An Interesting Mathematical Fallacy That All Triangles Are Isosceles

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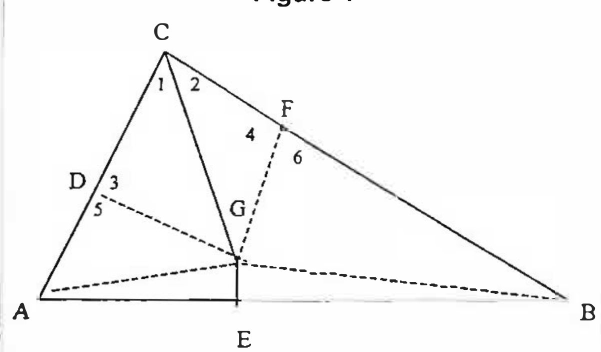
This fallacy is a remarkable proof that all triangles are isosceles.

Figure 1



Let ABC be any triangle. Construct the bisector of $\measuredangle$ C and the perpendicular bisector of side AB. From G, their point of intersection, drop perpendiculars GD and GF to AC and BC respectively and draw AG and BG. Now in triangles CGD and CGF, $\measuredangle 1 = \measuredangle 2$ by construction and $\measuredangle 3 = \measuredangle 4$ because all right angles are equal. CG is common to the two triangles. Therefore triangles CGD and CGF are congruent. (If two angles and a side of one triangle are equal respectively to two angles and a side of another, the triangles are congruent.)

Consequently, DG = GF. (Corresponding parts of congruent triangles are equal.) Then in triangles Δ GDA and Δ GFB, $\measuredangle 5$ and $\measuredangle 6$ are right angles and, because G lies on the perpendicular bisector of AB, AG = GB. (Any point on the perpendicular bisector of a segment is eqidistant from the ends of the segment.) Therefore triangles Δ GDA Δ and GFB are congruent. (If the hypotenuse and another side of one right angle triangle are equal respectively to that of a second, the triangles are congruent.)

From these two sets of congruent triangles, Δ CGD and CGF and GDA and GFB, we have, respectively, CD = CF and DA = FB. By addition, we conclude that CA = CB, so that Δ ABC is isosceles by definition.

Actually, we do not know that EG and CG meet within the triangle, so we shall examine all other possibilities. The above proof is valid in the cases where G coincides with E (Figure 2) *or* where G is outside the triangle but so near to AB that D and F fall on CA and CB (Figure 3).

Figure 2

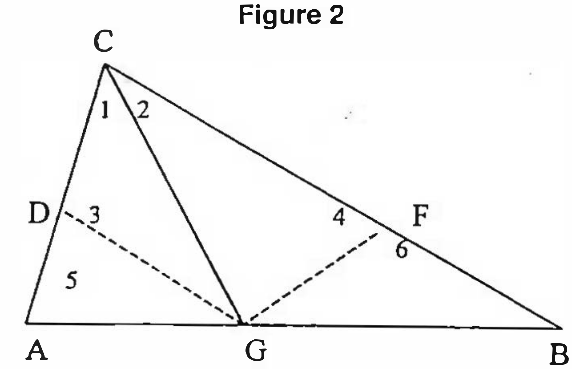
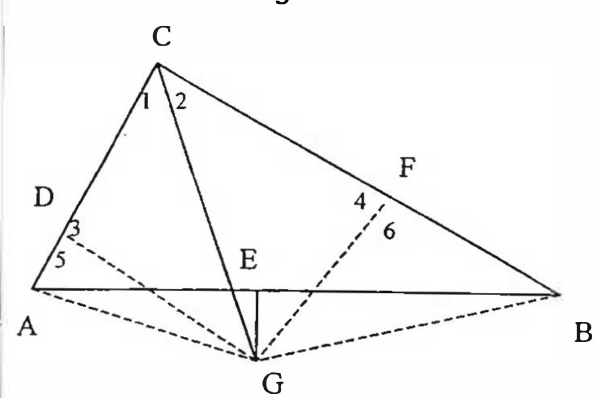
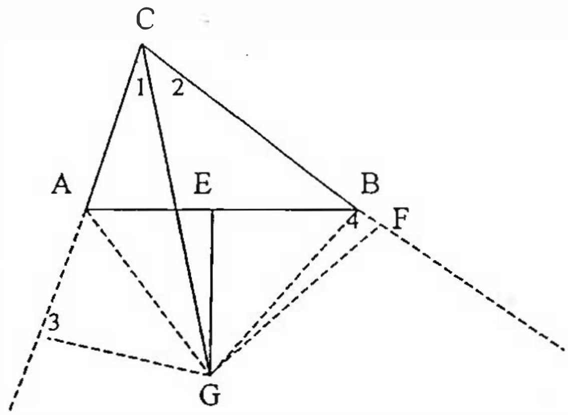


Figure 3



There is the possibility as in Figure 4 that G lies so far outside the triangle that D and F fall on CA and CB produced:

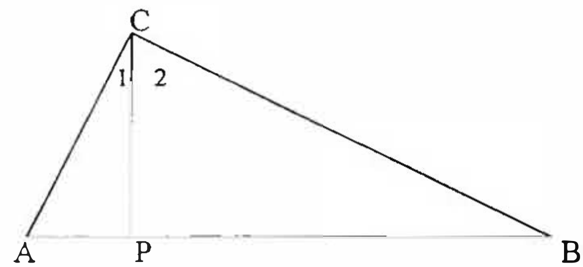
Figure 4



Again, as in the first case, Δ CGD and Δ CGF are congruent, as are Δ GDA and Δ GFB. And again, CD = CF and DA = FB. But in this case, we subtract these last two equations to get CA = CB.

Finally, it may be suggested that CG and EG do not meet in a single point G but either coincide or are parallel. Figure 5 shows that in either of these cases, the bisector CP of angle C will be perpendicular to AB, so that $\measuredangle 7 = \measuredangle 8$. Then $\measuredangle 1 = \measuredangle 2$, CP is common and Δ APC is congruent to Δ BPC. Again, CA = CB.

Figure 5



It appears that we have exhausted all possibilities and that we must accept the obviously absurd conclusion that all triangles are isosceles.

Actually, the possibility we failed to examine is the case where one of the points D and F falls outside the triangle and the other falls inside.

Bibliography

Gardner, M. *Mathematical Puzzles and Diversions*. New York: Simon & Schuster, 1959.

Meyer, J. S. *Fun with Mathematics*. New York: World, 1952.

National Council of Teachers of Mathematics (NCTM). *The Mathematics Student* 24, no. 1 (1976).

Northrup, E. *Riddles in Mathematics*. New York: Van Nostrand, 1957.

Tocquet, R. *The Magic of Numbers*. Trans. by D. Weaver. New York: A. S. Barnes, 1957.

Bees . . . by virtue of a certain geometrical forethought . . . know that the hexagon is greater than the square and the triangle and will hold more honey for the same expenditure of material.

Pappus of Alexandria