only one did not conclude that a relationship existed between passenger class and survival rates. Such results gave the students an opportunity to discuss possible sources of error. The other students wanted to conclude that the group made an error either in its calculation or in its sampling technique. This group, however, when given the opportunity to defend

Table 4

Example of a Student Sample

Class	Survived	Lost	Total
First	12	6	18
Second	14	14	28
Third	15	39	54
Total	41	59	100

its work, could justify the work and give the alternate possibility; the conclusion was incorrect because of a type II error. In this situation, a type II error represents the outcome for which the null hypothesis—passenger class and survival rates are independent—is not rejected even though it is incorrect. As a follow-up to this activity, I asked the students to perform a similar analysis to determine whether survival and gender or age were related. The list we used had titles (Mr., Mrs. and so on) and identified children, so the students could easily classify individuals in the sample as adult male, adult female or child. Students wrote up their full analysis, including relative-frequency histograms, calculations and the final conclusion.

The overall process was exciting and motivating for both my students and me. The students were very interested in the sociological aspects of the study and enjoyed talking about the movie and fact versus fiction. Several resources on the Internet allow further examination of the *Titanic* disaster. I used the following sites to obtain information on the *Titanic*. These sites also have links to additional sites for more information.

Internet Sites

- www2.nexus.edu.au/TeachStud/titanic2/home: This site is called "The Titanic in the Classroom." It gives activities for using the *Titanic* for classroom activities.
- www.anesi.com/titanic.htm: This site is called "The Titanic Casualty Figures." It examines the casualty figures in this article and provides sources of investigations performed at the time of the disaster.

References

Anesi, C. "The Titanic Casualty Figures." Available online <http://www.anesi.com/titanic.htm.>

"British Parlimentary Papers, Shipping Casualties (Loss of the Steamship 'Titanic'), 1912, cmd. 6352." In *Report of a Formal Investigation into the Circumstances Attending the Foundering on the 15th April, 1912, of the British Steamship "Titanic," of Liverpool, after Striking Ice in or near Latitude 41B 46' N., Longitude 50E 14' W. North Atlantic Ocean, Whereby Loss of Life Ensued,* 42. London: His Majesty's Stationery Office, 1912.

Department of Education Training and Employment (South Australia). "The Titanic in the Classroom." Available online <http://www2.nexus.edu.au/TeachStud/titanic2/home.>

Moore, D. S. *The Basic Practice of Statistics*. New York: Freeman, 1995.

Reprinted with permission from The Mathematics Teacher, Volume 92, Number 8 (November 1999), pages 660–64, an NCTM publication. Minor changes have been made to spelling and punctuation to fit ATA style.

Dodecahedra

I have an unknown number of regular dodecahedra, which I cannot differentiate from one another. If I take red and blue paint and paint every face of each dodecahedron either red or blue, how many different dodecahedra (based on color) can I create?

A Collection of Connections for Junior High Western Canadian Protocol Mathematics

Sol E. Sigurdson, Thomas E. Kieren, Terri-Lynn McLeod and Brenda Healing

We have put together "A Collection of Connections" that consists of 12 uses of junior high school mathematics. These activities support the communication and connections strands of the Western Canadian Protocol mathematics curriculum. In using them, teachers may adapt them extensively. They can serve as a basis of one to three mathematics periods. We have also found that teachers need to plan if they intend to incorporate them into their teaching units. They can be used as end-of-unit activities or as focal activities in the development of a unit. We would encourage teachers to use the activities as a means of bringing mathematical skills to life. These contexts provide an opportunity to enhance a student's view of mathematics. As one Grade 9 girl said at the conclusion of one activity: "That just proves that mathematics is everywhere."

The following are samples from the shape and space and the patterns and relations strands.

Shape and Space (Measurement)
The Volume of a Sphere
The Volume of a Sphere Student Activities
Patterns and Relations (Patterns)
Mail Carrier Routes
Mail Carrier Routes Student Activities

The Volume of a Sphere

Intent of the Lesson

The student gets experience in looking at the relationship between the volume of a cube, a cylinder and a sphere. The mathematics involved is practical measurements, volume formulas, data collection and straight line graphs.

General Question

The textbook tells us that the formula for the volume of a sphere is $^4/_3\ \pi r^3$. Why is this a reasonable formula? What meaning does this formula have for us? We know what the formula for the volume of a sphere is but do we understand it? Today we will take a procedure from science and use it to help us understand some mathematics. Usually we use mathematics to help us understand science but today we have the reverse. Why is the formula for the volume of a sphere $^4/_3\ \pi r^3$?

Discussion Questions

- What is the formula for the volume of a cube of side s? (s^3)
- What is the formula for the volume of a cylinder of radius, r, and height, h? (pr^2h)
- Describe what r^3 means physically. (The volume of a cube with each side of length r.)
- What is the area of a circle? (pr^2)
- Do you have a way of remembering the area of a circle?

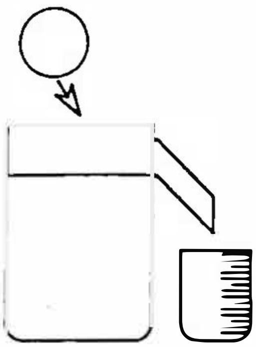
Preliminary Activities

Surface Area of a Sphere

Earlier in this collection, we found out how many stars we could see. We used the formula for the surface area of a dome (half of a sphere). How do you think that was calculated? The formula for the surface area of a half sphere is $2\pi r^2$. Let's compare it to something you already know. Imagine you had an elastic circular disk with radius r and it could be changed into half of a sphere by blowing air on it. The surface area of that disk when it is flat is πr^2. The surface of the curved dome should be about double the area of the flat disk upon which it rests. Although this is not a proof of the surface area of a hemisphere formula, it makes the formula reasonable. Using the same thinking, the surface area of a sphere is four times the area of the circle through its centre, $4\pi r^2$.

The area of a parallelogram can be compared to a rectangle on the same base and the same height. In geometry, it is often useful to compare one figure to another. In this lesson we will compare the volume of a sphere to the volume of the cube and the cylinder into which the sphere could be placed exactly.

A Lesson from Science



We can determine the volume of the sphere by water displacement. The science laboratory may provide us with a displacement jar. We fill the jar so that water flows out of the jar. This means that it is exactly full (as in the picture).

When the sphere is lowered into the water, some water will spill out of the spout. The amount that spills out the spout is exactly equal to the volume of the sphere. Measuring the volume of water that collects in the catch beaker is a way of determining the volume of the sphere. If 20 mL (millilitres) of water is collected, the volume of the sphere is 20 cc (cubic centimetres).

Discussion Questions

- What shape does the object have to be for this to work? (Shape is irrelevant. The method is most often used for irregularly shaped objects.)
- What if the object floats? (A method is needed to submerge the object without displacing extra water.)
- What if the object sinks to the bottom? (The experiment is easier to perform if the objects sink.)
- If our first object is in the displacement jar, can we put another object in? (Yes, as long as you empty the catch beaker before adding the second object, or if you are using a graduated cylinder, just record the volume before the second object is put in.)
- What condition must be met for the displacement test to work? (The object must displace water.)

To understand our lesson, students must be convinced that no matter how far down the sphere is in the displacement jar, the volume of water displaced will be equal to the volume of the sphere.

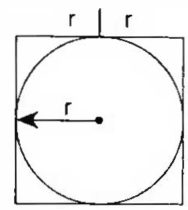
Verifying Volume by Water Displacement

To help students see that the displacement method of finding volume actually works, the teacher can demonstrate the method using an object of known volume. Connecting, for example, eight one-cubic centimetre blocks together would create an object of known volume and could be used for the demonstration. To show that the shape of the object is irrelevant, connect the blocks together to form different shapes and repeat the demonstration.

Answering the General Question

If we simply wanted to find the volume of the sphere, we could displace water and have our answer. However, we want to find the *formula* for volume. We do this by comparing this volume to the volume formula for something we already know. We have two such objects, a cube and a cylinder.

A Cube

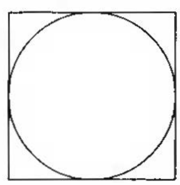


Imagine a cube just large enough so that our sphere fits into it exactly. The dimensions of the cube will be $2r \times 2r \times 2r$. The diagram of a circle in a square illustrates the point.

The diameter of the circle is $2r$. Therefore, each side of the square is $2r$. The volume of our cube is $8r^3$ and we can easily see that the volume of our sphere is less than this. A reasonable guess is that it is about half this amount, or $4r^3$. That is, the volume of the cube outside the sphere is about equal to the volume that is inside the sphere.

A Cylinder

Now let us imagine our sphere fitting exactly into a cylinder which is the same height as the sphere. When we look at this from the top of the cylinder, we see that the sphere touches at all points of the cylinder. It is a perfect fit. However, when we look at it from the side (if we could see through the cylinder) the picture would look like this.



The sphere would not be touching the cylinder in any of its corners. We need to recall that the formula for the volume of a cylinder is $V = \pi r^2 h$. In this case $h = 2r$ (the diameter of the sphere) so the formula becomes $2\pi r^3$. So now we can see that the formula for the volume of a sphere is less than $2\pi r^3$ because the sphere does not entirely fill the cylinder. A good guess might be one-half of $2\pi r^3$ or πr^3.

Looking back to our earlier guess of $4r^3$, based on the volume of a cube, we should notice that the estimate of πr^3 is about the same since this is about $3r^3$ (because $\pi = 3.1$). We might also reasonably guess that πr^3 is in the formula. The constant π does not appear in the cube formula since the cube is not in any way in the shape of a circle while the cylinder is.

These comparisons have yielded two insights:

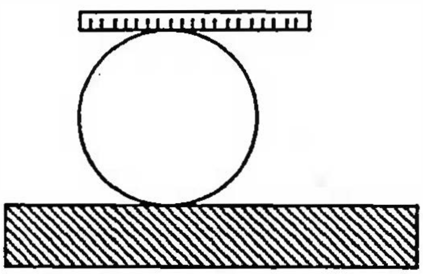
The formula for the volume of a sphere is close to $4r^3$ and πr^3.

The formula probably contains πr^3.

A Measurement Procedure

We now know that the formula for the volume of the sphere is some number K times r^3. We are also reasonably sure that π is part of the number K. We are going to use our science measurement to make a reasonable guess at that number. We know that it must be fairly close to 3 or 4.

Each group is given a different ball. All, however, need to fit in the displacement jar. If they sink on their own accord so much the better. Otherwise we will have to find ways to submerge them. However, before students find the displacement they should figure out the volume of the cube and cylinder that are related to their ball.



Exact measurements are a must, certainly to the nearest millimetre. One way of measuring the diameter of the sphere is to wrap adding machine tape around it and simply measure across the edges of the tape. Placing a ruler on top of the ball parallel to the table top is another way.

The students have four tasks:

1. Find the radius of the sphere.
2. Find the volume of the related cube.
3. Find the volume of the related cylinder.
4. Take the actual sphere and see how many cubic centimetres it displaces.

The Displacement Activity

As each group submerges its sphere in the displacement jar, the teacher will need to supervise the activity. The displacement jar must be full and the catch beaker must be empty. The tricky part is to fully submerge the sphere. If the sphere is heavier than water, this is easy. If it is lighter and floats, a device will need to be used to push the sphere down so that it just submerges. If the pushing device goes into the water, it will displace water giving inaccurate measurements.

The larger the ball is, the less percent error there will be. Small balls give large errors. The line of best fit method (given below) will make it clear when one of the measurements is erroneous. This measurement should be excluded.

Using a Table

In the chart we will include the radius, the volume of a cube, the volume of a cylinder and displacement. The last three numbers will give us a basis for comparison. Since the volume of the sphere is probably going to be some multiple of r^3, we will also put that number in a sixth column. This is a calculation based on the radius.

Once all the groups have contributed their numbers to this chart, the class can proceed to make a graph of r^3 on the x-axis and the displacement on the y-axis.

1	2	3	4	5	6	7 optional column
Name of group	radius (least to greatest)	volume of cube	volume of cylinder	displacement	value of r^3	displacement divided by r^3

Teaching Suggestion

The teacher should supervise the filling in of the chart. Once this is done, students can be assigned the task of drawing the graph of the displacement against r^3. Using the line of best fit method, they will try to determine the line on the graph. Those points that fall far from the line may be due to error and may be eliminated.

A discussion of error is a chance to reflect on the displacement activity and to focus on the physical nature of the measurements. This, of course, contrasts with the error-free use of mathematics that occurs in typical mathematics classes. The discussion of error brings out the real world aspects of this activity.

Determining the Equation

Points on the line can be used to determine the equation for the relation:
D(isplacement) = $K \times r^3$
The value of K should be close to 4. The actual value is $^4/_3\pi$ which is about 4.2. The discussion of the constant K should centre on whether $4r^3$ or $^4/_3\pi r^3$ is closer.

The formula of the volume of a sphere is $^4/_3\pi r^3$. Students can then be asked to find out what fraction the sphere is of the cube and what fraction is of the cylinder. It is ½ of the volume of a cube and $^2/_3$ the volume of the cylinder. A discussion of these results should help to confirm that the formula for the volume of a sphere is reasonable.

An Alternative Method of Data Analysis

A simpler, but less accurate way, of finding K is to create another column for "the displacement divided by r^3." The numbers in this column should all be close to 4.2, that is $^4/_3\pi$.

Discussion Questions

This activity is not a proof of the formula, but the activity shows how the volume of a sphere is related to the volume of a cube, the volume of a cylinder and πr^3.

Materials

Displacement jar, water, spheres of various sizes and grid paper.

Modifications

If a displacement jar is not available, a beaker or a large graduated cylinder may be used. In this case, the volume of the sphere is the difference between the volume reading while the sphere is submerged and the volume reading before the sphere is submerged in the water.

It is essential that the beaker or graduated cylinder be large enough to catch and measure all the water displaced by the ball.

The problem in this activity is accuracy. However, the main idea is to have students thinking of the cube, cylinder and the sphere in comparison terms and also identifying r^3 as the key variable in the change of volume with π being a unique and important constant.

The whole activity might be done as a teacher demonstration. In this case the students miss the multiple calculations and the discussion of error and how it arises. The teacher may point out that error in using a formula arises when the measurements for a variable used in the formula contain error.

Volume of a Sphere Student Activities

General Question

The textbook tells us that the formula for the volume of a sphere with diameter $2r$ is $^4/_3\pi r^3$. How does the volume of this sphere compare to the volume of a cube with side of length $2r$? How does the volume of a cylinder of diameter $2r$ and height $2r$ compare to the sphere? Is there a relationship between the volumes of these three shapes?

Activities

1. a) The surface area of a disk is πr^2. What is the surface area of a dome (hemisphere) placed on that disk? Why is this reasonable?
   b) Based on question 1a, what is the surface area of a sphere?
   c) What is the area of a disk with radius π cm? (Do not do the actual calculations.) What is the surface area of a hemisphere on that disk? Of a sphere formed by adding the other half of that hemisphere?
2. a) We can find the area of a parallelogram by relating it to a rectangle on the same base. What can a trapezoid be related to in order to find the area? (In the trapezoid we know the length of the two parallel lines and the height of it.)

   b) There are at least three options for finding the area of a trapezoid: break it into triangles, double its area and make it into a parallelogram, or make a rectangle on its longer base and one on its shorter base. (Hint: the width of the rectangles is half the height of the trapezoid.) Show one of these ways (one that you did not do in question 2a).

3. a) A cube measures 8 cm on a side. It has a lid. A cylinder and a sphere fit perfectly into it. Assuming the cylinder also has a lid, find the surface areas of all three of these shapes. What is the ratio of the surface area of the cylinder compared to the surface area of the cube and the sphere compared to the cube?
   b) Find the volume of the three shapes in Question 3a. What is the ratio of the volume of the cylinder compared to the volume of the cube and the sphere compared to the cube?
4. a) In measuring the diameter of a sphere, Jerry measured 4.5 when the actual measurement was 4.2. Jerry said a 0.3 error was not a problem. What percentage was Jerry's error of the actual measurement? Would you agree with Jerry's conclusion?
   b) Jerry used this measurement to determine the volume of the sphere. What percentage error occurs compared to the actual volume? Do you think Jerry would agree with his conclusion now?
   c) Why is there such a big difference in the percentage error in question 4a compared with 4b?
5. a) A displacement jar (which is a cylinder) is full of water and contains 100 cc of water. A sphere is submerged which fits perfectly into the jar so that the top of the sphere just touches the top of the water when submerged. What percent of the water in the displacement jar remains?
   b) When the sphere is lifted out, how many cc of water remain in the displacement jar?

Mail Carrier Routes

Intent of the Lesson

This lesson relates to operations research, which is an area of mathematics used to solve problems in engineering and city planning. It shows how mathematical reasoning can be applied to questions of logistics and efficiency. It relates to drawing networks of geometric figures and uses line segments and intersections.

The General Question

The general question asked here is, What is the most efficient way for a mail carrier to cover a route? Can we make a route that covers every street without going on the same street twice? This is not related to any part of the mathematics curriculum but can be engaged in as an exercise in a systematic approach to an abstract problem. It comes from an important application of mathematics called operations research. Operations research examines issues such as industrial efficiencies, transportation routings and pipeline oil flow to and from refineries.

Mail carriers want to cover every street in a certain town (or subdivision) but they want the most efficient route. For example, suppose they needed to deliver mail in a four-by-four town.

What is the most efficient way for a mail carrier to go over every street of the town? Several attempts will convince students that however they proceed, it will be necessary to backtrack in order to get to every street.

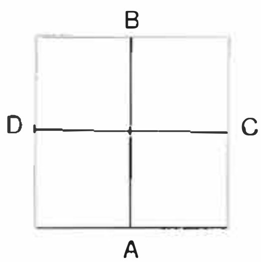
1	2	3	4	5	6	7 optional column
Name of group	radius (least to greatest)	volume of cube	volume of cylinder	displacement	value of r^3	displacement divided by r^3

Mail carriers, puzzled by the problem, have invited you, a mathematician, to help them solve the problem. You want to solve the problem for a four-by-four town but you would also like to find a general solution that would apply to any type of town or district. A general solution would be of help not only to mail carriers but also to paper carriers and snowplow drivers all across the country. Your few trials should convince you that the four-by-four town cannot be covered without going over some streets twice.

Discussion Questions

- Where do problems occur? (At intersections with an odd number of paths)
- Besides paths, what other feature is obvious on this town map? (Intersections)
- All paths are straight lines but are all intersections the same? (No, the number of paths leading to and from them may differ.)
- Which do you think we are going to be more interested in—paths or intersections?

Preliminary Activity



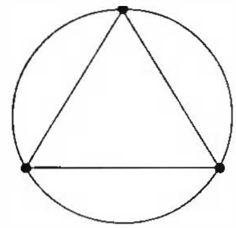
Let us take a very simple town, say three by three.

Since this town has a certain symmetry, we can only start at three mathematically "different" places: a corner, at point A or at the centre. Intersections A, B, C and D are similar in one respect. They all have three paths connecting them to other points. No matter where we start, we have to come to one of those points, let's say C.

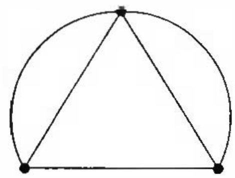
Having arrived at C, our journey will have to end at C because one of the remaining roads (from C) will take us away and the other will bring us back. Once back, we will be unable to leave again because all three roads will have been covered. Because A, B and D also have three paths, we can argue that the journey will have to end at each of the points, which is impossible since a journey can only end in one place. So, regardless of where we start our journey—a corner, at point A or the centre---once we arrive at any of the points B, C or D, our journey has to end there.

Discussion Questions

- Our small three-by-three town has how many intersections? (Nine)
- Does each intersection have the same number of paths? (No, two paths [corners], three paths [points A, B, C and D] and four paths [the centre)].
- What is the main problem in making a route?
- How many intersections does the town below have? (Three)



- How many paths does each intersection have? (Four)
- Can we make a route easily? (Yes)
- How many intersections does the town below have? (Three)
- How many paths are there to each intersection? (Two have three, one has four)



- There is only one way for making a path in this town. (We must start at one odd number intersection and end at the other.)
- What happens if a town has more than two intersections each with three paths such as our three-by-three town? (You cannot make a route without doubling back on some path.)

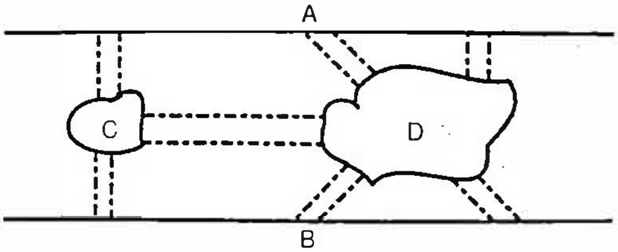
A Mathematician's Advice

Mathematics people speak of paths and nodes. Nodes are important. Furthermore, the important characteristic of nodes is whether they are even or odd nodes. A node that has an even number of paths is even, and a node that has an odd number of paths is odd. Based on this type of analysis, they have come up with four guidelines:

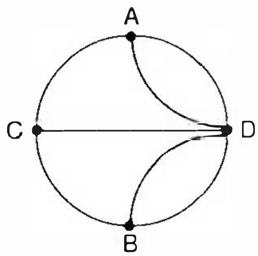
1. A town with *more than two* odd nodes cannot be covered without going over some roads twice.
2. Odd nodes always occur *in pairs*. You cannot make a town that has an odd number of odd nodes. We might try to make a town with three nodes that each have three paths.
3. If a town has *exactly two odd nodes*, the only route that can be made is one that begins at one odd node and ends at the other.
4. If a town has more than two odd nodes, we can make a route by making a path that *connects the pairs of two nodes*, thereby making them even nodes. Once we have a town with only even nodes, the route can begin at any point.

A Famous Problem—The Bridges of Konigsberg

The bridges of Konigsberg look like this. There are two islands in a river.



The Sunday afternoon strollers of Konigsberg wondered if they could make the walk by crossing over every bridge only once. With our knowledge we can help them, but first let's make the problem look like paths and nodes. We label the point A (one shore), B (the other shore), C (the small island) and D (the large island). Each of these four nodes is odd.

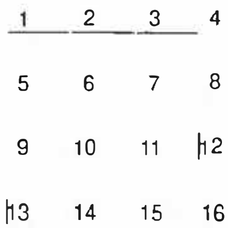


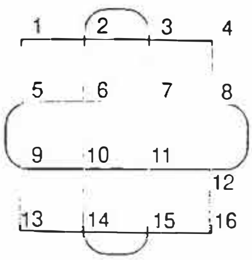
The route is impossible. The Konigsberg strollers did not have a serious practical problem, only a curious mind. However, our problem of the shortest route for mail carriers is practical.

Answering the General Question

Looking back at the four-by-four town, we can number the intersections which we now call nodes. There are 16 in all.

Nodes 2, 3, 5, 8, 9, 12, 14 and 15 are odd nodes. Each has three paths, so our rule says that a simple route is not possible. However, we can make it possible by making an extra path between pairs of odd nodes. "Making an extra path" simply means walking back over the same path.





Actually, if we can start at an odd node and arrange to end at its pair (another odd node), we will not have to make an extra path there. However, this solution is not acceptable if we want to end at the same point that we started. Usually mail carriers want to end where they started because they may have left their car there or they may be able to catch the bus home.

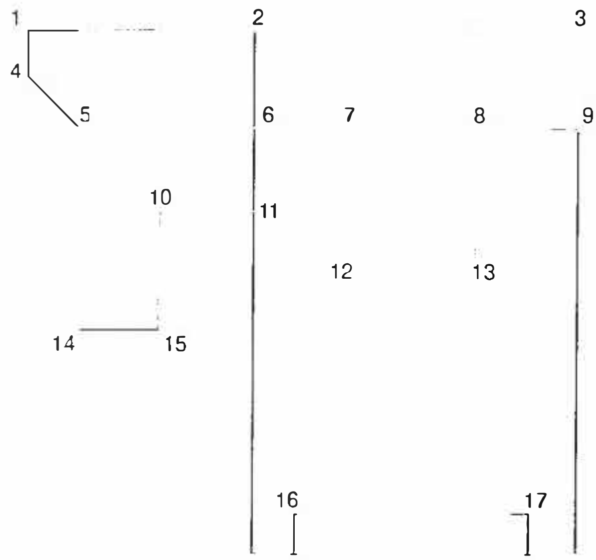
An Edmonton Problem Where Street and Avenues Are Not the Same Length

When making a route by joining two odd nodes to make extra paths, some choice is possible, particularly, in a city like Edmonton, where in certain districts blocks along an avenue are much longer than blocks along a street. Consider the following neighborhood.

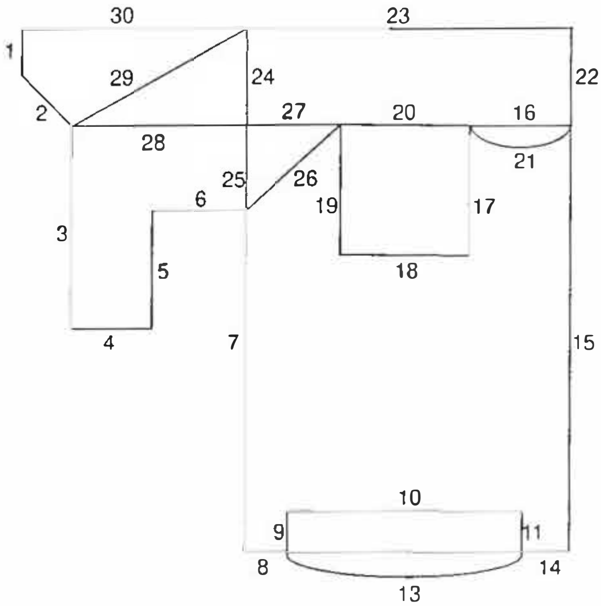
We have four odd nodes. Joining 3 to 4 and 5 to 6 is more efficient than joining 3 to 5 and 4 to 6. This is an extreme case to show that some choice of node joining is necessary.

Your Own Neighborhood

Students can be asked to find a city map and decide upon a paper route that they might have in a six-by-six block neighborhood. They should begin the journey at their house and make the shortest path possible. Remind the students that *all nodes should be numbered* and identified as odd or even nodes. They should then proceed to join nodes in an efficient manner. One such path is given below for the teacher to assign if desired.



One of many solutions is provided below. Additional paths were added between nodes 2 and 5, 7 and 11, 8 and 9, and 19 and 20. The numbers here represent the order in which the paths were followed to make the route.



An alternative manner of identifying the route is to list the numbered nodes in order. In this case it would be 1, 4, 14, 15, 10, 11, 18, 19, 16, 17, 20, 19, 20, 21, 9, 8, 13, 12, 7, 8, 9, 3, 2, 6, 11, 7, 6, 5, 2, 1.

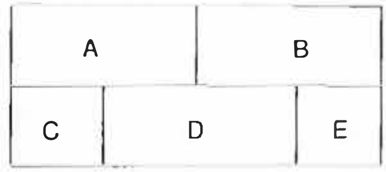
Materials

No special materials other than grid paper are needed for this lesson.

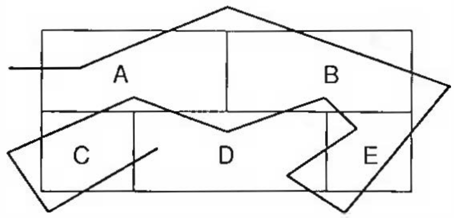
Modifications

The teacher can easily make up easy or difficult problems depending on the capability and interest of the class. The discussion opens up a large mathematics area of operations research that deals with efficiencies of systems often without numbers. Although these are logistical problems, mathematicians often solve them by attaching numbers and making the problem as concrete as possible. For example, we could assign fractional kilometre distances to the Edmonton street problem above. We could rephrase the question as, "What is the shortest distance a mail carrier needs to travel to cover every street?"

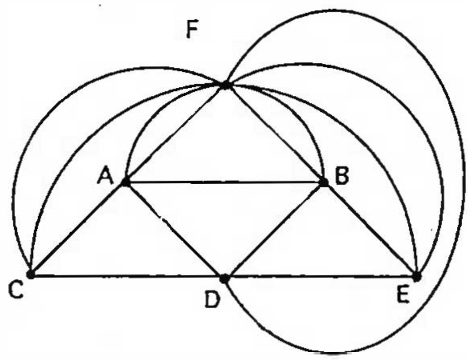
Another approach to these problems is to make a *path node diagram* out of problems that don't look that way, such as the Bridges of Konigsberg. Here is another such problem. The idea here is to draw a single line that cuts the side of each rectangle only once.



Here is one attempt. Which segment is not cut?



The node diagram looks like this: (F is the region outside the square.)

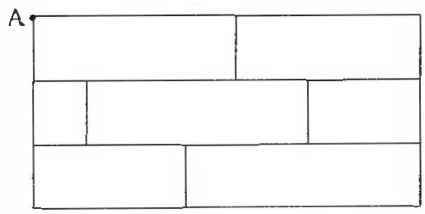


The drawing of the node diagram is fairly difficult but it is a nice example of taking the fun out of a puzzle by analyzing it mathematically.

Mail Carrier Routes Student Activities

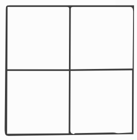
General Question

Imagine a small town (shown here). Can you travel the streets of this town, starting at A, covering each street only once? You will soon see that this is impossible. In other words, if you want to walk on every street of the town, you will have to use some of the streets twice. The question is, Which streets should you cover twice to be the most efficient (cover the shortest distance possible)?



Activities

1. a) For the three-by-three town shown below, number the intersections from 1 to 9 and show three paths starting at each of three points: a corner, a side intersection and the middle intersection. Describe these paths by writing down the sequence of intersections that they go through.



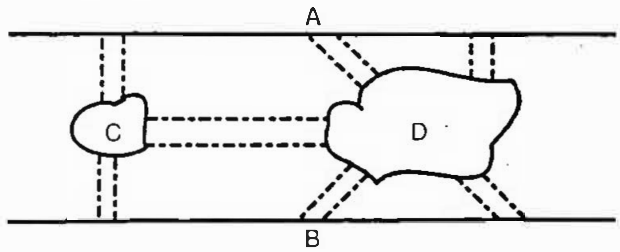
   b) In each of the three paths above (Questions 1a), at which of the three intersections (a corner, a side intersection or the middle intersection) do your paths end? To answer this question, complete these three sentences:
      i) When I start at a corner, I end at a ______ or______.
      ii) When I start at a side intersection, I end at a ______.
      iii) When I start at the middle intersection, I end at a ______ or ______.
   c) Explain your observations in question 1b? Why is a simple three-by-three town so complicated?

2. a) Make a ***three-intersection*** town in which each intersection has three paths connecting it to other intersections. Mathematicians say this cannot be done. Explain why towns work this way.
   b) Here's a task that is possible: make a three-intersection town with two intersections that have three paths leading to other intersections.
   c) In the three-intersection town that you made in question 2b, begin at the intersection that does *not* have three paths and make a route that doesn't involve going down one street twice. Why is this impossible?
   d) Where would a possible path start? Where would it end?

3. a) Below is a town. Label all the nodes. Begin at point A and try to make a route without thinking about where the nodes are odd or even. Which paths did you end up covering twice?

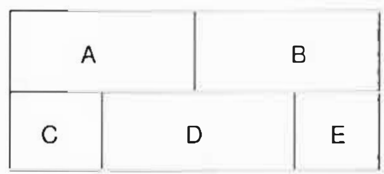
   b) Now use the rule of connecting odd nodes by noting which nodes are odd. Which odd nodes would you connect to make the *shortest* complete route?
   c) Was your route in 3a different than in 3b? Does this example illustrate why mail carriers can benefit from studying mathematics? Elaborate.

4. a) Here is a diagram of the Bridges of Konigsberg. The question is whether a person can make a path that crosses each bridge only once. A and B are the banks of the river and C and D are islands.

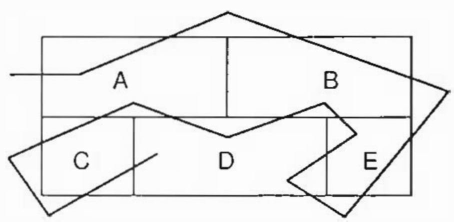


   Draw a node diagram of the Bridges of Konigsberg. Discuss your answer to the question based on the node diagram.
   b) How many bridges would need to be covered twice? Which bridges should these be to involve the least walking?

5. a) Here is a common puzzle which most of your parents will have seen. Draw a continuous path which intersects every segment of the diagram only once.

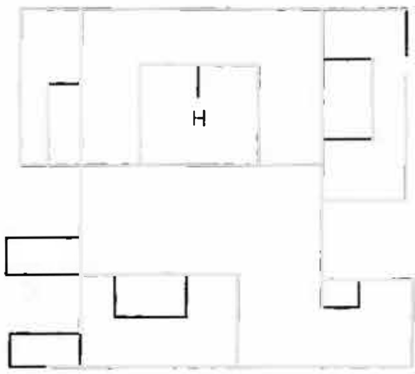


One attempt is shown below. Which segment remains uncrossed?



   b) Draw a node diagram for this problem. (This is not easy.) Label the regions A, B, C, D, E, F (exterior). Now the *paths* are the connections between regions. How many odd nodes do we have?
   c) We can now *change the problem* to make it possible. Add a box (internally) to this diagram to make it possible. Now it is possible but only if you start inside one box and end up inside another.

6. Here is a neighborhood with your home indicated. Number the nodes. Determine which of these are odd and indicate by number which nodes to connect so that you can deliver papers most efficiently.



7. From a city map (or from your own knowledge), draw a six-by-six block neighborhood around your house. Label the intersections and describe an efficient route indicating which streets would have to be covered twice.

Authors' Note: Readers interested in the entire volume of "A Collection of Connections" may contact Sol E. Sigurdson, Faculty of Education, Department of Secondary Education, University of Alberta, Edmonton T6G 2G5; phone 780-492-0753.

The authors would like to acknowledge the financial support for this project provided by the Alberta Advisory Committee for Educational Studies (AACES) and the Central Research Fund of the University of Alberta.

Two Bridge Tables

At a family party held at Green's place, bridge was being played at two tables. The players were Mr. Green, Mr. Pink, Mr. Black and Mr. White and their respective wives. The partner of White was his daughter. Pink played against his mother. Black's partner was his sister. Mrs. Green played against her mother. Pink and his partner have the same mother. Green's partner was his mother-in-law. No player had an uncle participating. Who plays with whom at what table?

Calendar Math

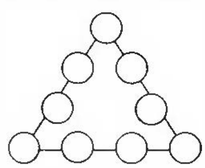
Art Jorgensen

Here are the math exercises for the month of June 2000.

1. The following is an addition problem.

   S E N D Each letter has a unique value.
   M O R E
   M O N E Y

2. Using the digits 1, 2, 3, 4, 5, 6 and 7 exactly once each, make an addition problem for which the sum is 100.
3. The length of a rectangle is increased by 10 percent and the width decreased by 20 percent. By what percent, either up or down, is the original area changed?
4. Tom has 19 checkers. How many ways can he arrange them into three piles so that each pile contains an odd number of checkers?
5. In her pocket Susan has fewer than 10 coins totaling 75 cents. What are all possible combinations of coins that she could have? Use only quarters, dimes, nickels and pennies.
6. Have the students list and then graph their favorite TV programs.
7. Bimba's birthday falls on a Monday. His friend Sibongile's birthday is 25 days later. On what day of the week is Sibongile's birthday?
8. Markerville and Centreville are 700 km apart. Tom leaves Markerville at 8 a.m. traveling at 75 km per hour. At the same time Leslie leaves Centreville traveling at 100 km per hour. At what time will they meet? How far from Centreville will they meet?
9. John and Mary are going for lunch. John who has $22 will pay for the lunch. If he wants to leave a 10 percent tip, what is the most he can pay for lunch?
10. What are the next three terms in the following sequence? 2, 9, 28, 65, ___, ___, ___.
11. Barbara buys 3 candy bars on sale for 99 cents. If she had paid the regular price, it would have cost her 39 cents more. What was the regular price of each candy bar?
12. Draw a triangle with sides of 5 cm, 7 cm and 13 cm. What did you discover?
13. In the barnyard there are 9 animals. They are either cows or chickens. Altogether there are 28 legs. How many cows and how many chickens are there?
14. How many times in a 24-hour period will a digital clock show 3 consecutive numbers, that is, 1:23?
15. The sun rises at 7:30 a.m. Sunday. It rises 3 minutes earlier each day. At what time will it rise the following Sunday?
16. A bag contains 75 candies composed of 30 red candies, 20 green candies and 25 white candies. If Martin draws 15 candies from the bag, how many of each color will there be if the ratio remains the same?
17. Place all the numbers from 1 to 9 in the enclosed circles, so as to obtain the same sum in each direction.



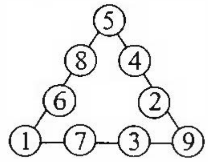
18. Mary is 8 years older than her brother. In 4 years she will be twice as old as her brother. How old is Mary?
19. Susan washes 75 windows in 7.5 hours. With Rose's help it takes 5 hours. How long would it take Rose alone?
20. 73.156 minus 28.499 = ________, rounded to the nearest tenth.
21. 1000, 600, 360, 216, ... Rounded to the nearest whole number, what is the seventh term in the sequence?
22. Find a perfect square that when divided by 5 has a remainder of one.
23. $1^{0.25} + 16^{0.5} =$
24. A and B together possess $570. If A's money were three times what it really is, and B's five times what it really is, the sum would be $2,350. What is the amount that each possesses?
25. One fifth of the class wanted oranges, 40 percent wanted apples and the rest wanted bananas. What percent of the class wanted bananas?

26. On a scaled map, 1 cm = 60 km. Using this scale, what distance will a line 105 mm long represent?
27. Find the product of the cube roots of 8 and 64.
28. Two cyclists start from the same point at the same time and travel in opposite directions. One cyclist travels at 16 km/h. After 1.5 hours they are 60 km apart. What is the speed of the second cyclist?
29. Find the quotient when the largest factor of 12 is divided by an even prime.
30. _?_30 on a 24 hour clock is equal to 2:30 p.m.?

Teachers are encouraged to have students find different answers and problem-solving approaches to the problems. With minimal changes, many of the problems can be adapted to students at various grade levels.

Answers

1.
```
  9,567
 +1,085
 10,652
```
2. 52 + 34 + 1 + 6 + 7. There may be more solutions.
3. Reduced by 12 percent
4. I found 9 ways. Can you find more?
5. I found 10 solutions. Can you find any more? I made a table.
6. Graphs will vary.
7. Friday
8. a. noon
   b. 400 km
9. $20
10. 126, 217, 344
11. 46 cents
12. The triangle can't be drawn. The combined length of the two shortest sides must be longer than the length of the third side.
13. 5 cows, 4 chickens
14. 14 times
15. 7:09 a.m.
16. 6 red, 4 green, 5 white
17.



18. Mary is 12 years old.
19. 11.25 hours
20. 44.7
21. 47
22. 36
23. 5
24. A has $250 and B has $320.
25. 40 percent
26. 630 km
27. 8
28. 24 km/hr
29. 6. Using 12 as a factor of itself. Two is the only even prime.
30. 14

Square Numbers

Is it possible to determine, without a calculator, whether the number 3 141 592 653 589 793 is a square number?

An Enigmatic E-Mail Riddle with an Explanation Using Modular Arithmetic

Murray Lauber

The following riddle was circulated by e-mail at Augustana University College by Tom Bateman of the political studies department. Perhaps you have already encountered it.

This is pretty cool, but you must follow directions!

Scroll down slowly!

Do not skip ahead. Read this message *one line at a time* and just do what it says. You will be glad you did. If not, you will be sorry and you wished you had listened.

1. Pick a number from 1 to 9
2. Subtract 5
3. Multiply by 3
4. Square the number (multiply by the same number, not square root)
5. Add the digits until you get only one digit (that is, $64 = 6 + 4 = 10 \rightarrow 1 + 0 = 1$)
6. If the number is less than 5, add 5. Otherwise, subtract 4
7. Multiply by 2
8. Subtract 6
9. Map the digit to a letter in the alphabet 1 = A, 2 = B, 3 = C and so on
10. Pick a name of a country that begins with that letter
11. Take the second letter in the country name and think of a mammal that begins with that letter
12. Think of the color of that mammal

(Keep scrolling.)

DO NOT SCROLL DOWN UNTIL YOU HAVE DONE ALL OF THE ABOVE!

Here it comes, NO CHEATING or you will be sorry. Now scroll down.

You have a grey elephant from Denmark. Right?

The Enigma

Doesn't it seem rather mysterious that you should get the same answer as someone who started with a different number?

The Explanation

The following is an explanation based on responses sent to Tom Bateman by me and by Bill Hackborn, a colleague of mine in Augustana's math department.

After picking a number from 1 to 9 and subtracting 5 in steps 1 and 2, one has a number from -4 to 4. Then, after multiplying by 3 and squaring, one obtains one of 0, 9, 36, 81 or 144. The sum of the digit 0 is obviously 0. The remarkable thing about the other four numbers, and the fact that allows the riddle to work, is that in each case the sum of their digits is 9. The result of summing the digits in step 5 is thus either a 0 or a 9. After adding a 5 (or subtracting a 4) in step 6, everyone will then have the same answer, 5. From there on, everyone participating is on common ground, achieving the number 4 at step 8, and the corresponding letter D at step 9. For those with a modest knowledge of geography and mammalogy, the D suggests Denmark and the E an elephant. For most, the associated color is grey.

The Mathematical Basis

It turns out that the restriction on the size of the number in step 1 is not necessary. The riddle will work for any number. Steps 1 and 2 are used only to keep the computations simple.[1] The square of a multiple of 3 is always a multiple of 9. Further, if the digits of any multiple of 9, regardless of its size, are summed recursively, the sum will be a multiple of 9. For example, $3{,}888 = 9 \times 432$, a multiple of 9. Summing its digits recursively, one obtains $3 + 8 + 8 + 8 = 27 \rightarrow 2 + 7 = 9$.

Proof (for a 4-digit multiple of 9): the following is a proof of this property for any 4-digit multiple of 9.

Suppose that $N = 9m$, m an integer, is a multiple of 9. Let the decimal representation of N be $T\ h\ t\ u$, where T is the thousands digit, h is the hundreds digit, t the tens digit and u the unit digit.

Then $9m = 1000T + 100h + 10t + u$

$\Rightarrow \quad 9m = (999 + 1)T + (99 + 1)h + (9 + 1)t + u$

$\Rightarrow \quad 9m = 999T + T + 99h + h + 9t + t + u$

$\Rightarrow \quad T + h + t + u = 9m - 999T - 99h - 9t$

$\Rightarrow \quad T + h + t + u = 9(m - 111T - 11h - t)$

$\Rightarrow \quad T + h + t + u$ is a multiple of 9.

Thus, the sum of the digits of any 4-digit multiple of 9 is itself a multiple of 9. Summing the digits recursively will eventually result in one of the single-digit multiples of 9, namely, 0 or 9.

It is easy to see how this proof could be adapted to a multiple of 9 with any number of digits. There is a more general theorem that extends these results even to numbers in other bases, but it involves concepts of modular arithmetic that are too extensive to include here.

Note

1. In my response to Tom Bateman, I suggested that restricting the size of the number in step 1 and subtracting 5 from it in step 2 were useful in keeping the computations within the single digit capabilities of today's generation of students and professors.

References

Lauber, M. R. "Casting Out Nines—An Explanation and Extensions." *The Mathematics Teacher* 83, no. 8: 661–65.

——. "Checking Polynomial Arithmetic: Casting Out Nines Reincarnated." *delta-K* 31, no. 2 (April 1993): 18–22.

Meserve, B. E., et al. *Contemporary Mathematics*. 4th ed. Englewood Cliffs, N.J.: Prentice Hall, 1987, 160–67.

Setek, W. M. *Fundamentals of Mathematics*. 7th ed. Englewood Cliffs, N.J.: Prentice Hall, 1996, 276–83.

Fireworks

My grandfather really likes fireworks. This year he bought as many rockets as his age in years to really celebrate his birthday. However, half the rockets became wet, the grandchildren borrowed one third of the rockets for their bush party and 21 rockets were without the ignition device. "It does not matter," said my grandfather, "that means that instead of one rocket for each year, each rocket has to count 10 years." This would really be a poor show. We will have a better fireworks show when my grandfather turns 100. When will that be?

The Birthday Problem Extended

Sandra M. Pulver

It is interesting to see how the probability of at least two people in a group having the same birthday can be determined by using the classic birthday problem. The probability of success is much greater than most people would expect.

But how do we figure out more specific questions like, What are the chances of getting exactly three or four or any other number of parts of matches? This problem can be solved through an extension of the basic birthday problem to k matches.

The birthday problem is a classic example used widely in classrooms to demonstrate the principles of probability. The earliest version of this problem was developed in 1939 by Von Mises. The standard problem asks to find the probability, in a group of n individuals, that at least two of them will have the same birthday. The outcome is quite surprising because the probability of success is way beyond what most people would guess.

We compute P (at least 2 of n persons share the same birthday) as 1 − P (no people share the same birthday).

Assuming there are 365 equally likely possibilities and n people, the number of possible birthday combinations is 365. To solve for the probability of nonmatches, we calculate the probability that each individual does not match any proceeding person.

Considering the individuals arranged in order I_1, I_2, I_3...I_n, I_1 can have any of the 365 birthdays, with the probability of no matches being 365/365. I_2 can have any of the 364 birthdays and not match I_1. I_3, can have any one of 363 birthdays and ... I_n can have any of 365 − (n − 1) birth dates not used.

The probability that no two people in group size n share the same birthday is

$$= \frac{365}{365} \times \frac{364}{365} \times \dots \times \frac{(365 - (n - 1))}{365} = \frac{365!}{(365)^n (365 - n)!}$$

If n is 23, the probability of no matches is 0.4927, and so, the probability of the successful outcome is 1 − 0.4927 = 0.5073, which is much higher than what most people would expect.

The probability of at least one match for various n can be found in the table.

Number of People (n)	Probability of at Least One Match
5	0.0271
10	0.1169
15	0.2529
20	0.4114
23	0.5073
25	0.5687
80	0.9999

In the basic birthday problem, we solved for the probability of at least one match. This problem can be extended to a special case of Von Mises' problem:

> Determine the probability of exactly k pairs of matches in a group size n.

Let P (k) denote the probability of exactly k pairs of matches (no triple or higher matches) in a random group of size n.

Two basic principles of probability are used to develop the formula. First, the probability of an event equals the number of outcomes that meet the condition multiplied by the probability of a particular case that meet the required conditions. (This assumes that each outcome is equally likely and mutually exclusive.) For instance, the probability of 3 heads in 5 tosses of a coin that results in heads with probability p is $\binom{5}{3} p^3 (1 - p)^2$. The $p^3 (1 - p)^2$ is the probability of a specific sequence and the $\binom{5}{3}$ is the number of equally likely and mutually exclusive sequences.

Second, the probability of a sequence of events is the product of each event's conditional probability.

$$P(E_1 E_2 \dots E_k) = P(E_1) \cdot P(E_2/E_1) \cdots P(E_k/E_1 E_2 \dots E_{k-1})$$

We calculate the probability of P_0 (no matches) in a group of size n.

$$P_0 = P(I_1 \neq I_{k<1}) \cdot P((I_2 \neq I_1) \cdot P(I_3 \neq I_1 I_2) \cdots P(I_n \neq I_1 I_2 \ldots, I_{n-1})$$

$$= \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdots \frac{365-(n-1)}{365}$$

$$= \frac{365!}{(365)^n\,(365-n)!}$$

Next, we compute the probability P_1, of exactly one match. In a group of size n, there are $\binom{n}{2}$ ways of picking the individuals to have the one match and there are $\binom{365}{1}$ ways of picking the exact date to be matched.

Let P' be the probability that a particular date is matched by a specific pair in the group. For instance let P'_1 be the probability that the first two individuals were the only ones born on January 1. The probability that both I_1 and I_2 have birthdays January 1 is $(\frac{1}{365})(\frac{1}{365})$. The probability that the third person, I_3 does not match the first two individuals is $(\frac{364}{365})$, and the probability that the fourth person I_4 does not I_1 I_2 or I_3 is $(\frac{364}{365})$. Thus, the probability that the nth person does not match any of the preceding birth dates is $\frac{[365-(n-2)]}{365}$.

Therefore,

P' = P (I_1 born January 1) · P(I_2 born January 1) · P(I_3 does not match I_1 or I_2) · ... · P(I_n does not match the proceeding n − 1 individuals) =

$$= \frac{1}{365} \cdot \frac{1}{365} \cdot \frac{364}{365} \cdots \frac{[365-(n-2)]}{365}$$

We solve for P_1 by multiplying P'_1 by the number $\binom{n}{2}$ of ways of selecting the matching pair, then by $\binom{365}{1}$ of ways of choosing the matching date.

$$P_1 = \binom{n}{2}\binom{365}{1}\frac{364!}{(365-n+1)!\,(365)^n} = \frac{\binom{n}{2}\,365!}{(365-n+1)!\,(365)^n}$$

We use these same principles to solve for P_2 of exactly two matching pairs. There are $\binom{n}{2}$ ways of choosing the first matching pair, and from the remaining n − 2 individuals, there are $\binom{n-2}{2}$ ways of choosing the second matching pair. There are $\binom{365}{2}$ ways of choosing the dates to be matched. Let P'_2 be the probability that I_2 and I_5 have birthdays on January 1 and I_4 and I_7 have birthdays on January 2, and no other matches exist.

P' = P(I has birth date other than January 1 or 2) · P(I was born January 1) · P(I has birth date other than January 1 or 2 and different from I) · P(I was born January 2) · P(I was born January 1) · P(I was born January 1) · P(I was not born January 1 or 2 or on the birth dates of I or I) · P(I was born on January 2) · P(I has different birthday from preceding individuals) · ... · P(I has different birth date from the proceeding individuals)

$$= \frac{363}{365} \cdot \frac{1}{365} \cdot \frac{362}{365} \cdot \frac{1}{365} \cdot \frac{1}{365} \cdot \frac{361}{365} \cdot \frac{1}{365} \cdot \frac{360}{365} \cdots \frac{(365-n+3)}{365}$$

$$= \frac{365!}{(365)^n\,(365-n+2)!}$$

To compute P_2, we multiply P'_2 by the $\binom{n}{2}\binom{n-2}{2}$ ways of selecting the individual pairs that match and then by $\binom{365}{2}$ number of ways of choosing the dates of matches.

$$P_2 = \binom{n}{2}\binom{n-2}{2}\binom{365}{2} \cdot \frac{363!}{(365-n+2)!\,(365)^n} = \frac{\binom{n}{2}\binom{n-2}{2}\,365!}{2(365)^n\,(365-n+2)!}$$

Finally, to calculate the probability P_k of exactly k matching pairs, we first recall that there are $\binom{n}{2}$ ways of choosing the first individuals to match, $\binom{n-2}{2}$ ways of choosing the next matching pair form the remaining (n − 2) individuals, and $\binom{n-2(k-1)}{2}$ ways of picking the kth matching from the remaining n − 2(k − 1) individuals after the first (k − 1) pairs have been picked. Also, there are $\binom{365}{k}$ ways of picking the specific birthdays to be matched. Letting P'_k be the probability that I_1 and I_2 have birthdays on January 1, I_3 and I_4 have birthdays on January 2 and so on, and none of the remaining n − 2k individuals match anyone else.

P'_k = P(I_1 born January 1) · P(I_2 born January I) · P(I_3 born January 2) · P(I_4 born January 2) · ... · P(I_{2k-1} born the kth day of the year · P(I_{2k} born the kth day of the year) · P(I_{2k+1} does not match proceeding birthdays) · ... · (I_n does not match any of the proceeding birthdays)

$$= \frac{(365-k)!}{(365)^n\,(365-n+k)!}$$

There are $\binom{n}{2}\binom{n-2}{2} \ldots \binom{n-2(k-1)}{2}$ ways of picking the pairs of individuals to match birth dates and there are ways of choosing the specific dates that are matched. Thus,

$$P_k = \binom{n}{2}\binom{n-2}{2} \cdots \binom{n-2(k-1)}{2}\binom{365}{k} P'_k$$

$$= \frac{\binom{n}{2}\binom{n-2}{2} \cdots \binom{n-2(k-1)}{2}\binom{365}{k}(365-k)!}{(365-n+k)!\,(365)^n}$$

$$= \frac{n!\;365!}{(365)^n\,k!\,2^k\,(n-2k)!\,(365-n+k)}$$

= Probably of exactly k matching pairs of birthdays.

Thus, we can compute the probability of k pairs of numbers being exactly the same. For example, in a group of 100 people, the chances of having 1 match is:

$$P_1 = \frac{100!\;365!}{(365)^{100}\,1!\,2(98)!\,(365-100+1)!}$$

When this is computed, we can see that the chances of having exactly one match in a group of 100 people is extremely slim.

Bibliography

Abramson, M., and O. J. Moser. "More Birthday Surprises." *The American Mathematical Monthly* 77 (October 1970): 856–58.

Hirschhorn, M. D. "A Birthday Present for Ramanujan." *The American Mathematical Monthly* 97 (May 1990): 398–400.

Hocking, R. L., and N. C. Schwertman. "An Extension of the Birthday Problem to Exactly k Matches." *The College Mathematics Journal* 17 (January 1986): 315–21.

Jones, K. "The Birthday Problem Again?" *The Mathematics Teacher* 86 (May 1993): 373–77.

Parzen, E. *Modern Probability Theory and Its Applications*. Toronto: John Wiley, 1960.

Spencer, N. "Celebrating the Birthday Problem." *Mathematics Teacher* 70 (April 1997): 348–53.

The Circle on the Chessboard

What is the radius of the largest circle constructed on a chessboard in such a way that the perimeter of the circle lies entirely in the white squares? Where is the location of the circle's centre? The sides of the squares on the chessboard are one unit long.

Using a Graphing Calculator to Compute Interest Rates

David R. Duncan and Bonnie H. Litwiller

Teachers are always alert to real-world situations in which interest rates are involved. It is advantageous if a graphing calculator can be employed.

Suppose that Joel is considering purchasing a yard tractor with a cash price of $23,650. A financing plan is available: a down payment of $5,500 and five subsequent yearly payments of $5,500. If Joel takes advantage of this financing opportunity, what annual interest rate would he be paying?

To analyze this problem let

P = the original price after the down payment. In this case P = $18,150.
R = the yearly payment. In this case R = $5,500.
r = the yearly interest rate.

Next we must express all of the money at the same point in time; that is, we must move each yearly payment to the time of purchase immediately after the down payment. The present values of the five yearly payments are, respectively,

$$\frac{R}{(1+r)};\ \frac{R}{(1+r)^2};\ \frac{R}{(1+r)^3};\ \frac{R}{(1+r)^4};\ \frac{R}{(1+r)^5}.$$

The original price after the down payment then is equal to the sum of the five present values; that is,

$$P = \frac{R}{(1+r)} + \frac{R}{(1+r)^2} + \frac{R}{(1+r)^3} + \frac{R}{(1+r)^4} + \frac{R}{(1+r)^5}.$$

Call this equation 1.

Multiplying both sides of equation 1 by (1+r) yields:

$$P(1+r) = R + \frac{R}{(1+r)} + \frac{R}{(1+r)^2} + \frac{R}{(1+r)^3} + \frac{R}{(1+r)^4}.$$

Call this equation 2.

Subtracting equation 2 from equation 1 yields:

$$P[1-(1+r)] = \frac{R}{(1+r)^5} - R$$

or $$P(-r) = \frac{R}{(1+r)^5} - R.$$

Then: $$\frac{R}{(1+r)^5} - R + P(r) = 0$$

And: $$R - R(1+r)^5 + P(r)(1+r)^5 = 0$$

Thus: $$R[1-(1+r)^5] + P(r)(1+r)^5 = 0.$$

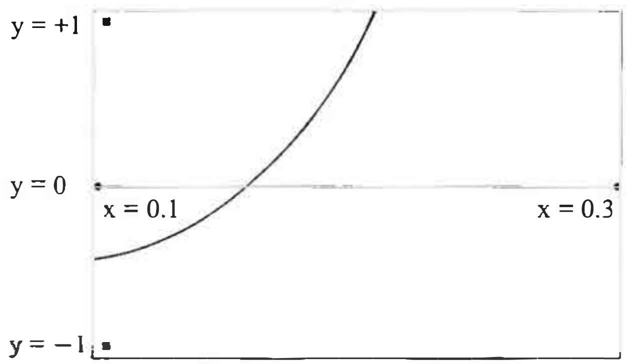
Since P and R are known, it remains only to solve this equation for r. Since this is a 6th degree equation in r, we used the TI-85 to graph:

$y = 5500[1-(1+x)^5] + 18150\,x(1+x)^5$.

To reduce the range, we actually graphed

$y = 5.5[1-(1+x)^5] + 18.15\,x(1+x)^5$.

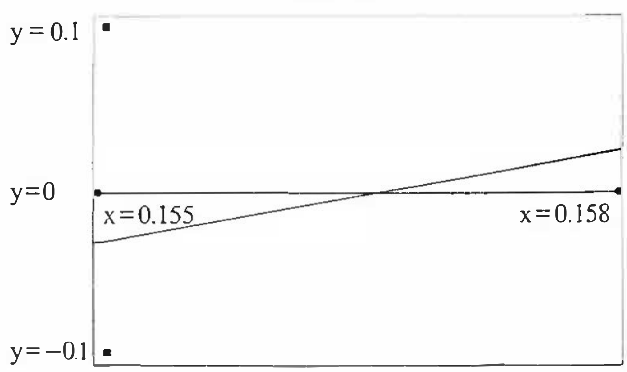
Graph 1



Window $0.1 \le x \le 0.3$
$-1 \le y \le 1$

The x-intercept of this graph will be the interest rate in decimal form. Using the trace key, x is between 0.1556 and 0.1571. To determine a closer approximation, redefine the window. Let $0.155 \le x \le 0.158$
$-0.1 \le y \le 0.1$

Graph 2



The x-intercept is then between .15664 and .15667. The interest rate is thus approximately 15 ²/₃ percent.

To illustrate the operation of this periodic interest process, let us build an amortization table for Joel. The first year proceeds as follows:

1. At the beginning of the first year, the principal owed is $18,150.
2. At the end of the first year, the annual payment of $5,500 is made.
3. At the end of the first year, the interest due is 15 $^{2}/_{3}$ percent of $18,150 or $2,843.50. This part of the first annual payment is used to pay interest and does not reduce the principal.
4. The difference between the annual payment ($5,500) and the interest due ($2,843.50) is then used to reduce the principal. This difference is $2,656.50.
5. Reducing the original balance by $2,656.50 leaves a closing balance after the first year of $15,493.50.
6. This closing balance from year 1 becomes the original balance for year 2. These steps are then repeated for years 2, 3, 4 and 5.

Challenges

1. Is the 15 $^{2}/_{3}$ percent a good interest rate for Joel to pay?
2. Apply the same process to home mortgages. If you know the initial price of a home and the monthly payment of a friend's mortgage, calculate the interest rate that your friend is paying.

Table 1
The Entire Amortization Process

Year	Original Balance	Total Annual Payment	Interest Paid	Principal Paid	Closing Balance
1	$18,150.00	$5,500.00	$2,843.50	$2,656.50	$15,493.50
2	15,493.50	5,500.00	2,427.32	3,072.68	12,420.82
3	12,420.82	5,500.00	1,945.93	3,554.07	8,866.75
4	8,866.75	5,500.00	1,389.12	4,110.88	4,755.87
5	4,755.87	5,500.00	745.09	4,754.91	0.96*

* Note the rounding errors which result from the rounded value of r.

A Strange Sequence

What is the rule by which the following sequence of numbers has been created? What are the next three elements of this sequence?

1, 11, 21, 1211, 111221, 312211, 13112221, 1113213211, ...

On Centroids, Medians and Lines of Symmetry

David E. Dobbs

To make room for the innovations incurred by calculus reform, some traditional topics have been de-emphasized in recent curricula. One such topic concerns centres of mass, more specifically, centroids and moments, although this material remains important in areas such as engineering and statistics. The gradual de-emphasis of this material is apparent if one compares some leading calculus texts of different eras. In this regard, consider the result that justifies the "centroid" terminology: the centroid of any triangular region in the plane is the classical centroid (that is, the intersection of the medians) of the triangle bounding the region. This result was proved in the leading text, *Calculus with Analytic Geometry* (Johnson and Kiokemeister 1969, 404, example 2) 30 years ago, relegated to the exercises in another leading text, *Calculus* (Stewart 1991, 506, exercise 23) a decade ago, but omitted from the reform version *Calculus (Concepts and Contexts, Single Variable)* (Stewart 1998) of *Calculus* (Stewart 1991). One purpose of the present article is to sketch a self-contained proof of this theorem (see following section). The main purpose is to investigate a possible explanation for this result on triangles in terms of principles applicable to more general planar regions. In particular, consider the Symmetry Principle (which was stated in Stewart 1991 and 1998 and was often proved in the more rigorous texts of yesteryear [Leithold 1972, theorem 7.8.1]: if L is a line of symmetry of a planar region R, then the centroid of R lies on L. In view of the result in the next section, it is natural to ask whether the medians of a triangle Δ are lines of symmetry of the planar region bounded by Δ. The answer is in the negative (and so the result in the next section is still needed), for we show in the subsequent section that the medians of a triangle Δ are lines of symmetry for the region bounded by Δ if and only if Δ is equilateral. The material in the subsequent section can be used to enrich the precalculus unit on symmetry, as well as the precalculus topics of equations of lines and solution of linear inequalities and systems of linear equations. It would also reinforce the geometric/graphical approach in any first-year reform calculus sequence. The second section can be read independently of the third section. The material in the second section can be used to enrich the calculus unit on applications of the definite integral; this material also reinforces the solution of linear systems.

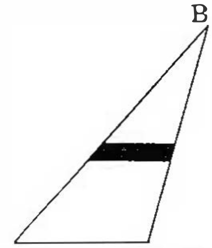
Centroids Are Centroids

The centroid $(\bar{x}, \bar{y})$ of a planar region R is given by $\bar{x} = M_y / A$ and $\bar{y} = M_x / A$, where M_y (resp., M_x) is the moment of R with respect to the $y-$ (resp., $x-$) axis and A is the area of R. If the boundary of R consists of a triangle Δ, Theorem 1 establishes that $(\bar{x}, \bar{y})$ is the intersection of the medians of Δ. A relatively short proof of Theorem 1 is available by using facts about similar triangles [(Johnson and Kiokemeister 1969, 404, example 2, 404), but the proof sketched below uses only equations of lines and calculus.

Theorem 1. *If a triangle Δ is the boundary of a planar region R, then the centroid of R is the intersection of the medians of Δ.*

Proof. For simplicity, we suppose that $\Delta = \Delta ABC$ has a vertex at the origin and a horizontal side, as in Figure 1.

Figure 1



If $D(a/2,0)$ denotes the midpoint of AC, then the median of Δ which passes through B is the line BD, whose point-slope equation is

$$y - c = \left(\frac{c - 0}{b - \frac{a}{2}}\right)(x - b).$$

Similarly, the median of Δ which passes through A has equation $y = (c/(b+a))x$. Solving these equations simultaneously, we obtain that the classical centroid of Δ has coordinates $((a+b)/3, c/3)$. For a class needing assurance that the medians of Δ are concurrent, note that the given coordinates also satisfy $y = [(-c/2)/(a-b/2)](x-a)$, an equation for the median of Δ which passes through C. On the other hand, for a class that knows that the medians of Δ meet at the point $^2/_3$ of the way from B to D, one need only observe that

$$\left(\frac{1}{3}b + \frac{2}{3}\frac{a}{2}, \frac{1}{3}c + \frac{2}{3}\cdot 0\right) = \left(\frac{a+b}{3}, \frac{c}{3}\right).$$

The area of R (or Δ) is $A = ac/2$. Since $(\bar{x}, \bar{y}) = (M_y/A, M_x/A)$, a proof that $(\bar{x}, \bar{y}) = ((a+b)/3, c/3)$ reduces to showing that

$$M_y = \frac{ac(a+b)}{6} \text{ and } M_x = \frac{ac^2}{6}.$$

Consider a horizontal strip, as in Figure 1. The area of the strip is $dA = (x_2 - x_1)\,dy$, where x_2 (resp., x_1) is the expression for x in terms of y obtained by solving for x in a point-slope equation of BC (resp., BA). One readily finds that

$$x_2 = \left(\frac{b-a}{c}\right)\left(y + \frac{ca}{b-a}\right) \text{ and } x_1 = \frac{by}{c}.$$

Now, M_y is the definite integral of $x'\,dA$, where $x' = (x_2 + x_1)/2$ is the $x-$ coordinate of the geometric centre of the horizontal strip. Thus,

$$M_y = \int x'\,dA = \int \left(\frac{x_2 + x_1}{2}\right)(x_2 - x_1)\,dy$$

$$= \int_0^c \left(\frac{\frac{(b-a)}{c}\left(y + \frac{ca}{b-a}\right) + \frac{by}{c}}{c}\right)\left(\left(\frac{b-a}{c}\right)\left(y + \frac{c\alpha}{b-\alpha}\right) - \frac{by}{c}\right)dy.$$

After the integrand is algebraically simplified as a polynomial in y, a routine integration via the Fundamental Theorem of Calculus yields that $M_y = (abc+a^2c)/6 = ac(a+b)/6$, as required.

Finally, $M_x = \int y'\,dA$ where y', the second coordinate of the geometric centre of the horizontal strip, is essentially $y' = y'$. Thus,

$$M_x = \int_0^c y\left(\left(\frac{b-a}{c}\right)\left(y + \frac{c\alpha}{b-\alpha}\right) - \frac{by}{c}\right) dy = ac^2/6$$

where the final equality follows by another routine application of the Fundamental Theorem of Calculus. The proof is complete.

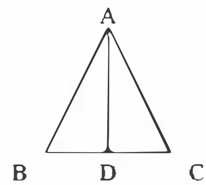
When Medians Are Lines Of Symmetry

Unfortunately, Theorem 1 is not a consequence of the Symmetry Principle since the medians of a triangle Δ are, in general, not lines of symmetry of the region bounded by Δ. As Theorem 4 and Corollary 5 document, requiring such symmetry restricts the nature of Δ severely. First, we isolate two pieces of the argument.

Lemma 2. *Let Δ = Δ ABC be an isosceles triangle, with AB=AC. Let D be the midpoint of BC, and let R be the planar region bounded by Δ. Then the median AD is perpendicular to BC, and AD is a line of symmetry of R.*

Proof. The data are summarized in Figure 2.

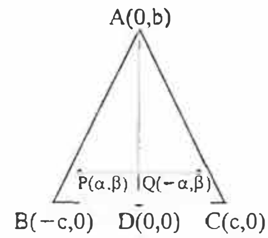
Figure 2



Since $AB=AC$ by hypothesis and $BD=CD$ by the definition of midpoint, it follows from the SSS (Side-Side-Side) congruence criterion that $\Delta ABD \cong ACD$. Therefore, $\angle ADB \cong \angle ADC$, and so $AD \perp BC$.

It remains to show that R is symmetric about AD. For convenience, locate the coordinate axes so that D is the origin and C is on the positive $x-$ axis. (Then AD falls along the $y-$ axis.) Consider an arbitrary point $P(\alpha,\beta)$ in R. Then $Q(-\alpha,\beta)$ is the point symmetric to P with respect to AD. Our task is to show that Q is in R. The data are summarized in Figure 3.

Figure 3



We show that if P is inside ΔABD, then Q is inside ΔACD. While this may seem clear "pictorially," an analytic proof depends on the meaning of "inside" and the fact that the solution sets of linear inequalities are half-planes. Observe that an equation for AB (resp., AC) is $x/(-c)+y/b = 1$ (resp., $x/c + y/b = 1$). The hypothesis that P is in the interior of ΔABD means that P is in the appropriate three half-planes determined by the sides of ΔABD, as follows:

$\alpha/(-c) + \beta/b < 1$, $\beta > 0$, and $\alpha < 0$.

The assertion that Q is in the interior of ΔACD means that $(-\alpha,\beta)$ satisfies the inequalities describing the appropriate three half-planes determined by the sides of ΔACD, as follows:

$(-\alpha)/c + \beta/b < 1$, $\beta > 0$, and $-\alpha > 0$.

These conditions are implied by (in fact, equivalent to) the inequalities imposed by the hypothesis on P, and so the proof is complete.

Lemma 3. Let $\Delta = \Delta ABC$ be a triangle, let D be the midpoint of BC, and let R be the planar region bounded by Δ. Suppose that the median AD is perpendicular to BC. Then Δ is isosceles, with $AB=AC$, and AD is a line of symmetry of R.

Proof. The data are summarized in Figure 2. Since $AD \perp BC$, the angles $\angle ADB$ and $\angle ADC$ are right angles, and hence are congruent to one another. Moreover, $DB=DC$ by the definition of midpoint. It now follows from the SAS (Side-Angle-Side) congruence criterion that $\Delta ADB \cong \Delta ADC$, and so $AB=AC$. An application of Lemma 2 completes the proof.

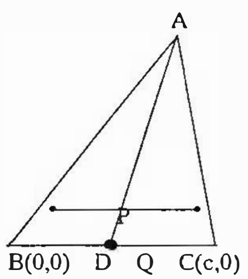
We next present our main result.

Theorem 4. *Let $\Delta = \Delta ABC$ be a triangle, D the midpoint of BC, and R the planar region bounded by Δ. Then the following three statements are equivalent:*

1. *The median AD is a line of symmetry of R;*
2. *The median AD is perpendicular to BC;*
3. *Δ is an isosceles triangle, with $AB=AC$.*

Proof. (2)⇒(3) by Lemma 3, while (3)⇒(1) by Lemma 2. It remains only to prove that (1)⇒(2). We shall prove the contrapositive. Assume, then, that AD has well-defined slope. We shall show that (1) fails by producing a point P in the interior of ΔABD (and hence in R) such that Q, the point symmetric to P with respect to AD, is not in R. There is no harm in locating the coordinate axes so that B is the origin and $C(c,0)$ is on the positive $x-$axis. We may also suppose that $L=AD$ has positive slope. (The proof in the case of negative slope is similar; alternatively, turn the page over!) Finally, let S denote the point of intersection of L and PQ. The data are summarized in Figure 4.

Figure 4



It will be enough to take $S=D$. First, let M denote the ray emanating from D such that M is perpendicular to L and M enters the half-plane determined by L and B. We shall find a suitable P on M. Indeed, since $\tan(\angle ADC)=\text{slope}(L) > 0$, $\angle ADC$ is an acute angle and so its supplement, $\angle ADB$, must be an obtuse angle. Hence, M enters the interior of $\angle ADB$. In particular, M enters the interior of $\angle ABD$. Choose P to be any of the infinitely many points of M which lie in the interior of $\angle ADB$. Then Q, the point symmetric to P with respect to L, has negative $y-$ coordinate, and so Q is not in R. This completes the proof of the contrapositive of (1)⇒(2).

Next, we state a result which was announced in the introduction. The proof of Corollary 5 is immediate from Theorem 4.

Corollary 5. *Let Δ be a triangle and let R be the planar region bounded by Δ. Then the following three statements are equivalent:*

1. At least two of the medians of Δ are lines of symmetry of R;
2. All three medians of Δ are lines of symmetry of R;
3. Δ is an equilateral triangle.

In closing, we indicate an exercise that goes beyond what was established in the proof of Theorem 4. By filling in the details of what is sketched in Remark 6, one would further reinforce the topic of graphs of linear inequalities, as well as basic facts about limits.

Remark 6. Let us return to the context addressed in Figure 4, namely, the proof of the contrapositive of (1)⇒(2) in Theorem 4. Here, we expand upon the assertion that infinitely many points $P(\alpha,\beta)$ in the interior of ΔABD are such their corresponding symmetric points $Q(\gamma,\delta)$ are not in R. Let A have coordinates (a,b). One can prove that there exists ε_1, $0<\varepsilon_1< a-c/2$ with the following properties. If $0<\varepsilon\leq\varepsilon_1$ then $S(c/2+\varepsilon, k)$ is such that P is in the interior of ΔABD (and hence in R) and $\delta < 0$ (so that Q is not in R).

The reasonableness of the preceding assertion can be indicated heuristically by applying a paper-folding experiment to Figure 4. A proof can be fashioned by considering the function of ε, for $0 < \varepsilon < a-c/2$, given by

$$\mu = \frac{\frac{bc}{2a} - \frac{2b\varepsilon}{2a-c} - \frac{(2a-c)\varepsilon}{2b}}{\frac{(2a-c)\varepsilon}{2b} + \frac{b\varepsilon}{a}}.$$

Put $\alpha = c/2 - \lambda\varepsilon$, where λ is a positive real number to be further specified later; for the moment, we require that $0 < \lambda < c/(2\varepsilon)$. Since $k = 2b\varepsilon/(2a-c)$ and $PS \perp L$, the "negative reciprocal" result leads to

$$\beta = \frac{(\lambda+1)\varepsilon(2a-c)}{2b} + \frac{2b\varepsilon}{2a-c}.$$

By considering appropriate half-planes, one shows that P is in the interior of ΔABD if and only if

$$0 < \beta < \frac{b}{a}\left(\frac{c}{2} - \lambda\varepsilon\right);$$

equivalently, if and only if $\lambda < \mu$. Observe via standard limit theorems that

$$\lim_{\varepsilon\to 0^+} \mu = \infty \text{ and } \lim_{\varepsilon\to 0^+} \frac{\mu}{\left(\frac{c}{\varepsilon}\right)} = \frac{\frac{b}{2a}}{\frac{2a-c}{2b} + \frac{b}{c}} < \frac{1}{2}.$$

Thus, putting $\lambda = \mu/2$, we can choose ε (and implicitly ε_1) so that

$$\max\left(\frac{8b^2}{(2a-c)^2}-1,0\right) < \lambda < \min\left(\frac{c}{2\varepsilon},\mu\right).$$

One then verifies the assertions concerning P and Q.

References

Johnson, R. E., and F. L. Kiokemeister. *Calculus with Analytic Geometry*. 4th ed. Boston: Allyn & Bacon, 1969.

Leithold, L. *The Calculus with Analytic Geometry*. Part I, 2d ed. New York: Harper & Row, 1972.

Stewart, J. *Calculus*. 2d ed. Pacific Grove, Calif.: Brooks/Cole, 1991.

———. *Calculus (Concepts and Contexts, Single Variable)*. Pacific Grove, Calif.: Brooks/Cole, 1998.

Area Between Two Circles

What is the area between two circles, one of which circumscribes a regular heptagon and the other of which is inscribed in a regular heptagon? The length of each side of the heptagon is 1 cm.

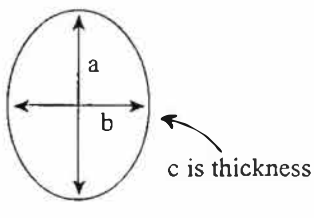
Pebble Power: Investigating Stone Shapes

Richard T. Edgerton

My children, ages 4 and 7, have a habit of collecting rocks wherever we go. To my kids, the rocks have qualities I cannot begin to understand. Their collections include a variety of colors, designs and shapes—the more unusual the better! I know my children's affection for rock collecting is shared by many other kids, much to the exasperation of their teachers and parents. I am pleased I have finally found a mathematical connection by which we can organize and discuss their collection without getting into intensive geological discussions—an area I prefer to avoid for this part of my life.

A geologist acquaintance told me about a method to classify rocks according to calculations made regarding their dimensions. Rocks are placed into one of four categories by making three measurements and computing two simple ratios. In short, the measurements are the three longest diameters of the rock that are perpendicular to each other.

Figure 1



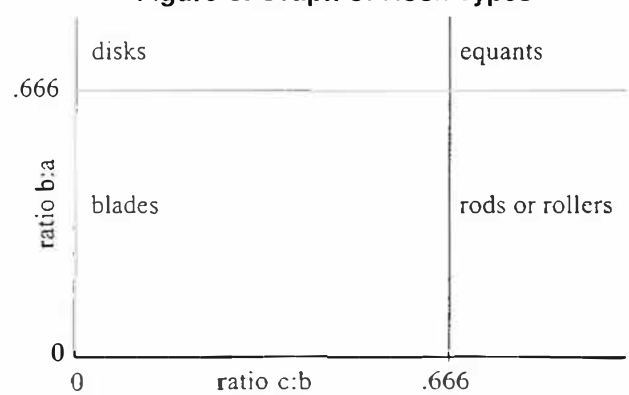
Use calipers, if possible, to do the measurements. They can even be made from a ruler and sticks or you can substitute a drawing compass. A standard ruler can be used for the measurements after students understand how to avoid parallax error by getting close to the ruler and looking perpendicular to its surface. The greatest distance between any two points on the rock is the largest diameter. Call this distance *a*. The measurement is done by moving the caliper around the rock until it shows the greatest span. Record *a*. Note the endpoints of *a* with pencil marks or by returning the caliper to the rock. Measure *b*, the next greatest diameter of the rock that is perpendicular to the segment that made *a*. Record *b*. Try to visualize a plane made up by the intersection of segments *a* and *b*. Measure *c* as the largest diameter perpendicular to that plane. Record *c*. Divide, preferably by calculator, to find the ratios c:b and b:a. Place the rock on the graph according to its coordinates (c:b, b:a) where you go across the first value then up the second value. Identify the rock's type by its placement on the graph. The table in Figure 1 shows how data can be organized and offers a few examples. The measurements are in millimetres.

Figure 2. Sample Data Table

Rock	a	b	c	c:b	b:a	Rock Type
1	63	44	16	.36	.70	disk
2	109	64	41	.64	.59	roller
3						
4						
5						
and so on						

Draw a large graph like the one in Figure 2 on the classroom floor. Students will begin to make connections between a rock's dimensions and its properties. Rock 1 would make a good skipping stone but rock 2 would not. Students will notice patterns in the shapes as the rocks are distributed across the graph. Students also speculate about why the names given the groups are descriptive and why the ratios never exceed 1. Students also see interesting relationships as they explore the reasons why some rocks are placed on the boundaries between categories, why overall size is unrelated to a rock's classification and how certain characteristics change as one moves from the extremes of one category to another.

Figure 3. Graph of Rock Types



This activity can be done easily and cheaply with students of virtually any age. All they need is a basic understanding of how to measure and what division means. The activity enhances discussions of stone tool use, the forces and products of nature and why we pick up the rocks we choose to collect.

Rocks seem to provide a compelling interest to kids. I believe this activity provides a new dimension to something they already like and provides a way of connecting mathematics to the real world while kids practise skills and develop understanding.

The Last Digits

What are the last two digits of the exponential number 7^{7^7} when written as a whole number?

A Fence for the Community Pasture

Klaus Puhlmann

The county council meeting was about to adjourn when, yet again, counsellor Smith requested the floor and said, "We are facing a serious problem with the use of our community pasture. The farmers wanted the cattle, horses and sheep in separate areas. It is therefore necessary that the existing pasture be cross-fenced to create three separate areas for the animals." Everyone seemed to agree with the suggestion.

"To avoid problems later on, it is best that the three subdivisions have the same area," said the reeve.

"That's probably a good idea," replied counsellor Smith. "Since the community pasture is a perfect square, we simply have to divide it into three rectangular pastures of equal area."

"That's not possible," remarked counsellor Jones, who is also a teacher. "The circular pond is located right at the northeast corner of the pasture. If we subdivide the community pasture into three rectangular pastures, there will be no access to the pond from two of these subdivided pastures." After further discussion of the problem, Jones suggested, "One fence should be built from the northeast corner of the pasture to the west side and the second fence should be built from the northeast corner to the south side of the community pasture. If we do it right, all three subdivisions would be of equal area and all three would have access to the pond." The county council voted unanimously in favor of this suggestion.

"How many metres of fence do we need to build?" asked counsellor Prat.

"Well," said the reeve, "I have not done any geometry for a long time. I am clueless when it comes to these kinds of problems." He then asked, "How long is the fence around the entire community pasture?"

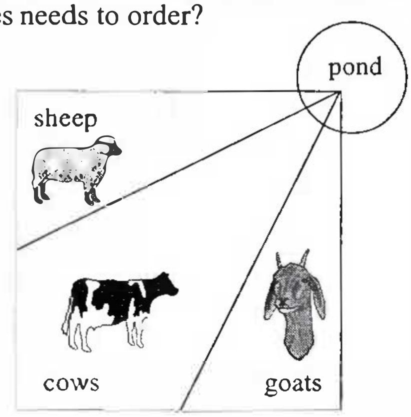
"Twelve hundred metres," was the response from Jones.

"Why don't we simply buy 1,200 metres of fencing? That would likely be enough," said the reeve.

"It would be better if we calculate more accurately what we need," suggested Smith.

Now all eyes were suddenly focused on counsellor Jones, the teacher, to provide them with the answer. After a brief pause, counsellor Jones said, "This is not a difficult problem. I will order exactly what we need and no more." Everyone agreed and the meeting was adjourned.

Do you know how many metres of fencing counsellor Jones needs to order?



Four-Color Map Problem

Klaus Puhlmann

The renowned conjecture in topology that arose circa 1852 asserts that four colors are both sufficient and necessary for coloring all maps drawn on a plane or sphere so that no two regions that touch (that is, share a segment of a boundary) are the same color. While this conjecture has always been an accepted fact for cartographers, for mathematicians it remained an unproved supposition. What has been known since 1890 is that five colors suffice and that there are maps for which three colors are insufficient. The chromatic number of a surface is the least number of colors that suffice to color any map on that surface. Except for the Klein bottle, the chromatic number of a surface is equal to the greatest integer not greater than, $\frac{1}{2}(7+\sqrt{49-24\chi})$, where χ is the Euler characteristic of the surface. Since $\chi = 0$ for the cylinder, Möbius strip and torus (doughnut), their chromatic numbers are 7. But $\chi = 0$ for the Klein bottle and its chromatic number is 6. For the projective plane, $\chi = 1$ and the chromatic number is 6. For the plane or sphere, $\chi = 2$ and the chromatic number is 4.

All of this has driven topologists to endless exasperation. They have been able to prove that only six colors are needed on a Möbius strip and seven on a torus, but no one has been able to prove what mapmakers have known for ages—that four colors are enough for any flat map or sphere. Topologists since Möbius' time have tried to draw a flat map on which five colors are needed; no one had done it, but neither had anyone proved that it cannot be done.

It remained a great-unsolved problem of topology until 1976 when two American mathematicians, Wolfgang Haken and Kenneth Appel, from the University of Illinois, presented a proof for this famous conjecture. Their method of proving this conjecture made unprecedented use of computers. Although this proof is an extraordinary achievement, many mathematicians found it to be deeply unsatisfying. This feeling was largely driven by their suspicion that a counter-example could still be found and that a proof in the rigorous mathematical sense does not exist. What mathematicians have seen, they claim, is a program for attacking the problem by computer along with the results of an "experiment" performed on a computer. Furthermore, the Haken-Appel proof of the four-color theorem is unsatisfying to mathematicians because it is not simple, beautiful or elegant. Haken and Appel both think that it is unlikely that a proof can be found that does not require an equally intensive application of computers, but there is no way of knowing this for sure. So one may ask, have Haken and Appel really proved the four-color-map conjecture? Many mathematicians who have examined the proof say yes, but the trial period is still not over. One thing is certain though: the four-color theorem has introduced a new era in mathematics—proof by using computers—and no one knows where it will lead.

Bibliography

Barr, S. *Experiments in Topology*. New York: Crowell, 1964.

Bergamini, D., and the Editors of *Life*. *Mathematics*. New York: Time, 1967, 184–85.

Bild der Wissenschaft. *Die Lösung des Vierfarbenproblems*. July 1977, 124–26.

Gardner, M. *The Last Recreations*. New York: Springer-Verlag, 1997.

MCATA Executive 2000–2001

President
Sandra Unrau — Res. (403) 284-2642
11 Hartford Place NW — Bus. (403) 777-6390
Calgary T2K 2A9 — Fax (403) 777-6393
sunrau@cbe.ab.ca *or* sunrau@col.ca

Past President and Newsletter Editor
Cynthia Ballheim — Res. (403) 278-2991
612 Lake Bonavista Drive SE — Bus. (403) 228-5810
Calgary T2J 0M5 — Fax (403) 229-9280
ballheimcj@aol.com

Vice President
Elaine Manzer — Res. (780) 624-3988
9502 79 Avenue — Bus. (780) 624-4221
Peace River T8S 1E6 — Fax (780) 624-4048
manzere@prsd.ab.ca

Vice President, 2000 Conference Chair and NCTM Representative
Graham Keogh — Res. (403) 347-5113
37568 Rge Rd 275 — Bus. (403) 342-4800
Red Deer County T4S 2B2 — Fax (403) 343-2249
gkeogh@rdcrd.ab.ca *or* gjkeogh@telusplanet.net

Secretary
Donna Chanasyk — Res. (780) 455-3562
13307 110 Avenue NW — Bus. (780) 459-4405
Edmonton T5M 2M1 — Fax (780) 459-0187
donnac@connect.ab.ca

Treasurer
Doug Weisbeck — Res. (780) 459-8464
66 Dorchester Drive — Bus. (780) 413-2211
St. Albert T8N 5T6 — Fax (780) 434-4467
weisguy@v-wave.com

Journal Editor
Klaus Puhlmann — Res. (780) 795-2568
PO Box 6482 — Bus. (780) 723-4471, ext103
Edson T7E 1T9 — Fax (780) 723-2414
klaupuhl@gyrd.ab.ca

Webmaster
TBA

2001 Conference Chair
TBA

Membership Director
Daryl Chichak — Res. (780) 450-1813
1826 51 Street NW — Bus. (780) 427-0010
Edmonton T6L 1K1 — Fax (780) 469-0414
mathguy@telusplanet.net

Mathematics Representative
Indy Lagu — Res. (403) 249-9901
Dept. of Mathematics — Bus. (403) 240-6154
Mount Royal College — Fax (403) 240-6505
4825 Richard Road SW — ilagu@mtroyal.ab.ca
Calgary T3E 6K6

Faculty of Education Representative
Dale Burnett — Res. (403) 381-1281
Faculty of Education — Bus. (403) 329-2416
University of Lethbridge — Fax (403) 329-2252
Lethbridge T1K 3M4 — dale.burnett@uleth.ca

Alberta Learning Representative
TBA

Directors
Len Bonifacio — Res. (780) 434-0504
4107 124 Street NW — Bus. (780) 462-5777
Edmonton T6J 2A1 — Fax (780) 460-5820

Rick Johnson — Res. (780) 458-5670
8 Grandin Wood Estates — Bus. (780) 492-4273, ext 293
St Albert T8N 2Y4 — Fax (780) 459-0512
rjohnson@edmc.net

Richard Kopan — Res. (403) 254-9106
72 Sunrise Crescent SE — Bus. (403) 256-9618
Calgary T2X 2Z9 — kopan@cadvision.com

Helen McIntyre — Res. (403) 652-7545
PO Box 5486 — Bus. (403) 652-7540
High River T1V 1M6 — Fax (403) 601-2511
mcintyrh@cadvision.com

Evelyn Sawicki — Res. (403) 242-4000
RR1, Site 6, Box 6 — Bus. (403) 298-1476
Calgary T2P 2G4 — Fax (403) 298-1480
evelyn.sawicki@crcssdl.calgary.ab.ca

Lorraine Taylor — Res. (403) 329-4401
10 Heather Place — Bus. (403) 328-4723
Lethbridge T1H 4L5 — lorraine.taylor@lethsd.ab.ca

PEC Liaison
Carol D. Henderson — Res. (403) 256-3946
860 Midridge Drive SE Suite 521 — Bus. (403) 938-6666
Calgary T2X 1K1 — Fax (403) 256-3508
hendersonc@fsd38.ab.ca

ATA Staff Advisor
David L. Jeary — Bus. (403) 265-2672
SARO — or 1-800-332-1280
540 12 Avenue SW Suite 200 — Fax (403) 266-6190
Calgary T2R 0H4 — djeary@teachers.ab.ca

ISSN 0319-8367
Barnett House
11010 142 Street NW
Edmonton AB T5N 2R1