Exemplifying If-Then Principles in Logical Reasoning

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Teachers are always seeking real-world settings that exemplify symbolic logic principles. The implication P -> Q (if P then Q) often causes student concern. We shall present an example that may be used to illustrate the many facets of this implication.

Consider the following two statements concerning a calendar month:

S = the month is September.
T = the month has exactly 30 days.

The implication $S \rightarrow T$ is clearly valid because every month of September has exactly 30 days; in other words, if the month is September, then it has 30 days.

Let us consider the other implications associated with $S \rightarrow T$.

1. *Converse:* $T \rightarrow S$
   This implication asserts that if the month has 30 days, then it must be September. This is clearly not valid reasoning because the month could also be April, June or November. The example supports the fundamental idea that a proposition is *not* equivalent to its converse.
2. *Inverse:* $\sim S \rightarrow \sim T$
   This implication asserts that if the month is not September, then the month cannot have 30 days. This is also not valid for the same reason cited in (1) above. As in (1), a proposition is not equivalent to its inverse.
3. *Contrapositive:* $\sim T \rightarrow \sim S$.
   Because this differs so much from the original statement, students often do not believe that they can be equivalent; however, the contrapositive reasserts the original implication. In this case, if the month does not have 30 days, then it cannot be September. This is clearly valid. A non--30-day month might be any of January, February, March, May, July, August, October or December; it cannot be September.

Another facet of the implication process involves the use of the terms *sufficient* and *necessary*. These terms are used as alternate ways of expressing P -> Q:

1. P is sufficient for Q.

This means that if we are trying to establish the truth of Q, then P provides sufficient information (all that is needed to do so). Although P is sufficient it may provide more information than is needed.

2. Q is necessary for P.

This means that if we are trying to establish the truth for P, then Q is essential in this process. Although Q is needed it might itself not be enough evidence to establish P.

How can the September-30 days example exemplify the sufficient and necessary settings? Recall that $S \rightarrow T$.

1. S is sufficient for T. If you had a calendar page with the last week torn off and you were trying to establish that the missing section ended with 30 days, the presence of the September label at the top of the page would be all the evidence you would need. The evidence is sufficient.
2. T is necessary for S. If you had a calendar page with the month label missing and you were trying to establish that the month was September, the first thing that you would check would be the number of days in the month. For your inquiry to continue there must be exactly 30 days. If not, admit defeat. So 30 days is necessary for September.

Can sufficient and necessary be interchanged in examples 1 and 2 just preceding?

1. Is S necessary for T?
   No, some other months also have 30 days. The absence of this piece of evidence is not critical to your case.
2. Is T sufficient for S?
   Again, the answer is no. Finding that the month has 30 days may be helpful in proving that it is September, but it is not conclusive (sufficient).

Thus, the terms *sufficient* and *necessary* are not interchangeable.

Challenge: Find other settings to illustrate the implication principles of this article.