Geometry Problems Promoting Reasoning and Understanding

Alfred B. Manaster and Beth M. Schlesinger

Current mathematics curricula emphasize "truth rather than reasons for truth" (Harel 1998, 497). This observation was supported in the TIMSS videotape study of Grade 8 mathematics classes, which found few occurrences of explicit mathematical reasoning in any courses other than geometry (Manaster 1998). The four related problems presented in this article provide examples of ways to include justifications of interesting mathematics in courses taught before a geometry course. At different times during their study of quadratic functions, students can solve these problems and can fully understand their solutions. This understanding requires that the students follow chains of reasoning that furnish convincing justifications of the correctness of the general results. The reasoning involves both algebra and geometry, but all the problems can be done before the student takes a formal geometry course.

Problem A. Find the dimensions of a rectangle with a perimeter of 30 inches and sides of integral length that has the largest possible area.

A straightforward solution to this problem, which is appropriate for students in middle school and above, involves constructing a table containing dimensions and areas of all rectangles with sides of integral length and a perimeter of 30 inches (see Table 1).

When they examine the complete table, students discover that a rectangle—7 inches by 8 inches or 8 inches by 7 inches—exists with maximum area of 56 square inches. Since the table lists all possible rectangles satisfying the given conditions, a brief discussion completes a proof that the problem has been solved.

Table 1

Side_1 (In.)	Side_2 (In.)	Area (In.^2)
1	14	14
2	13	26
3	12	36
4	11	44
5	10	50
6	9	54
7	8	56

Problem B. Find the dimensions of a rectangle with a perimeter of 30 inches that has the largest possible area.

When we remove the constraint that the sides have integral length, the nature of the problem changes dramatically. The student cannot make a complete table of all possible rectangles.

By extending the table to include some rectangles with fractional lengths, students can observe symmetry in the table and may suspect or believe that the solution is the square with sides of 7.5 inches. The issue that we address in this article concerns the role of proof in developing a deeper understanding that this conjecture is indeed correct.

A next step in developing a more complete table might be to find the formulas for the height and area in terms of the base of the rectangle. Since $2h + 2b = 30$,

$$h = \frac{30}{2} - b$$

and

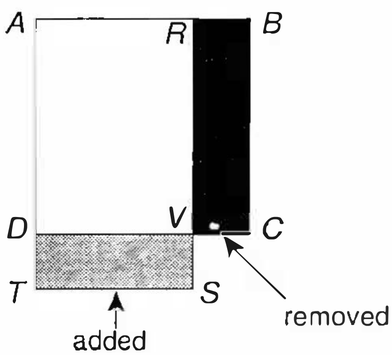
$$A = b \cdot h = b \cdot \left(\frac{30}{2} - b\right).$$

Students can use the formula and a graphing calculator to graph the area as a function of the length of the base. They can see by inspection that a maximum value of 56.25 square inches appears to exist when the base is 7.5 inches. By zooming in on the graph or by constructing tables with smaller and smaller step sizes, they can gather more evidence in support of this conjecture. Sophisticated students might use the "Maximum" function of a calculator to obtain the value 7.5. Teachers must be aware that this procedure may supply information, but it cannot lead to full understanding unless the student also knows how the maximum was found and why that algorithm works.

At some point, some students will notice that the apparent solution is a square. On the basis of that insight, it is possible to construct a beautiful geometric proof that a square always has the largest area of all rectangles with a given perimeter.

In Figure 1, rectangle $ARST$ was formed from the square $ABCD$ by first shortening the square's base an arbitrary amount equal to VC. The height of the new rectangle must be increased by the same amount, DT, to keep the perimeter constant. Rectangles $RBCV$ and $DVST$ have the same width, that is, $VC = DT$. Since RV is equal to a side of the original square and DV is shorter than a side of the original square, the area of $RBCV$ is greater than the area of $DVST$. The area that we removed is greater than the area that we added; therefore, the area of the square $ABCD$ is greater than the area of the new rectangle $ARST$. Because VC could represent any length less than AB, the new rectangle could be any rectangle with the same perimeter as square $ABCD$. Since $ABCD$ was any square, we have completed a geometric justification that the square has the largest area of all rectangles with a given perimeter.

Figure 1



Since the perimeter of *ABCD* equals that of *ARST*, the area added is less than the area removed.

Not all students are likely to have the insight that leads to the preceding argument; it is therefore worthwhile to explore other ways to understand why the square is the solution. One approach is to analyze the formula for the area as a function of the base.

Since $A = b(15 - b)$, then $A = -b^2 + 15b$. We want to find the largest possible value for A and the value of b at which it occurs. Because these values are difficult to ascertain from this formula, we use algebraic identities to rewrite it in a form that we can analyze to find these values:

$$\begin{aligned} A &= -b^2 + 15b \\ &= -(b^2 - 15b) \\ &= -\left(b^2 - 15b + \left(\tfrac{15}{2}\right)^2 - \left(\tfrac{15}{2}\right)^2\right) \\ &= -\left(\left(b - \tfrac{15}{2}\right)^2 - \left(\tfrac{15}{2}\right)^2\right) \\ &= \left(\tfrac{15}{2}\right)^2 - \left(b - \tfrac{15}{2}\right)^2 \end{aligned}$$

The final expression helps us find the largest value of A fairly easily. The first term is simply a constant. The maximum value of A occurs when we subtract the smallest possible value. Because we subtract a perfect square, its value is always greater than or equal to zero. If we make this term zero, we subtract as little as possible and make A as large as possible. Since the second term can equal zero only when $b = 15/2$, we see that the largest value of A is 56.25, which occurs only when $b = 7.5$.

Problem C. Consider all rectangles with perimeter equal to the circumference of a circle with radius 1 m. Find the dimensions of the rectangle that has the largest possible area.

The first step is to note that the circumference of the circle is 2π m. The next step depends on the student's knowledge about a solution to problem B. A student who understands the general principle that the square has the largest area can apply that result to this problem to see that the sides of the square have length $\pi/2$ m and that the area of the square is $(\pi/2)^2$ square metres. Otherwise, the student can use the same approaches that were used for problem B. A graphing calculator will help students find several slightly different values by observation, depending on the window chosen to view the graph. An algebraic approach gives

$$\begin{aligned} A &= -\left(b - \tfrac{\pi}{2}\right)^2 + \left(\tfrac{\pi}{2}\right)^2 \\ &= \left(\tfrac{\pi}{2}\right)^2 - \left(b - \tfrac{\pi}{2}\right)^2, \end{aligned}$$

so that the maximum occurs when $b = \pi/2$ and has the value $(\pi/2)^2$. It might be a good pedagogical strategy to present problem C some weeks after problem B so that students need to rethink their solution to a problem of this type. Redoing the algebra is likely to strengthen their understanding of the usefulness of underlying algebraic techniques. One advantage of seeing that both approaches give the same result is that students can observe that more than one good approach exists and that each reinforces the other.

Problem D. Is the ratio of the area of a square to the area of the circle whose circumference is equal to the perimeter of the square always the same? Why or why not?

Since the students have two examples, they might begin to find an answer by computing the requested ratios for each. For the square and circle in problem C, the area of the circle should be familiar to the students and is π square metres. The students have already found that the area of corresponding square is $(\pi/2)^2$ square metres; therefore, the ratio of the area of the square to the area of the circle is

$$\frac{\left(\frac{\pi}{2}\right)^2}{\pi} = \frac{\pi}{4}.$$

Our classroom experience indicates that high school students are more likely to use decimals to compute numerical approximations than to use formulas to compute exact values. In problem B, the area of the square was found to be 56.25 square inches. The circumference of the corresponding circle is 30 inches, so the radius is approximately 4.77 inches and the area is approximately 71.48 square inches. The ratio is approximately 0.79, which is also the two-decimal-place approximation of $\pi/4$.

We have seen that both ratios are $\pi/4$, or approximately 0.79. One question that calculators cannot answer conclusively is whether the two ratios are exactly the same. When π is used throughout the calculations, some calculators will show the difference between the two computed ratios as 0, whereas others will display a very small number. Other variations will depend on the rounding that students use in computing or estimating the areas in problem C. It might be helpful for students to look at other examples. They should eventually realize that the ratio is always about 0.79, which should lead them to ask whether the ratios are exactly the same and, even more important, why.

Fortunately, the algebraic solutions for problems B and C can lead to a proof that the ratios are all the same. Building on the result of problem C, if we call the perimeter of the square p, we see that the area of the square is

$$\left(\frac{p}{4}\right)^2 = \frac{p^2}{16}$$

and that the radius of the circle is $p/2\pi$, so the area of the circle is

$$\pi\left(\frac{p}{2\pi}\right)^2 = \frac{p^2}{4\pi}.$$

For any value of p, then, the ratio of the area of the square to the area of the circle is

$$\frac{\frac{p^2}{16}}{\frac{p^2}{4\pi}} = \frac{4\pi}{16} = \frac{\pi}{4}.$$

This result confirms our previous observations, and since no p exists in the final expression, $\pi/4$, the ratio is always the same.

Another explanation for the answer to problem D uses properties of similar figures and proportional reasoning. Let P_1 and P_2 be any two values for the perimeters of the circle and square. The ratio between any corresponding lengths in the similar figures, such as the radius of the first circle to the radius of second or the side of the first square to the side of the second. is P_1/P_2; and the ratio between any corresponding areas, for example, the semicircle of the first circle to the semicircle of the second, is

$$\left(\frac{P_1}{P_2}\right)^2.$$

Let C_1 and C_2 represent the areas of the circles with perimeters P_1 and P_2 respectively, and let S_1 and S_2 represent the areas of the squares with those perimeters. Since

$$\frac{S_1}{C_2} = \left(\frac{P_1}{P_2}\right)^2 = \frac{C_1}{C_2},$$

we see that

$$\frac{S_1}{S_2} = \frac{C_1}{C_2}$$

and

$$\frac{S_1}{C_1} = \frac{S_2}{C_2},$$

Therefore, the ratio of the area of the square to that of the circle does not depend on their common perimeter.

These four related problems and their solutions are accessible to students in courses other than geometry. They invite exploration. They encourage the meaningful use of technology. They call for writing. They lead to interesting and deep mathematics. They blend algebra and geometry. The variety of methods used to establish the mathematical results builds the students' appreciation of the power of the techniques they have developed and helps them recognize that mathematics is at least as much about "why" as about "how."

References

Harel, G. "Two Dual Assertions: The First on Learning and the Second on Teaching (or Vice Versa)." *American Mathematical Monthly* 105 (June–July 1998): 497–507.

Manaster, A. B. "Some Characteristics of Eighth Grade Mathematics Classes in the TIMSS Videotape Study." *American Mathematical Monthly* 105 (November 1998): 793–805.

Reprinted with permission from The Mathematics Teacher, *Volume 92, Number 2, February 1999, pages 114–16, an NCTM publication. Minor changes have been made to spelling and punctuation to fit ATA style.*