Using ROOTine Problems for Group Work in Geometry

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How can mathematics teachers structure classroom activities so that students will be intellectually challenged? How can they create a learning environment that encourages students to communicate and reason mathematically, make decisions collaboratively and acquire mathematics skills and concepts that they thoroughly understand? The *Professional Standards for Teaching Mathematics* (NCTM 1991) suggests that mathematics teachers need to focus on the major components of teaching: worthwhile tasks, discourse and students' active participation and involvement. Cooperative-learning approaches offer practical classroom techniques that teachers can use to motivate all their students to learn and appreciate mathematics (Davidson 1990).

Many factors influence and affect group work, since the method of instruction is inseparable from curricular content. One crucial factor is the activity that the teacher selects or designs and offers to the students for group work. Van Hiele (1986, 39) points out that students learn not by direct teacher talking but through "a suitable choice of exercises."

In this article we introduce a technique for developing tasks suitable for group investigation. The format of the task is simple: students work from sets of numerical examples toward making generalizations. The teacher identifies the generalization—the root problem—and prepares a set of subproblems for each group. A distinctive feature of the problems, however, is that they lend themselves to different strategies for solving the same problem. This aspect promotes group discussion, problem solving and inquiry.

Problem Set 1: Angle Sum of a Triangle

The objective of the first set of problems is to prepare students to construct the concept of the sum of the angle measures of a triangle. We assume that students are familiar with the definitions of a linear pair of angles and an exterior angle and with the property that the sum of the angles in a linear pair is equal to 180 degrees. For example, the root, or general, problem is shown in Figure 1. The subproblems for the groups are shown in Figure 2. For Figure 2, it is important that the letters on the pictures be the same for the different groups and that the depiction of the angles should correctly reflect the given magnitude of the angles.

Groups receive the "same" generic problem but with different parameters, and each group works with several of the subproblems, answering the same set of questions. Operating with different given parameters, the groups should each arrive at the same result for the second question in Figure 1, the sum of the interior angles. Then the teacher leads a class discussion in which groups share their results on each of the subproblems, so that students all observe that the result for question 2 is the same in each case.

The students are asked to explain this phenomenon. They are able to make a generalization after observing and analyzing other groups' work. Is this generalization merely a coincidence, or does it reflect conformity with a mathematical law? Depending on the developmental level and the students' ability, the teacher

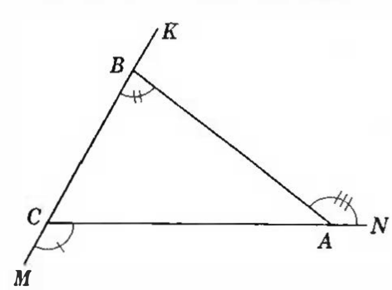
Figure 1
Sum of the Angles of a Triangle: Root Problem

Given: $\triangle ABC$

$m\angle BAN$, $m\angle ABC$, $m\angle MCA$

Find: 1. $m\angle BCA$, $m\angle BAC$, $m\angle KBA$

2. $m\angle BAC + m\angle ABC + m\angle ACB$



may provide a rigorous proof of the theorem of the sum of the angle measures in a triangle or just state the theorem, highlighting that this theorem can be proved. This problem is accessible to students at all levels. Students who are not ready for abstract proofs can investigate it by using a paper model of a triangle, cutting the angles and then lining them up on a straight line.

This problem offers excellent exploration opportunities with The Geometer's Sketchpad (Jackiw 1990) or Cabri Géomètre (Baulac, Bellemain and Laborde 1992). The student can use one of these tools to construct a triangle and determine the individual angle measures and the sum of the three angles. Then, taking any vertex and moving it, the student observes that although the individual angle measures of a triangle may change, the sum remains constant. The software allows the investigation of many triangles, since each movement of a vertex creates a new triangle. This demonstration offers a convincing argument for most students. The experimentation should occur before any formal proof is attempted. The result of this work may be organized as in Table 1.

As an additional question or as an extension of the problem, the teacher asks students to find the sum of the measures of ∠BAN, ∠KBA and ∠ACM. The students discover that 360 degrees is the sum of the measures of the three exterior angles of triangle ABC. The proof is easy if the teacher provides a generic picture as a hint (see Figure 3).

The arithmetic of the problem is simple:

$m\angle 1 + m\angle 2 = 180°$,

$m\angle 3 + m\angle 4 = 180°$,

and

$m\angle 5 + m\angle 6 = 180°$;

therefore, adding these three equations and moving some terms from the left side to the right side, we get

$$m\angle 2 + m\angle 4 + m\angle 6 = 540° - (m\angle 1 + m\angle 3 + m\angle 5) = 540° - 180° = 360°.$$

As a follow-up activity, students can then measure the exterior angle of a triangle and compare it with the sum of the measures of the two remote interior angles of the triangle. Again using technology, the students move any vertex of the triangle and observe the pattern.

The approach described gives students three angles in the diagram—one interior angle and two exterior angles. For a diagram with these angles to exist, the sum of the two given exterior angles minus the given interior angle must be 180 degrees. Thus, the information given in this approach "forces" students to the desired conclusion. The students' main task is to apply the principle that the sum of the angles of a linear pair is 180 degrees.

A more open-ended approach gives students less information and has them develop the remaining information by actual measurement. For example, in Figure 2a, one might show the same diagram

Figure 2
Sum of the Angles of a Triangle: Subproblems

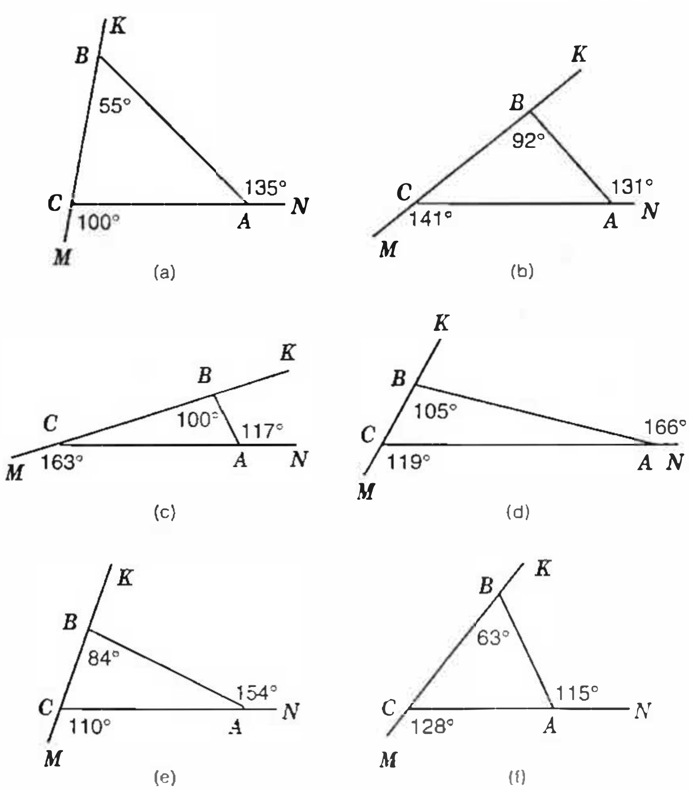
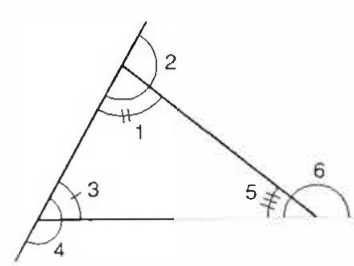


Figure 3
Sum of the Exterior Angles of a Triangle



but mark only the angles of 55 degrees and 100 degrees—and perhaps give a length for BC. Students would then draw a line and mark off a segment representing BC. They would draw rays starting from B and C to create the desired angles of 55 degrees and 100 degrees and label the point where these rays meet as A. They could then measure the remaining angles and find both the sum of the interior angles and the sum of the exterior angles.

Problem Set 2: Area of Equivalent Figures

A second set of problems connects several ideas: the Pythagorean theorem; properties of parallelograms and rectangles; and area of rectangles, triangles and parallelograms. The objective of this problem set is to strengthen the concept of area of equivalent figures. The root problem is shown in Figure 4a: ABCD is a parallelogram, and F is a point on AD. The goal is to have students compare the sum of the areas of

Figure 4
Areas of Triangles and Parallelograms: Root Problem

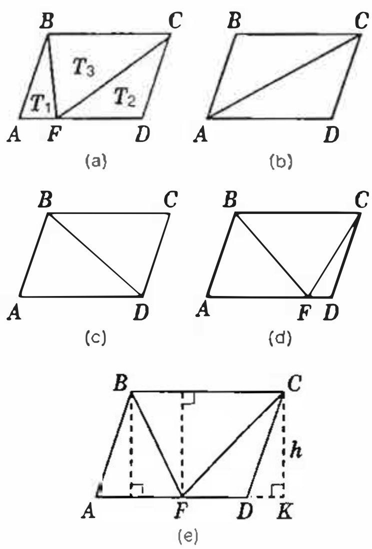


Figure 5
Areas of Triangles and Parallelogram: Subproblems

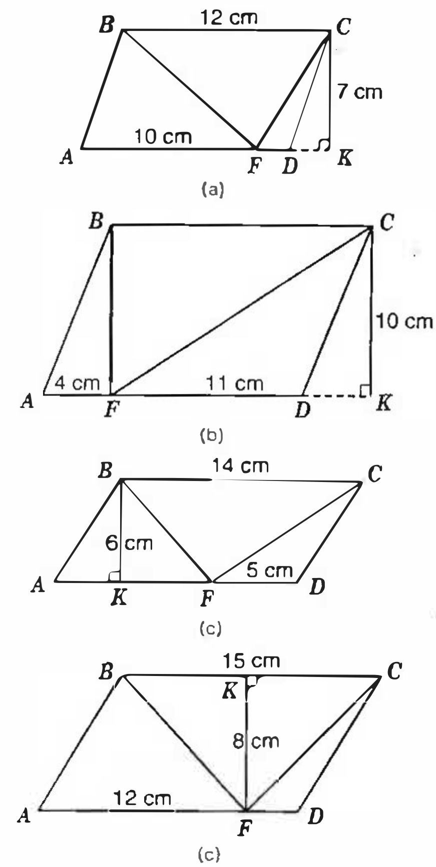
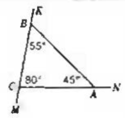
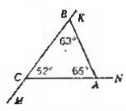


Table 1
Sum of the Angles of a Triangle

Problem		Observation	Hypothesis	Explanation
1.		55° + 45° + 80° = 180°	The sum of the angles of any triangle is equal to 180 degrees.	
2.		52° + 63° + 65° = 180°		

triangles T_1 and T_2 with the area of triangle T_3 and with the area of parallelogram ABCD. Figures 4b and 4c show the cases in which F coincides with either A or D. Students may already have proved that the two triangles are congruent in these two instances, so their areas are equal and the sum of their areas is equal to the area of the parallelogram.

The teacher develops a set of subproblems by fixing the "moving" point and giving a numerical value to the selected segments of the parallelogram (see Figure 5).

The geoboard is a very useful tool for demonstrating the "moving" point and helping students understand the effect of the motion of the point on the shapes within the parallelogram. However, although geoboards allow the systematic study of shapes, area, perimeter and so on, they represent a discrete structure and are limited by the number of pegs. If available, software can be used to construct shapes, thus reinforcing the defining properties of geometric figures.

An analogous problem can be considered using a rectangle instead of a parallelogram. In this case, the root problem is as shown in Figure 6. The subproblems are shown in Figure 7.

Students first apply the Pythagorean theorem and obtain the lengths of segments MF and FD. They can then organize their search for areas in different ways. They may see the segment BC as the sum of the two segments, MF and FD, and find the area of triangle BFC by using the formula; or they may see the area of triangle BFC as the difference between the area of rectangle MBCD and the sum of the areas of triangle MBF and triangle FCD. Students discover that

area ΔMBF + area ΔFCD = area ΔBFC = ½ area MBCD.

In general, as shown in Figure 8,

area $\Delta MBF = \frac{ax}{2}$,

area $\Delta FCD = \frac{a(b-x)}{2}$,

and

$$\text{area } \Delta MBF + \text{area } \Delta FCD = \frac{ax}{2} + \frac{a(b-x)}{2} = \frac{ax + ab - ax}{2} = \frac{ab}{2}.$$

Since the area of MBCD equals *ab*, it then follows that the area of triangle BFC equals one-half the area of MBCD. The most important conclusion is that the foregoing inference does not depend on the location of the "moving" point F.

Figure 6
Areas of Triangles and Rectangles: Root Problem

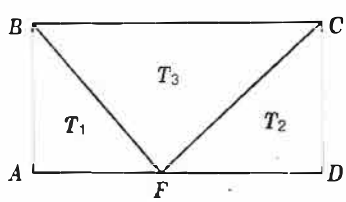


Figure 7
Areas of Triangles and Rectangles: Subproblems

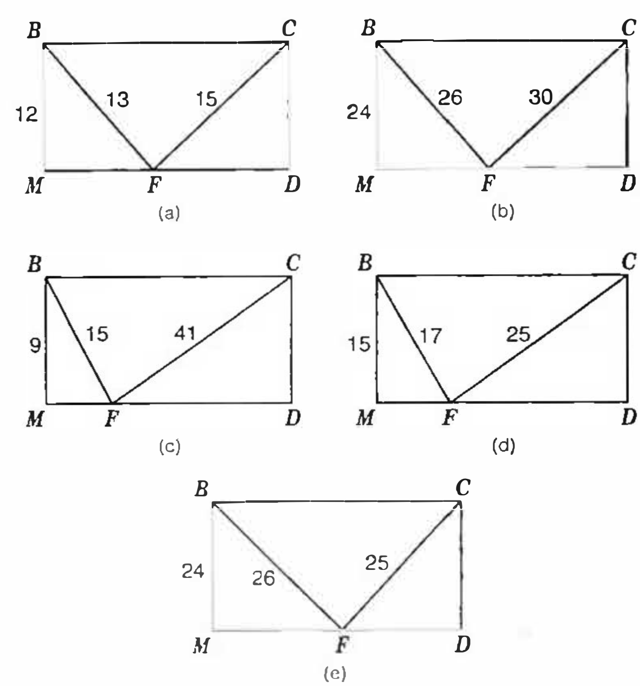
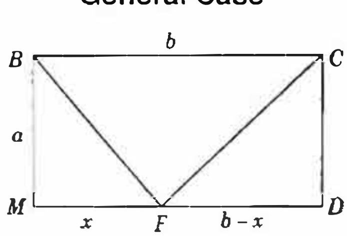


Figure 8
Areas of Triangles and Rectangles: General Case



Planning Instruction

This approach is based on van Hiele's (1986, 177) phases of learning, which can be used as a scheme when planning instruction. Do not confuse the phases with the levels of thought. According to van Hiele, the learning process leading to understanding at the next higher level has five phases, approximately, but not strictly, sequential; in other words, a student goes through various phases in proceeding from one level to the next. The transition from one phase to the following takes place "under influence of a teaching-learning program" (p. 50). Geometry activities should focus on techniques that stimulate students to move from one level of thought to the next and encourage more than one level of thought to provide learning opportunities for the range of students' abilities.

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Mathematicians create by acts of insight and intuition. Logic then sanctions the conquests of intuition.

Morris Kline

I have discovered a truly marvelous proof of this (the result known as Fermat's last theorem), which however the margin is not large enough to contain.

Pierre de Fermat