Interactive Technology and Classic Geometry Problems

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Interactive geometry software connects visual justification and empirical thinking to higher levels of geometric thinking with logical justification in formal proof. Battista and Clements (1995, 53) recommend the use in the secondary school geometry curriculum of software that "should guide students to learn significant and interesting concepts."

Some of the more engaging concepts are embedded in the classic works of the Greek geometers, yet interest can be killed if students struggle with the emphasis on the formal aspects of the Euclidean model of mathematical inquiry (Perkins et al. 1995). The enjoyment and challenge come from making ideas that were conceived with the limited tools of the ancient Greeks come alive for our students with the help of the computer. The task for the teacher is not so much to present the constructions and theorems in a colorful display on the computer screen; instead, the challenge is to design activities that engage students in the dynamic and interactive features of the software so that they can demonstrate real understanding.

An example of a classic construction problem is the circle of Appolonius: the locus of all points, P, in a plane such that given any two points A and B in the plane, $PA : PB$ will be a constant ratio k where $k \neq 1$. Students usually do not know where to begin with this problem. Traditional algorithmic instruction and our top-down knowledge usually produce an activity with an efficient set of steps for the construction. Consider the steps for the activity in Figure 1, and ask yourself how involved your students would be in understanding the construction. Note that The Geometer's Sketchpad (Jackiw 1993) is used for all work in this article.

Figure 1
Activity Directions for the Traditional Approach

Find the locus of all points, P, in a plane such that given any two points A and B in the plane, PA : PB will be a constant ratio k where $k \neq 1$.

1. Choose any two points; label them A and B. Choose any other point in the plane, and label it P. Note that $PA \neq PB$.
2. Construct segments for $\triangle PAB$. Select segments PA and PB and find the ratio, $PA : PB$, by using the Measure menu. This ratio will be k, the constant, $k \neq 1$.
3. Construct the angle bisector of $\angle APB$ and its intersection with side AB. Label the intersection X.
4. Construct the line through A and B. Construct the exterior-angle bisector at P and its intersection with line AB. Label the intersection Y.
5. Construct segment XY; label its midpoint O. (see Figure 2)
6. Construct the circle with center O and radius OX.
7. Select any other point on the circle; label it Q. Construct and select segments QA and QB. Find the ratio $QA : QB$. Is it the same as $PA : PB$? (see Figure 3)
8. Select Q and the circle, and under the Display menu, choose "animate" to move Q around the circle. Does the value of the ratio $QA : QB$ change? Select a point T not on the circle and find the value of $TA : TB$. Drag T to make a conjecture about points not on this circle.
9. Have you found the locus of points? Describe the locus of points.

What happens when $PA : PB = 1$? (see Figure 4)

This activity uses many of the features of Sketchpad—especially the animation feature—to convince students that the circle meets the criterion of the constant ratio. Also, the use of the interior- and exterior-angle bisectors at P is an essential hint in the proof that the locus of points is a circle. But do students have any insight into the connection between the bisectors and the constant ratio? Although students are asked to summarize their work with a description and to drag P to find out what happens with k equal to 1, they really have little ownership in this construction.

The steps in the traditional activity are very efficient and almost eliminate any guesswork and questions on the students' part. Students may have a shallow surface level of understanding of the problem and can fail to make connections among the important concepts behind the construction. Because they have not worked through the difficulty of the problem, they are not ready to make formal arguments that the locus is a circle.

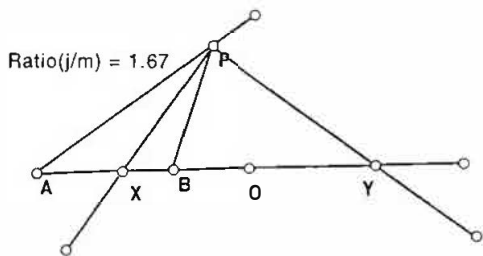Students often have a perception that when a mathematician sits down to work a problem or prove a conjecture, he or she very quickly produces a flawless paper. Actually, the paper is the final clarification of a lot of hard work with some dead ends, a few frustrations and the rewarding moments of "aha."

A more interactive approach to this activity would take students through the complete process of what a mathematician does: play with an idea, make a conjecture, make a formal argument. Schwartz (1995, 98) describes the software as an intellectual mirror in which "users explore their own understanding of a mathematical domain. One feature of such software environments is that while the user explores a particular question, the environment can display a logical universe of inquiries to which the user's particular inquiry belongs."
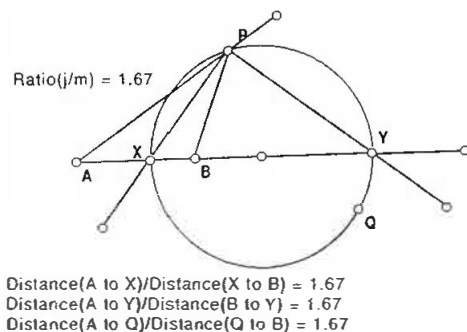
A connection is made between playing with a sketch and making a formal argument because the software requires a syntax that is similar to formal geometric thinking. For example, to draw a circle, think "Given points O and X, construct a circle with center O and radius OX" as you select O and X to activate the Construct menu. As teachers, we can capitalize on the connection between the words and the action to smooth the students' transition from lower to higher levels of geometric thinking.

Instead of using the dragging feature at the end of the activity to verify a prescribed process, use it in the beginning to foster discovery. Consider how the steps in the interactive activity in Figure 5 have evolved from the original list of steps. Unfortunately, the power of the animation cannot be demonstrated here. However, you can notice that $Q2$ is now above line AB. The ratio, s/t, corresponds to $Q2A: Q2B$, and it remains unchanged as $Q2$ moves around the circle.

Figure 2
Student's Sketch of Steps 1–5

Ratio(j/m) = 1.67

Figure 3
Student's Sketch of Steps 6 and 7

Ratio(j/m) = 1.67

Distance(A to X)/Distance(X to B) = 1.67
Distance(A to Y)/Distance(B to Y) = 1.67
Distance(A to Q)/Distance(Q to B) = 1.67

Figure 4
Student's Sketch of Steps 6 and 7

Ratio(j/m) = 1.00
Distance(A to X) = 1.13 cm
Distance(X to B) = 1.13 cm

$PA \cdot PB = 1$.
When $PA : PB = 1$, the locus is not a circle.
The exterior-angle bisector is now parallel to line AB.
The locus of points is the perpendicular bisector of segment AB
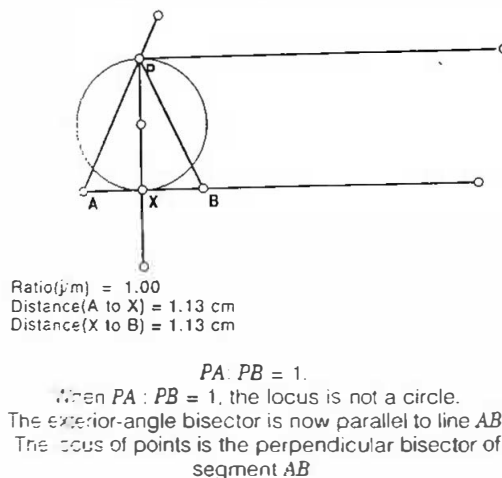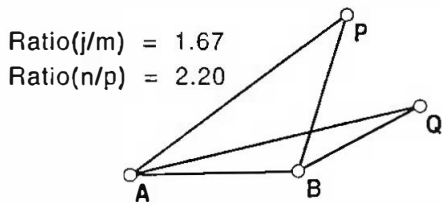
Figure 5
Activity Directions for the Interactive Approach

Find the locus of all points, P, in a plane such that given any two points A and B in the plane, $PA : PB$ will be a constant ratio k where $k \neq 1$.

1. Choose any two points; label them A and B. Choose any other point in the plane, and label it P. Choose P so that $PA \neq PB$.
2. Construct segments PA and PB and line AB. Select $\overline{PA}$ and $\overline{PB}$, and find the ratio, $PA : PB$, by using the Measure menu. This ratio will be k, $k \neq 1$. __
3. Choose another point Q. Construct and select $\overline{QA}$ and $\overline{QB}$. Find ratio, $QA : QB$. Compare this ratio with k. (see Figure 6)
4a. Drag Q as needed until $QA : QB = k$. (see Figure 7)
4b. Repeat step 4a at least five times. Can you identify a region on the plane where you would not choose Q? Can you choose a point above line AB? Below line AB? On segment AB? On line AB to the left of A? On line AB to right of B?
5. What figure does the locus of points suggest? If you were not guessing points, how would you describe the locus? Be very specific. Are any points other than A, B and P essential to your description? (see Figure 8)
6. Use your description to construct the locus, or path, of points. Does this path go through all the points you chose for step 4b? Note: the round-off error may make some of the points slightly off the locus. (see Figure 9)
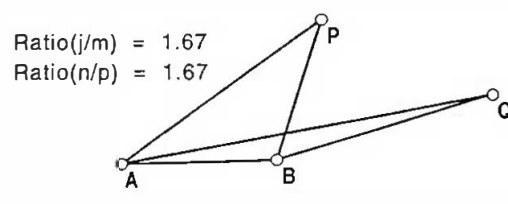7. As you animate P around the locus, verify that the ratio, k, remains constant. (see Figure 10)

Figure 6
Student's Sketch of Steps 1–3

Ratio(j/m) = 1.67
Ratio(n/p) = 2.20

At this point, $QA : QB > PA : PB$.

Figure 7
Student's Sketch of Step 4a.

Ratio(j/m) = 1.67
Ratio(n/p) = 1.67

The ratios are the same.

Figure 8
Student's Sketch of Steps 4 and 5

Ratio(j/m) = 1.67
Ratio(n/p) = 1.67

Ratio(q/r) = 1.67
Ratio(s/t) = 1.67
Ratio(x/y) = 1.67
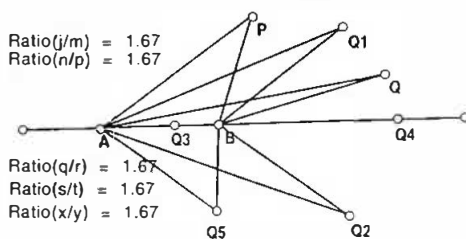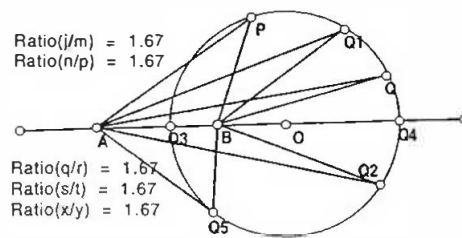
Distance(A to Q3)/Distance(Q3 to B) = 1.67
Distance(Q4 to A)/Distance(Q4 to B) = 1.67

The points suggest a circle, and Q3 and Q4, the points on line AB, are on the diameter of this circle.

Figure 9

Ratio(j/m) = 1.67
Ratio(n/p) = 1.67

Ratio(q/r) = 1.67
Ratio(s/t) = 1.67
Ratio(x/y) = 1.67

Distance(A to Q3)/Distance(Q3 to B) = 1.67
Distance(Q4 to A)/Distance(Q4 to B) = 1.67

The locus of points is the circle with centre at the midpoint of segment Q3Q4.

This version of the activity does not immediately give away the fact that the locus is a circle or immediately identify the essential points X and Y for the construction. I have found that in the original activity, students lose sight of the role of the constant ratio. In this revised activity, the ratio is essential to their choice of points. However, the activity is still very empirical. The final steps connect this work to formal proof (see Figure 11).
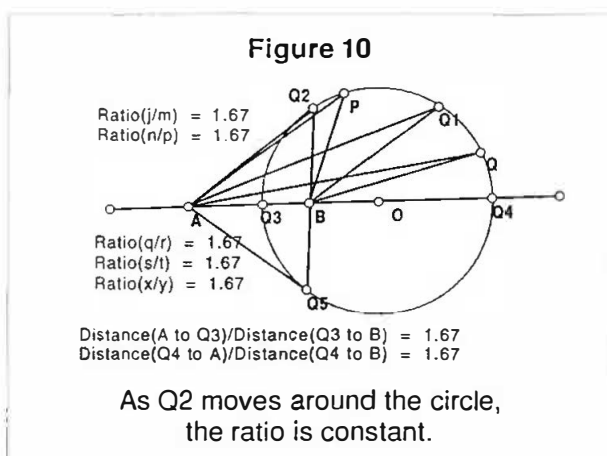
Step 12 allows students to verify the construction of the circle of Appolonius for points A, B and P. In the more traditional activity, students could drag P and preserve the circle. In the interactive activity, P appears to be on the circle, but the computer may not recognize it as a point of the circle. All points in step 4b were fixed according to their measurement relationships to A and B, and we know that numerical measure is not a part of legitimate construction. Steps 1–11 give students the opportunity to discover the construction and the important relationships among the parts of the process. Step 12 lets students use induction to test the generalizability of the process. In writing the script, they should produce a list of commands that are essentially like the steps in the original activity. They are now ready for deductive proof.

At this point, students should have made three observations that can be restated as lemmas to prove that P, X and Y determine a circle. It is important that in groups or as a class, the students discuss the restriction that $k \neq 1$. If $k = 1$, then $\triangle APB$ is either isosceles or equilateral. Point X would be the foot of the perpendicular bisector to $\overline{AB}$ from P, and locus of points would be this perpendicular bisector. The exterior-angle bisector at P is parallel to $\overline{AB}$ (see Figure 4).

A whole-class discussion can bring the class to a consensus regarding steps 8–12. Focus students' attention on the connection between the bisectors and the constant ratio so that the following lemmas can be stated:

I.   Point X is the point where the interior-angle bisector at P intersects $\overline{AB}$ in $\triangle ABP$ if and only if $XA: XB = PA: PB$.
II. If $AP \neq BP$, Y is the point where the exterior-angle bisector at P intersects line AB and only if $YA: YB = PA: PB$.
III. The interior-angle bisector and the exterior-angle bisector from a vertex of a triangle are perpendicular.

Using these lemmas, students can describe and justify the construction of the circle of Appolonius. You may or may not want to go into the details of the lemmas. Both parts of the "if and only if" statements in lemmas I and II can be proved with the law of sines. Lemma III is a simple proof for geometry

Figure 10

Ratio(j/m) = 1.67
Ratio(n/p) = 1.67
Ratio(q/r) = 1.67
Ratio(s/t) = 1.67
Ratio(x/y) = 1.67

Distance(A to Q3)/Distance(Q3 to B) = 1.67
Distance(Q4 to A)/Distance(Q4 to B) = 1.67

As Q2 moves around the circle,
the ratio is constant.

Figure 11

The Final Steps of the Activity Directions for the Interactive Approach

8. Relabel the point of the circle on segment AB as X. Relabel the point on the circle that is on line AB, but not on segment AB, as Y.
9. Choose on the display menu a color that has not been used. Construct segment PX. Measure $\angle APB$, $\angle APX$ and $\angle XPB$. What can you conclude?
   Measure $\angle AQX$ and $\angle XQB$. What can you conclude? Can you generalize for all the points that you chose in step 4b?
10. Extend lines to form the exterior angles at P and Q for $\triangle APB$ and $\triangle AQB$. Use angle measures to find how segments PY and QY are related to the exterior angles. What is your conjecture?
11. How are segments PX and PY related? How are segments QX and QY related? Can you make a generalization for the other points you chose in step 4b? (see Figure 12)
12. Using your generalization in steps 8–11, describe the locus of points. Test your description by making a new sketch of the locus in which you record a script—be sure to include measures in the script.

classes and involves the supplementary exterior and interior angles at vertex P. Then half of an interior angle and half of an exterior angle will be complementary, so that $\overline{PY} \perp \overline{PX}$.

Description. Given any two points A and B and a point, P, not on $\overline{AB}$: let $PA : PB$ be a constant ratio k where $k \neq 1$. The locus of points in the plane with constant ratio k is the circle with diameter $\overline{XY}$ where Y is the intersection of the interior-angle bisector at angle P with side AB and Y is the intersection of the exterior-angle bisector at P with line AB.

Proof. Consider the given points A and B, and let P be a point not on $\overline{AB}$ such that $PA : PB = k$ and $k \neq 1$. Construct $\triangle APB$.

By lemma I, the intersection of the interior-angle bisector at angle P with side AB is a point X such that $XA : XB = PA : PB = k$. Then X satisfies the condition of the locus.

Since $k \neq 1$, the intersection, Y, of the exterior-angle bisector at P exists. By lemma II, $YA : YB = PA : PB = k$. Y satisfies the condition of the locus.

By lemma III, $\triangle XPY$ is a right triangle in which P is the right angle of $\overline{XY}$ is the hypotenuse. The midpoint, O, of hypotenus XY is equidistant from points X, P and Y. Therefore, for any point P such that $PA : PB = k$, P lies on the circle with center O and diameter $\overline{XY}$.

From the introduction of the problem on the computer to the final discussion, the objective is to help students focus on the conditions of the locus and to use the technology to explore and discover the essential features of the construction. Students seem easily to get to the point where they see the circle. Their work breaks down when they have to clarify how to find and describe that particular circle, but the lesson is not over until the class has communicated in logical mathematical language what has been learned. It should always be emphasized that although the Sketchpad is an effective and convincing tool for conjecture, it creates sketches that are incomplete without descriptions and justification.

Other classic problems that have been good topics for interactive computer activities are applications of Menelaus and Ceva's theorem, Steimer's theorem, Napoleon's theorem, problems with the nine-point circle, and re-creations of Euclid's propositions. As the students sketch to make sense of the statements of these problems or theorems, encourage them to make text notes and measures on the sketches to prepare them for any formal work that will be assigned.
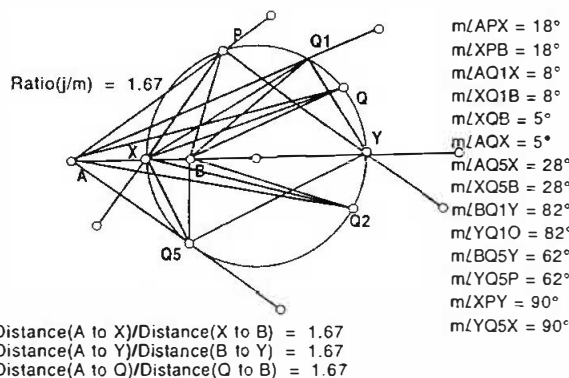
A goal for these activities is to use the classic problems as a context with computer applications to connect students to a richer experience of "what mathematics is, what it means to know and do mathematics" (NCTM 1989, 16). The development of these activities will be a work in progress that will apply principles that are consistent with teaching for understanding. As you work with students, the computer-laboratory tasks will evolve from procedural activities to interactive activities, and you will begin to see the maximum benefits for students.

References

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Schwartz, J. L. "Shuttling Between the Particular and the General: Reflections on the Role of Conjecture and Hypothesis in the Generation of Knowledge in Science and Mathematics." In *Software Goes to School: Teaching for Understanding with New Technologies*, edited by D. N. Perkins et al. New York: Oxford University Press, 1995.

Figure 12

Ratio(j/m) = 1.67

$m\angle APX = 18°$
$m\angle XPB = 18°$
$m\angle AQ1X = 8°$
$m\angle XQ1B = 8°$
$m\angle XQB = 5°$
$m\angle AQX = 5°$
$m\angle AQ5X = 28°$
$m\angle XQ5B = 28°$
$m\angle BQ1Y = 82°$
$m\angle YQ1O = 82°$
$m\angle BQ5Y = 62°$
$m\angle YQ5P = 62°$
$m\angle XPY = 90°$
$m\angle YQ5X = 90°$

Distance(A to X)/Distance(X to B) = 1.67
Distance(A to Y)/Distance(B to Y) = 1.67
Distance(A to Q)/Distance(Q to B) = 1.67

Student's Explanation of the Relationships Between Segments and Angles and Why These Points Are on a Circle