n-Secting a Line Segment by Straightedge and Compass Methods (with an Inductive Proof Using Vector Theory)

Murray L. Lauber and Keith Harder

The motivation for this article grew out of a problem presented to me by an artist colleague at Augustana University College, Keith Harder. He had located the thirds, fourths, fifths and so on of the diagonals of a rectangle on a grid by methods in harmony with those described here and was looking for a mathematical justification of the methods. The grid illustrated a variety of methods of finding these thirds, fourths, fifths and so on. In the interests of simplicity, I have focused on just one.

The article consists of two parts: the first, a description of a method of n-secting a line segment; the second, a proof documenting why the method works using vector theory.

The Method

Construction Procedures Employed

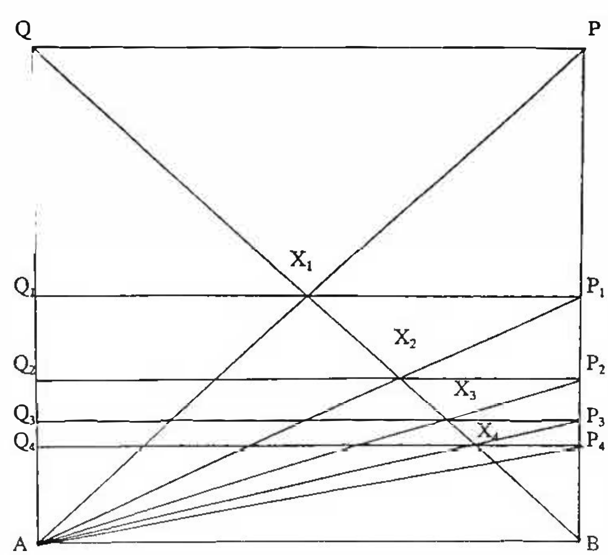
The method is based on two elementary straightedge and compass procedures:

- Constructing a rectangle with a given line segment as one of its diagonals (this can be done by bisecting the segment, constructing the circle with the segment as a diameter, then using that diameter along with a second diameter as the two diagonals of the rectangle).
- Constructing a line segment parallel to a given line segment through a given point.

Description of the Steps

The problem of n-secting a line segment is equivalent to the problem of n-secting a diagonal of a rectangle. This can be done on diagonal BQ of rectangle ABPQ as follows.

1. Let X_1 be the point of intersection of diagonals AP and BQ. Then X_1 bisects diagonal BQ.
2. Construct Q_1P_1 parallel to AB through X_1 as indicated in the diagram above. Then construct diagonal AP_1 of rectangle ABP_1Q_1. Let X_2 be the point of intersection of BQ and AP_1. Then X_2 is one-third of the way from B to Q. Call X_2 a "third" of BQ. Then the other third of diagonal BQ, the one nearest Q, can be located by an analogous process, completing the trisection of BQ.
3. Construct Q_2P_2 parallel to AB through X_2 as indicated. Then construct diagonal AP_2 of rectangle ABP_2Q_2. Let X_3 be the point of intersection of AP_2 and BQ. Then X_3 is one-fourth of the way from B to Q. Call X_3 a "fourth" of BQ. Then another fourth, the one nearest Q, can be located by an analogous process. These two fourths along with X_1 quadrasect, or 4-sect, BQ.
4. Construct Q_3P_3 parallel to AB through X_3 as indicated. Then construct diagonal AP_3 of rectangle ABP_3Q_3. Let X_4 be the point of intersection of AP_3 and BQ. Then X_4 is a "fifth" of BQ. Locate the fifth of BQ nearest Q, as well as the two-fifths of diagonal AP nearest A and P, by analagous processes. Construct the rectangle determined by these four-fifths. Then locate the thirds of the diagonal of this smaller rectangle coincident with diagonal BQ of rectangle ABPQ. These thirds, along with the fifths of BQ, 5-sect BQ.

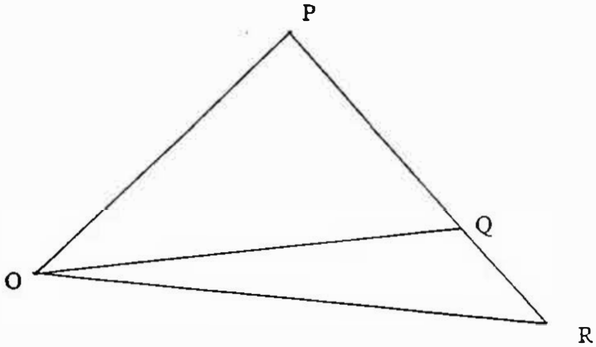


5. This process can be extended inductively, at least in theory, to n-sect BQ, for n any natural number larger than 5. For n prime, the last step in the process is exactly analogous to step 4 above. For n composite, it may be possible to reduce the number of steps in the process. For example, for $n = 6$, once the outer sixths are located, the inner sixths are the two-thirds and the bisector of BQ which have already been located.

Proof

The following is an inductive proof that a diagonal of a rectangle can be n-sected by the method described in the first section. It depends heavily on the rule that ratios of corresponding sides of similar triangles are equal and the following definition and theorem.

- **Definition**: Two vectors $\vec{u}$ and $\vec{v}$ are collinear if $\vec{u} = k\vec{v}$ where k is a nonzero real number.
- **Theorem** (Elliott et al. 1984, 302–03): If P, Q and R are collinear points such that $\vec{PQ} = k\vec{QR}$ and O is any other point, then $\vec{OQ} = \frac{k}{k+1}\vec{OR} + \frac{1}{k+1}\vec{OP}$



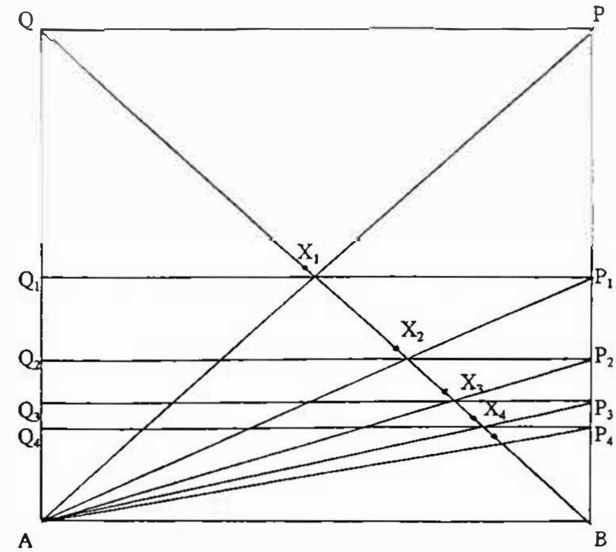
For example, if $\vec{PQ} = 2\vec{QR}$, *then* $\vec{OQ} = 2/3\ \vec{OR} + 1/3 + 1\ \vec{OP}$

The steps in the proof, based on the reference rectangle, are as follows.

1. Construct diagonals AP and BQ of rectangle ABPQ. Let X_1 be the point on diagonal BQ such that $\vec{QX_1} = X_1B$. Then, by the theorem cited above with $k = 1$,

   $\vec{AX_1} = ½\vec{AB} + ½\vec{AQ} = ½\vec{AB} + ½\vec{BP} = ½(\vec{AB}+\vec{BP}) = ½\vec{AP}$.

   Thus X_1 is on AP and is the point of bisection of diagonals AP and BQ.
2. Construct Q_1P_1 through X_1 parallel to AB as indicated in the diagram. Then, since $\vec{BX_1} = ½\vec{BQ}$, $\vec{BP_1} = ½\vec{BP}$ (similar triangles, aaa). Construct diagonal AP_1 of rectangle ABP_1Q_1. Let X_2 be the point on BQ such that $\vec{QX_2} = \overline{2X_2B}$. Then, by the

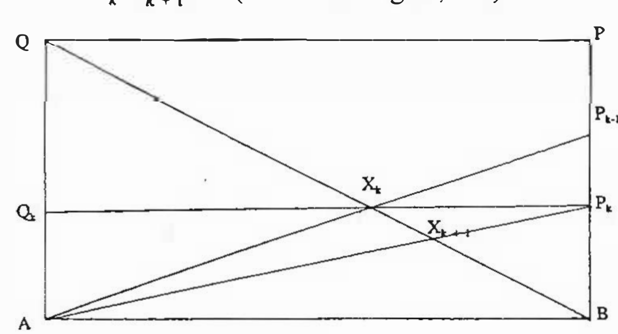


theorem with $K = 2$, $\vec{AX_2} = {}^2/_3\vec{AB} + {}^1/_3\vec{AQ} = {}^2/_3\vec{AB} + {}^1/_3\vec{BP} = {}^2/_3(\vec{AB} + {}^1/_2\vec{BP}) = {}^2/_3(\vec{AB} + \vec{BP_1}) = {}^2/_3\vec{AP_1}$. Thus X_2 is on AP_1, that is X_2 is the point of intersection of AP_1 and BQ. Further, X_2 is a third of BQ.

3. Construct Q_2P_2 through X_2 parallel to AB as indicated. Then, since $\vec{BX_2} = {}^1/_3\vec{BQ}$ (this follows from the conclusion in 2) that $\vec{QX_2} = 2X_2B$, $\vec{BP_2} = {}^1/_3\vec{BP}$ (similar triangles, aaa). Construct diagonal AP_2 of rectangle ABP_2Q_2. Let X_3 be the point on BQ such that $\vec{QX_3} = 3X_3B$. Then, by the theorem with $K = 3$,

   $\vec{AX_3} = {}^3/_4\vec{AB} + {}^1/_4\vec{AQ} = {}^3/_4\vec{AB} + {}^1/_4\vec{BP}$
   $= {}^3/_4(\vec{AB} + {}^1/_3\vec{BP}) = {}^3/_4(\vec{AB} + \vec{BP_2}) = {}^3/_4\vec{AP_2}$.

   Thus X_3 is on AP_2, is the point of intersection of AP_2 and BQ, and is a fourth of BQ.
4. Continue the above process inductively. Let X_k be the point of intersection of BQ and AP_{k-1}. Construct Q_kP_k through X_k parallel to AB and assume that $\vec{QX_k} = k\vec{X_kB}$. It follows that $\vec{BX_k} = \frac{1}{k+1}\vec{BQ}$ *and so* $\vec{BP_k} = \frac{1}{k+1}\vec{BP}$ (similar triangles, aaa).



Construct diagonal AP_k of rectangle ABP_kQ_k. Let X_{k+1} be the point on BQ such that

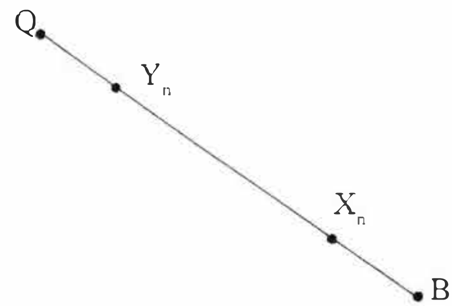
$\vec{QX_{k+1}} = (k + 1)\ X_{k+1}\ \bar{B}$. Then

$$\vec{AX}_{k+1} = \frac{k+1}{k+2}\vec{AB} + \frac{1}{k+2}\vec{AQ} = \frac{k+1}{k+2}(\vec{AB} + \frac{1}{k+1}\vec{AQ})$$
$$= \frac{k+1}{k+2}(\vec{AB} + \frac{1}{k+1}\vec{BP}) = \frac{k+1}{k+2}(\vec{AB} + \vec{BP}_k)$$
$$= \frac{k+1}{k+1}\vec{AP}_k$$

Thus X_{k+1} is on AP_k, is the point of intersection of BQ and AP_k, and is $\frac{1}{k+2}$ of the way from B to Q.

The preceding inductive proof verifies that the methods described in this article can be used to find a point that is an *n*th of the way from B to Q along diagonal BQ of rectangle ABPQ. What remains to be shown is that the methods can be used to partition BQ into *n* equal segments. The following sketch of an inductive argument, supported by the Strong Principle of Mathematical Induction, should convince the reader that this can be done.

Suppose that X_n is an *n*th of the way from B to Q. Then a point Y_n that is an *n*th of the way from Q to B can be found by an analogous process. The completion of the n-secting of BQ can then be accomplished by (n – 2)-secting X_nY_n.



Of course the n-secting of BQ could also be accomplished by the conventional method of using a compass to duplicate segments of length BX_n along BQ, but the inductive procedure described above seems somehow more satisfying.

Bibliography

Elliott, H., et al. *Math 31*. Toronto: Holt, Rhinehart & Winston of Canada, 1984, pp. 269–312.

Nicholson, W. K. *Linear Algebra with Applications*. 3d ed. Boston: PWS Publishing, 1990, pp. 141–55.

The tantalizing and compelling pursuit of mathematical problems offers mental absorption, peace of mind amid endless challenges, repose in activity, battle without conflict, refuge from the goading urgency of contingent happenings and the sort of beauty changeless mountains present to senses tried by the present-day kaleidoscope of events.

Morris Kline