1999 Calgary Junior Mathematics Contest

Robert E. Woodrow

Part A: Short Answer Problems

No part marks.

A1. How many zero digits does the whole number $\frac{2000}{0.000\ldots01}$ have, if the denominator has 1999 zeros after the decimal point?
Solution: 2003

A2. Which number in the box makes the equation true? $\sqrt{1+\sqrt{1+\sqrt{\square}}} = 2.$
Solution: 64

A3. A cube has sides of length of 1 metre. What is the largest number of corners you can choose so that none of the chosen corners are 1 metre apart?
Solution: 4

A4. How many numbers in the list 1, 2, 3, ..., 1999 are both perfect squares and perfect cubes of whole numbers?
Solution: 3

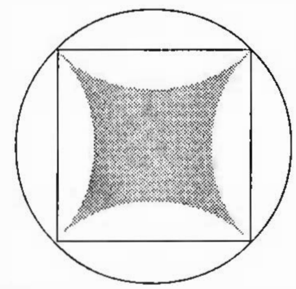
A5. Todd removes exactly 7/12 of the marbles from a jar of marbles. Then Tyler removes exactly ½ of the remaining marbles, and then Zita removes exactly 1/5 of the marbles that are left after that. What fraction of the original number of marbles is now left in the jar?
Solution: 1/6

A6. Find $(1 + \frac{1}{2}) \times (1 + \frac{1}{3}) \times (1 + \frac{1}{4}) \times \ldots \times (1 + \frac{1}{1999})$ in simplest form.
Solution: 1000

A7. Find three different positive integers a, b, c, all less than 10, so that $1/a + 1/b = 3/c$.
Solution: $(a, b, c) = (1, 4, 2)$ or $(2, 4, 3)$ or $(2, 8, 4)$ or $(4, 8, 6)$

A8. It takes Emily 20 minutes to walk to school. It takes Fran 15 minutes to ride her bicycle to school. Fran cycles three times as quickly as Emily walks. If Emily lives 2 km from the school, how far (in km) does Fran live from the school?
Solution: $^9/_2$ or 4.5 km

A9. In the figure, the circle has radius 1 metre. A square is drawn inside it as shown and the four flaps of the circle are folded over the square. What is the area in square metres of the uncovered (shaded) region?



Solution: $4 - \pi$

Part B: Long Answer Problems

Work must be shown to earn full credit. Part marks may be earned for partially correct solutions.

B1. A certain junior high school has Grades 7, 8 and 9. Last year the average grade of all the students in the school was exactly 8. Then all 99 of the Grade 9 students passed and left to go to high school, all the Grade 8 students passed into Grade 9, all the Grade 7 students passed into Grade 8 and 77 new Grade 7 students entered the school. The average grade of all the students in the school this year is still exactly 8. How many students are now in the school?
Solution: The only way the average grade of all the students in the school can be exactly 8 is if there are the same number of Grade 7 students as Grade 9 students. Since there were 99 Grade 9 students last year, there must have been 99 Grade 7 students as well. They have all become Grade 8 students this year. Since there are 77 Grade 7 students this year, and the average Grade is still 8, there must be 77 Grade 9 students this year too (these were the Grade 8 students last year). So the total number of students in the school this year is 77 + 99 + 77 = 253.

B2. Amanda and Cam are traveling to Calgary from Regina. Amanda took a plane while Cam drove. The plane's average speed was 550 km per hour; the average speed of Cam's car was 110 km per hour. Also, the road from Regina to Calgary is 100 km longer than the flight path of the plane, and Cam took 6 hours longer to make the trip than Amanda did. How far in km is it by plane from Regina to Calgary?

Solution: If we let x be the distance in km by plane from Regina to Calgary, then $x + 100$ will be the number of kilometres from Regina to Calgary by road. Since Amanda flies at 550 km/h, her trip will take $x/550$ hours. Since Cam drives at 110 km/h, his trip will take $(x + 100)/110$ hours. Thus we get the equation

$$\frac{x}{550} + 6 = \frac{x + 100}{110},$$

which simplifies to

$$x + 6(550) = (x + 100)5$$

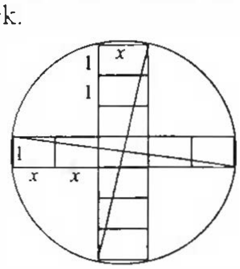
and then

$$x + 3300 = 5x + 500,$$

so $4x = 2800$ or $x = 700$ km.

Of course, this problem can also be done by "guess and test."

B3. A circular garden has two perpendicular paths across it, formed by equal-sized rectangular concrete blocks. In the east-west direction exactly five blocks fit across the garden, and in the north-south direction exactly seven blocks fit, as shown in the diagram. The shorter side of each block is exactly 1 metre. Find the length of the longer side of each block.



Solution: Let x be the length of the longer side of each block, in metres. The east-west path across the garden is a rectangle of length $5x$ metres and width 1 metre, so by the Pythagorean theorem the square of the diagonal of this rectangle will be $(5x)^2 + 1^2 = 25x^2 + 1$. The north-south path is a rectangle of length 7 and width x, so the square of its diagonal will be $49 + x^2$. Both of these diagonals are diameters of the circle, so their squares must be equal. Thus $25x^2 + 1 = x^2 + 49$, which says $24x^2 = 48$, so $x^2 = 2$. Therefore $x = \sqrt{2}$, since the distance x is positive.

B4. Yin and Zack went to a restaurant. When they got their bills, Yin added a 10 percent tip onto her bill while Zack added a 15 percent tip onto his, and the total amount they paid was \$41. If instead Yin had given a 15 percent tip and Zack had only given a 10 percent tip, the total amount they paid would have been \$40. What were the original amounts of their two bills?

Solution: If we add together the two amounts paid, the resulting total of \$81 can be thought of as the original (total) bill doubled, with a tip added on which is 15% + 10% = 25% of the undoubled bill and so 12.5% or 1/8 of the doubled bill. Thus \$81 must be 9/8 of the original doubled bill, so the original bill (doubled) must have been 8/9 of \$81 which is \$72, and the original bill must have been \$36.

On the other hand, if we subtract the two amounts paid we get \$1, and this amounts to 15% – 10% = 5% = 1/20 of the difference (Zack's bill minus Yin's bill). So the original bills must differ by exactly \$20. Therefore we want two amounts whose sum is \$36 and whose difference is \$20, and it is easy to see that they are \$28 and \$8. So Yin's bill was \$8 and Zack's was \$28.

Of course you could also do this problem by algebra or by "guess and test."

B5. Each day Len puts on his socks and shoes and laces up his shoes. Of course he must put his left sock on before he can put on his left shoe, and the same for his right sock and shoe, and he must also put each shoe on before he can lace it up. But otherwise he can put his socks and shoes on and lace them up in any order. In how many different ways can he do these (six) things?

Solution: The first thing Len must do is put on one of his socks. Let's say he puts on his left sock first; if we count the number of ways he can do the other five things after this, then the answer we want is exactly twice this, because there will be same number of ways to do the six things by starting with his right sock (just exchange left and right).

After putting on his left sock, Len has two choices: put on his right sock next or put on his left shoe next. And so forth. Here are all the ways Len can proceed:

- Left sock, right sock, left shoe, lace left shoe, right shoe, lace right shoe.
- Left sock, right sock, left shoe, right shoe, lace left shoe, lace right shoe.
- Left sock, right sock, left shoe, right shoe, lace right shoe, lace left shoe.

- Left sock, right sock, right shoe, lace right shoe, left shoe, lace left shoe.
- Left sock, right sock, right shoe, left shoe, lace right shoe, lace left shoe.
- Left sock, right sock, right shoe, left shoe, lace left shoe, lace right shoe.
- Left sock, left shoe, lace left shoe, right sock, right shoe, lace right shoe.
- Left sock, left shoe, right sock, lace left shoe, right shoe, lace right shoe.
- Left sock, left shoe, right sock, right shoe, lace left shoe, lace right shoe.
- Left sock, left shoe, right sock, right shoe, lace right shoe, lace left shoe.

These are 10 ways, so multiplying by 2, we get a total of 20 ways for Len to put on his socks and shoes.

There is a much easier way to do this problem, if you use combinations (which is not a Grade 9 topic, of course). There are six things to do, in some order, so imagine having a row of six blanks in which you write the six tasks. If you choose three of these six blanks to be where you will write in the three things Len does to his left foot, this determines the entire order, because he must do the left-foot things in a certain order, and the right-foot things must be done in a certain order in the other three blanks. So the number of orders is just the number of ways of choosing three blanks from a row of six blanks, which is

$$\binom{6}{3} = \frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1} = 5 \cdot 4 = 20.$$

B6. Karen has five cats. They each sleep 15 hours each day, though not necessarily at the same time, and not necessarily 15 consecutive hours; they could sleep several times a day for just a short period each time, as long as the total amount they sleep each day is 15 hours.

(a) Does there have to be a time each day in which all five cats are asleep at the same time? Explain your answer.

Solution: The answer to (a) is No, because, for instance, the sleeping schedules for the five cats could be this:

Cat 1: Asleep from midnight to 3 p.m. each day, awake otherwise.
Cat 2: Asleep from 4 a.m. to 7 p.m. each day, awake otherwise.
Cat 3: Asleep from 8 a.m. to 11 p.m. each day, awake otherwise.
Cat 4: Asleep from noon to 3 a.m. each day, awake otherwise.
Cat 5: Asleep from 4 p.m. to 7 a.m. each day, awake otherwise.

Then there are never more than four cats asleep at the same time.

(b) Find the least possible amount of time each day during which there are at least three cats asleep at the same time.

Solution: Suppose that for x hours in a day there are at least three cats asleep, and for the other $24 - x$ hours there are at most two cats asleep. Then the total number of sleeping hours this accounts for is at most $5x$ (which would happen if all five cats were asleep during the x hours) plus $2(24 - x)$ (if during the $24 - x$ hours there were always two cats asleep). This adds up to at most $5x + 2(24 - x) = 3x + 48$ sleeping hours. But the five cats together have a total of $5 \cdot 15 = 75$ sleeping hours, so $3x + 48$ must be at least 75. Solving this inequality we get that x is at least 9.

In fact we claim that $x = 9$ is possible. To prove this we must find a daily sleeping schedule for the cats which has only 9 hours during which at least three cats are asleep. To do this we will need that all five cats are asleep during these 9 hours, and that at other time there are never three cats asleep at once. Here is one way to do this. All cats are asleep from midnight to 9 a.m. each day, and also:

	9 a.m. to noon	noon to 3 p.m.	3 p.m. to 6 p.m.	6 p.m. to 9 p.m.	9 p.m. to midnight
Cat 1	asleep	asleep			
Cat 2		asleep	asleep		
Cat 3			asleep	asleep	
Cat 4				asleep	asleep
Cat 5	asleep				asleep

So the answer is 9 hours.

Another way to do this problem would be to find such a daily schedule that for nine hours all five cats are asleep while during the other 15 hours only two cats are asleep, and then argue that since we've packed the maximum number of sleeping cats into these 9 hours, and at all other times there are always two cats asleep, there is no "room" to have an answer of less than 9 hours.