Rational Fractions: Combine and Simplify

Ron Persky

This article is motivated by the instructor's task in an algebra course to find test questions on the topic commonly labeled "adding and subtracting rational expressions." An example is $x + 8/x - 3 - x - 14/3 - x$ which reduces to 2. But the denominators are multiples of each other. The objective here is to create this type of problem in which factors are not multiples of each other and which, after combining, have a numerator/denominator common factor simplification. We show that if we restrict ourselves to problems not using polynomials of degree higher than two in any calculations, this objective is not possible without having common factors in the denominators. Then we show how to find problems when one allows common factors.

The expressions we shall examine involve just two terms. In the following list, Form 1 is a starting point. The forms progress in complexity, each one providing an opportunity for numerator/denominator cancellation that was not afforded previously. The variable is x; all other letters denote constants. Assume factors written with different constants, for example, $ax + b$ and $cx + d$, are not multiples of one another (relatively prime).

Form 1. $\frac{u}{ax+b} + \frac{v}{cx+d}$

Form 2. $\frac{ux}{ax+b} + \frac{v}{cx+d}$ Note, $\frac{ux}{ax+b} + \frac{vx}{cx+d}$ would revert this back to form 1 for our purpose.

Form 3. $\frac{ux+e}{ax+b} + \frac{vx+f}{cx+d}$

Form 4. $\frac{u}{ax+b} + \frac{v}{(cx+d)(ex+f)}$ This can be modified by replacing v by vx or by $vx + w$.

Form 5. $\frac{u}{(ax+b)(cx+d)} + \frac{v}{(ex+f)(qx+r)}$

Form 5 is as complex as will be considered. If we change, for example, u to ux, the form 5 numerator combines to a third degree polynomial. This is too much for a test question.

None of the forms above will result in numerator/denominator cancellation when combined providing that the restriction of having factors relatively prime is kept. Showing this is basically the same for all forms, as exemplified for form 4:

$$\frac{u}{ax+b} + \frac{v}{(cx+d)(ex+f)} = \frac{u(cx+d)(ex+f)+v(ax+b)}{(ax+b)(cx+d)(ex+f)}.$$

Suppose we wish to have a numerator/denominator cancellation of the factor $ax + b$. Thinking of the numerator as a polynomial, $p(x)$, it would require that $p(\frac{-b}{a}) = 0$ for $ax + b$ to be a factor of $p(x)$. For that to happen, either $c(\frac{-b}{a}) + d = 0$ or $e(\frac{-b}{a}) + f = 0$. Suppose $c(\frac{-b}{a}) + d = 0$. This is equivalent to $\frac{a}{c} = \frac{b}{d}$ which is equivalent to $ax + b = k(cx + d)$. In other words, for $ax + b$ to be a common factor in the numerator and denominator, it would not be relatively prime to either $cx + d$ or $ex + f$. Similarly for either $cx + d$ or $ex + f$ to be a factor of the combined numerator.

To achieve a numerator/denominator cancellation, we relax the requirement of having all factors relatively prime. We will look at forms 4 and 5 in this light.

Consider form 4 modified to $\frac{u}{ax+b} + \frac{v}{k(ax+b)(ex+f)}$. Write this as $\frac{1}{ax+b}[u + \frac{v}{k(ex+f)}]$. Assign any values we like for u, v, k, e and f and combine what we have inside the bracket. The resulting numerator determines the common factor $ax + b$. For example, $k = 2$, $e = 3$, $f = 1$, $u = 1$ and $v = -4$ results in $ax + b = 3x - 1$. The problem presents itself as $\frac{1}{3x-1} - \frac{4}{(6x-2)(3x+1)}$.

For this same form 4, starting from $\frac{1}{ax+b}[u + \frac{v}{k(ex+f)}]$ $= \frac{1}{ax+b}[\frac{ukex+(ukf+v)}{k(ex+f)}]$, we can predetermine $ax + b$. For example, if we want $a = 6$ and $b = 5$, set $uke = 6$ and $ukf + v = 5$. If we make $k = 2$, $e = 1$ and $f = 1$, then $u = 3$ and $v = -1$. The problem presents itself as $\frac{3}{6x+5} + \frac{1}{(12x+10)(x+1)}$.

Form 5 can be modified to

$$\frac{u}{(ax+b)(cx+d)} + \frac{v}{k(ax+b)(gx+r)} = \frac{1}{ax+b}[\frac{u}{cx+d} + \frac{v}{k(gx+r)}].$$

Assign any values to the constants inside the bracket, combine and the resulting numerator determines the common factor. $ax + b$. Example: $c = 2$, $d = 1$, $k = 1$, $g = 5$, $r = 2$, $u = 3$ and $v = -7$ results in $ax + b = x - 1$.

The problem presents itself as $\frac{3}{(x-1)(2x+1)} - \frac{7}{(x-1)(5x+2)}$.

While changing u to ux in form 5 results in a third degree polynomial, this is not the case when there is a common factor. The form becomes

$$\frac{ux}{(ax+b)(cx+d)} + \frac{v}{k(ax+b)(ex+f)}$$

$$=\frac{1}{ax+b}\left[\frac{ukex^2+(ukf+vc)x+vd}{k(cx+d)(ex+f)}\right].$$

The numerator inside the bracket can be factored as $(uex + v)(kx + d)$ providing

$ukf + vc = ued + vk$

To make this happen, we can begin, for example, by picking, somewhat arbitrarily, $u = 2$, $e = 1$, $v = 3$ and $k = 2$. This gives

$4f + 3c = 2d + 6$

Set $d = -3$ (arbitrary).

$4f + 3c = 0$.

Find a solution: $f = -3$, $c = 4$. If $4f + 3c = 0$ had no solution, we could reassign a value to d.

Now the numerator inside the bracket is $(2x + 3)(2x - 3)$. Assign either factor to $ax + b$. If we use $ax + b = 2x + 3$, the problem presents itself as

$$\frac{2x}{(2x+3)(4x-3)} - \frac{3}{(4x+6)(x-3)}$$

The moving power of mathematical invention is not reasoning but imagination.

Augustus de Morgan

The pleasure we obtain from music comes from counting, but counting unconsciously. Music is nothing but unconscious arithmetic.

Gottfried Wilhelm von Leibniz