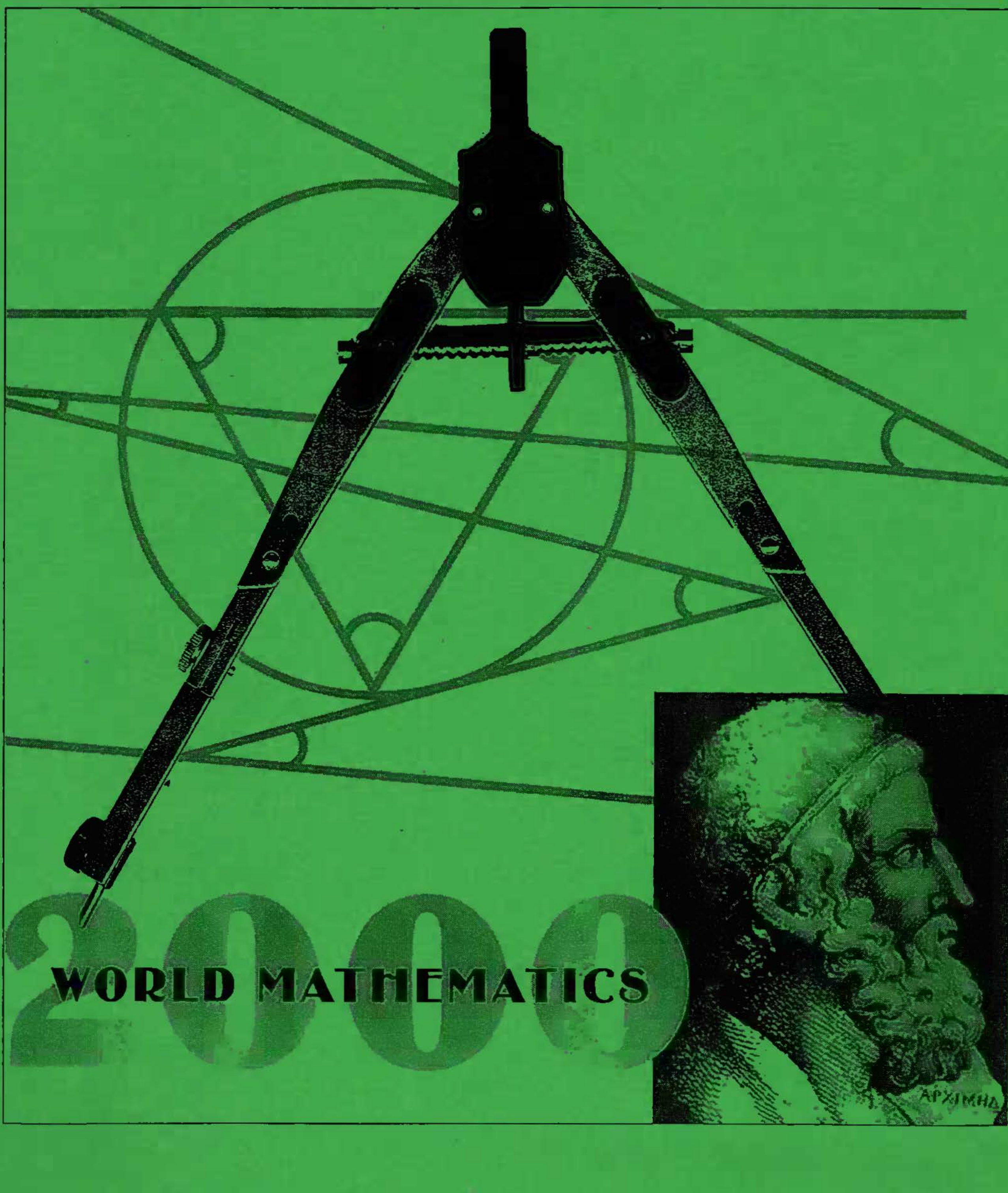
delta-k

JOURNAL OF THE MATHEMATICS COUNCIL OF THE ALBERTA TEACHERS' ASSOCIATION

Volume 38, Number 1 — December 2000



GUIDELINES FOR MANUSCRIPTS

delta-K is a professional journal for mathematics teachers in Alberta. It is published to

- promote the professional development of mathematics educators, and
- stimulate thinking, explore new ideas and offer various viewpoints.

Submissions are requested that have a classroom as well as a scholarly focus. They may include

- personal explorations of significant classroom experiences;
- descriptions of innovative classroom and school practices;
- reviews or evaluations of instructional and curricular methods, programs or materials;
- discussions of trends, issues or policies;
- a specific focus on technology in the classroom; and
- a focus on the curriculum, professional and assessment standards of the NCTM.

Manuscript Guidelines

1. All manuscripts should be typewritten, double-spaced and properly referenced.
2. Preference will be given to manuscripts submitted on 3.5-inch disks using WordPerfect 5.1 or 6.0 or a generic ASCII file. Microsoft Word and AmiPro are also acceptable formats.
3. Pictures or illustrations should be clearly labeled and placed where you want them to appear in the article. A caption and photo credit should accompany each photograph.
4. If any student sample work is included, please provide a release letter from the student's parent allowing publication in the journal.
5. Limit your manuscripts to no more than eight pages double-spaced.
6. A 250–350-word abstract should accompany your manuscript for inclusion on the Mathematics Council's Web page.
7. Letters to the editor or reviews of curriculum materials are welcome.
8. *delta-K* is not refereed. Contributions are reviewed by the editor(s) who reserve the right to edit for clarity and space. **The editor shall have the final decision to publish any article.** Send manuscripts to Klaus Puhlmann, Editor, PO Box 6482, Edson, Alberta T7E 1T9; fax 723-2414, e-mail klaupuhl@gyrd.ab.ca.

Submission Deadlines

delta-K is published twice a year. Submissions must be received by August 31 for the fall issue and December 15 for the spring issue.

MCATA Mission Statement

Providing leadership to encourage the continuing enhancement of teaching, learning and understanding mathematics.

delta-K

Volume 38, Number 1 December 2000

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 delta-K is published by the ATA for the Mathematics Council (MCATA). EDITOR: Klaus Puhlmann, PO Box 6482, Edson, Alberta T7E 1T9. EDITORIAL AND PRODUCTION SERVICES: Central Word Services staff, ATA. Opinions expressed herein are not necessarily those of MCATA or of the ATA. Address correspondence regarding this publication to the editor. *delta-K* is indexed in the Canadian Education Index. ISSN 0319-8367

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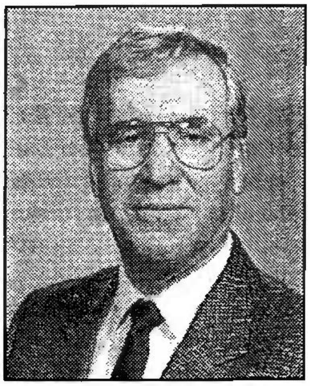
The essence of mathematics is in its freedom.

George Cantor

EDITORIAL



When you receive this issue of *delta-K* in early 2001, World Mathematics Year 2000 (WMY 2000) and our annual conference that focused on World Mathematics Year 2000 will be events of the past. Should we allow these events to simply fade away or should we continue to celebrate, acknowledge and work in the spirit of the Declaration of Rio de Janeiro on Mathematics of May 6, 1992, when the International Mathematical Union (IMU), supported by the United Nations Educational, Scientific, and Cultural Organization (UNESCO), declared the Year 2000 to be World Mathematics Year? To me, the answer is obvious because the declaration embraced three important goals that are worthy of our focus for many years to come:

- The determination of great mathematical challenges of the 21st century
- The promulgation of mathematics, both pure and applied, as one of the main keys for development
- The recognition of the systematic presence of mathematics in the information society (the image of mathematics)

In planning for WMY 2000, The Turn of the Century Committee was established and was chaired by Jacob Palis (IMU secretary, Brazil), with nine additional members from around the world. This committee envisioned what the great mathematical challenges of the 21st century will be. The computer presence for World Mathematics Year 2000 was established in Paris with the Internet address, http://wmy2000.math.jussieu.fr.

The resolution that declared the year 2000 as World Mathematics Year was driven by a belief that the discipline of mathematics is playing an increasingly crucial role as our society makes the transition from the Industrial Age to the Information Age. Mathematical knowledge and skills are essential for the growth of information and knowledge-based industries, and they are necessary for Canada to be competitive in the global economy. In its support for the declaration, UNESCO highlighted the central importance of mathematics and its applications in today's world with regard to science, technology, communications, economics and numerous other fields.

UNESCO is also aware that mathematics has deep roots in many cultures and that the most outstanding thinkers over several thousand years contributed significantly to its development. As well, the language and the values of mathematics are universal, thus making it ideally suited for international cooperation. UNESCO in its resolution to support the declaration also stressed the key role of mathematics education, in particular at the primary and secondary levels, for both the understanding of basic mathematical concepts and the development of rational thinking.

World Mathematics Year 2000 projects and activities included the following:

- A logo for WMY 2000
- WMY 2000 posters
- National WMY 2000 website
- WMY 2000 newsletter (started in 1993 with one per year and two per year in 1999)
- WMY 2000 homepage included global links
- Global reports about WMY 2000 activities

The world truly celebrated WMY 2000. The many countries that participated had comprehensive programs, which were shared on the website.

Canada, too, celebrated. The Canadian Mathematical Society (CMS) created a committee for WMY 2000 in 1997 to develop proposals for events during the year 2000 to make mathematics more visible in Canada. In addition to this CMS initiative, other Canadian mathematical societies and institutes participated and proposed activities to celebrate WMY 2000.

In celebration of WMY 2000, the CMS and CAIMS (Canadian Applied and Industrial Mathematics Society) met for the first time in a joint annual meeting, June 10–14, in Hamilton, Ontario. These societies were joined by the Canadian Operations Research Society, the Canadian Society for the History and Philosophy of Mathematics, the Canadian Symposium on Fluid Dynamics and the Canadian Undergraduate Mathematics Conference. This joint meeting brought together the largest number of Canadian mathematical scientists ever assembled in one place. Mathematicians from around the world attended as well.

In June 2000 at the Royal Ontario Museum in Toronto, the Fields Institute hosted a symposium to inform all Canadians of our unsung hero in the mathematical sciences, the visionary John Charles Fields, and his exceptional contributions to the world of mathematics. He established the world's highest award for achievement in mathematics, now known internationally as the Fields Medal (and often referred to as the "Nobel Prize of Mathematics"). It is struck by the Royal Canadian Mint of Canadian gold and shows the head of the ancient Greek mathematician Archimedes on the face. As a lasting record of this unique event, a documentary video and book have been produced.

In Montreal, Operation Metro-2000 was organized with support from CRM (Centre de recherches mathématiques), CMS and other sources. Posters were placed in the Montreal subway system to raise public awareness, particularly among students, of the importance and omnipresence of mathematics in the sciences and technology.

In western Canada, the WMY 2000 Museum of Mathematics Project brought the highly acclaimed travelling public exhibition, "Museum of Mathematics" [Mathematikmuseum], to visit the Winnipeg Children's Museum and the Saskatchewan Science Centre in Regina for two weeks each in May.

The Pacific Institute for the Mathematical Sciences (PIMS) promoted mathematics awareness by holding public lectures, presentations and hands-on workshops. Like Operation Metro-2000 in Montreal, for its "Mathematics Is Everywhere" campaign, PIMS placed posters on all public transportation systems in British Columbia and Alberta to increase public and student awareness of the importance and omnipresence of mathematics in the sciences and technology.

A number of British Columbia and Alberta elementary schools presented fun methods for doing mathematics and computer science with children and their parents.

The MCATA conference in Red Deer was itself a major event organized around the theme "World Mathematics Year 2000."

I believe that WMY 2000 celebrations need to continue well beyond this year. I also believe that if we want to achieve the goals set out in the Rio de Janeiro declaration, schools and school systems must take an active part in addressing them. It is at the school level where it all begins and where a love for mathematics develops. It is really here where the seeds of a mathematical culture are sown, so that one day mathematics, like music, is worth doing for its own sake. This is not to deny the great usefulness of mathematics; this very usefulness, however, tends to conceal and disguise the cultural aspect of mathematics. The role of music suffers no such distortion because it is clearly an art whose exercise enriches composer, performer and audience; music does not need to be justified by its contributions to some other aspect of human existence. Nobody asks after listening to a Beethoven symphony, "What is the use of that?" Moreover, mathematics does not gain in utility by having its inherent worth ignored—on the contrary, an appreciation of mathematics and an understanding of its inherent quality and dynamic are necessary to be able to apply it effectively.

World Mathematics Year 2000 was more than just engaging in a few mathematical activities. It was and continues to be about making a case for mathematics as something that is deeply rooted in our culture, as something that grows and develops in many ways unrelated to science and that therefore plays a crucial role in the history of human thought.

I long for the day when mathematics will be appreciated and enjoyed by students and educated lay people as an art and also respected as the mainstay of science. It has been so in the past, but it is not so now. World Mathematics Year 2000 allowed us to work with our students and communities toward that goal, but it must not end here.

Klaus Puhlmann

From the President's Pen

As a new president, I was nervous about writing this first From the President's Pen message, so I decided to look back through some issues of *delta-K* to see what others had written. After the first few, I found myself looking at the journal articles as well as examining messages from previous presidents.

When I receive a new issue of the journal, I look through it, read an eye-catching article, perhaps show a teaching colleague and then store it on a shelf. As I sat and looked through several in a row, I began to get a different perspective and could easily see why our journal has been internationally recognized through the National Council of Teachers of Mathematics (NCTM). NCTM recognized *delta-K* during the editors' session at its conference in San Francisco in 1999. Nancy Hawthorne described *delta-K* as a stellar mathematical journal. Our journal was also cited as an excellent model for other editors to emulate.

Klaus Puhlmann, *delta-K* editor, has done an outstanding job of writing, phoning, e-mailing, collecting and following up on articles that he wants in each issue. I can only guess at the amount of time he has put into this endeavor over the years. He works diligently on members' behalf to produce a quality publication. In the January 1999 issue, Klaus states his hope that "all readers find something interesting, useful and challenging" in every issue of *delta-K*. That's what makes this an award-winning journal. He ensures quality of articles, variety of topics, variety of levels and currency of mathematical issues for Alberta teachers, and he connects to the larger mathematical teaching and learning community through attention to the NCTM.

It is inspiring to see contributions from practising teachers as well as researchers and leaders in the field of mathematics education. The researchers inform our classroom practice, but it is the classroom teachers, through their practical knowledge and daily work with students, who make the love and learning of mathematics happen.

The Student Corner needs you. Have you come across that student explanation of a problem that made you say, "Wow!?" We want to see it. Encourage your students to send in some of their work. Klaus has given some good ideas for you to consider. Let's add the student voice to our journal. Let's create a culture of mathematics thinkers who reach out to the larger community through this fine journal.

I look forward to my term as MCATA president and encourage you to contact other executive members or me so that we know how you feel about the issues in math education in the province.

Sandra Unrau

Whatever your difficulties in mathematics,
I can assure you mine are far greater.

Albert Einstein

MCATA Executive in Action

Klaus Puhlmann

The pictures below depict your MCATA executive members during their meeting September 8–9. Executive members are primarily classroom teachers but also include representatives from Alberta Learning, the Faculty of Education, the Department of Mathematics, the ATA staff, the Provincial Executive Council and the National Council of Teachers of Mathematics (Canadian director). Executive meetings are always held Friday evening and all-day Saturday. Often, subcommittees continue until late Saturday evening following the executive meeting. Addressing the many pertinent issues surrounding the teaching and learning of mathematics is always the main focus of business. As a long-term goal the executive continues its strong focus on our mission, "Providing leadership to encourage the continuing enhancement of teaching, learning and understanding mathematics."

Note: Executive members Graham Keogh, Red Deer County, vice president, NCTM representative and conference director; Donna Chanasyk, Edmonton, secretary; Daryl Chichak, Edmonton, membership director; Lorraine Taylor, Lethbridge, director; Carol Henderson, PEC liaison; and Michael Fulton, Fort Qu'Appelle, Saskatchewan, NCTM Canadian director, are not shown.

Seven new members join the executive! (l–r) Shauna Boyce, Edmonton, Alberta Learning representative; Len Bonifacio, Edmonton, conference director; Evelyn Sawicki, Calgary, director; Robert Wong, Edmonton, webmaster; Helen McIntyre, High River, director; Richard Kopan, Calgary, director; Indy Lagu, Calgary, mathematics representative.

About the New Executive Members

(l–r) Len Bonifacio, Edmonton, director; Rick Johnson, St. Albert, director.

Len Bonifacio has been a teacher with Edmonton Catholic Schools for 25 years, including a secondment term at the Learner Assessment Branch. He is coordinator of mathematics and science at Holy Trinity High School.

Shauna Boyce has been a high school mathematics teacher at Spruce Grove Composite High School since 1990, where she also headed the mathematics department from 1995 to 1998. She has worked on various committees for Alberta Learning and has written support materials for numerous school boards and private businesses. In September 1998, she accepted a secondment position with Alberta Learning to work with the diploma examinations for Mathematics 33 and Applied Mathematics 30. Shauna is the examination manager for these courses and our Alberta Learning representative.

Richard Kopan served on the executive in the past and just couldn't stay away. He retired from the Calgary Board of Education in June 1997. Richard recently completed a three-year term on the board of directors of the National Council of Teachers of Mathematics. He is the Alberta representative for Spectrum Educational Supplies.

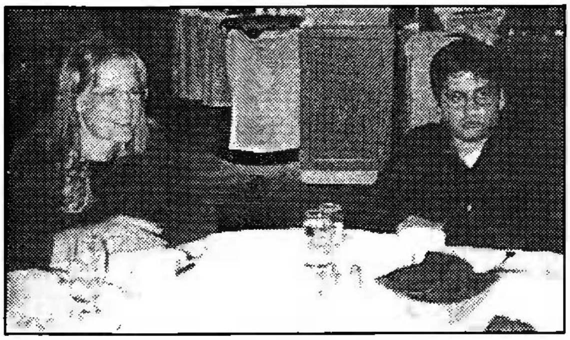
Indy Lagu received his Ph.D. in mathematics from the University of Calgary in 1996 and has since worked at Mount Royal College, Calgary. He is also education coordinator for the Calgary site of the Pacific Institute for the Mathematical Sciences (PIMS).

Helen McIntyre spent many years teaching for Foothills School Division No. 38. During the 1999–2000 school year, she had the unique opportunity to work as the junior high mathematics professional development coordinator for the Calgary Regional Consortium. This year, she has embarked on a new challenge as AISI project consultant for secondary mathematics for Rocky View School Division No. 41.

Evelyn Sawicki is mathematics supervisor for the Calgary RCSSD No. 1. Her goals as a mathematics educator and a MCATA member are to support the teaching and learning of the prescribed program of studies and to promote excellence in mathematics education for all students.

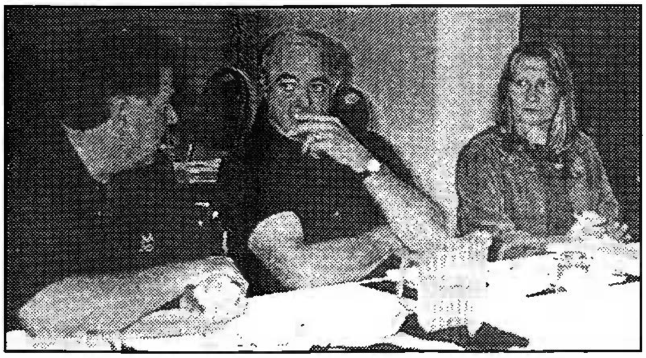
Robert Wong teaches at Vernon Barford Junior High School in Edmonton. He served on the Gauss Contest committee for nine years and on the Edmonton Junior High Math Contest committee for eleven years. Robert has looked after the school website since 1996 and coauthored a unit on number patterns for *The Learning Equation 7.*

(l–r) Dale Burnett, Lethbridge, Faculty of Education representative; Richard Kopan, Calgary, director

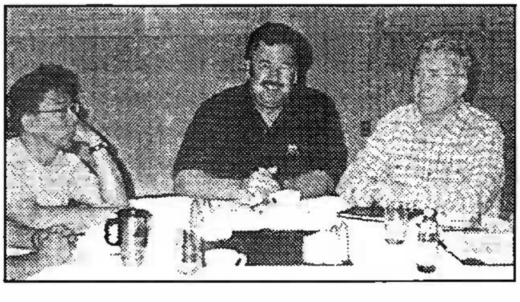


(l–r) Sandra Unrau, Calgary, president; Indy Lagu, Calgary, mathematics representative

(l–r) Cynthia Ballheim, Calgary, past president and newsletter editor; Elaine Manzer, Peace River, vice president



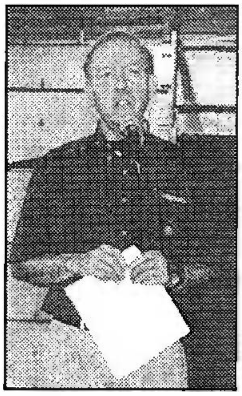
(l–r) Doug Weisbeck, St. Albert, treasurer; David Jeary, ATA staff advisor; Sandra Unrau, Calgary, president



(l–r) Robert Wong, Edmonton, webmaster; Doug Weisbeck, St. Albert, treasurer; Klaus Puhlmann, Edson, delta-K *editor*

MCATA 2000 Annual Conference

Message from the Conference Chair



Graham Keogh

The 2000 annual conference, "World Mathematics Year 2000," was held October 27–28 at the Red Deer Capri Centre. Four hundred and twenty registrants took the opportunity to mix with colleagues, share ideas, take in the sessions and visit the exhibits.

On Thursday, October 26, about 100 delegates attended the annual fall symposium sponsored by MCATA and Alberta Learning. The symposium featured the Applied Math program, with Alberta educators providing their experiences at implementing the course. Alberta Learning brought us up to date on the acceptance of Applied Math 30 at the postsecondary level. Zalman Usiskin discussed the possibilities of a universal math curriculum.

Conference delegates began arriving Thursday evening. Friday morning they started to take in sessions, 65 of which were offered over the two days. The majority of sessions were offered on site, with the technology sessions at Notre Dame High School and a practical elementary session at Kerrywood Nature Centre. Due to the implementation of the Western Canadian Protocol at the Grade 12 level, the conference included a large proportion of high school math teachers and sessions. These teachers especially enjoyed sharing ideas about course content, textbooks, implementation, resources and so on. In many sessions, presenters encouraged active participation and sharing with the delegates, which was much appreciated.

Friday's luncheon featured Zalman Usiskin, University of Chicago, talking about educating the public about school mathematics. At the luncheon, the Math Educator of the Year Award was presented to Cathy McCabe, Len Bonifacio, Shauna Boyce and Evan Fleetwood. Dale Karpluk, Don Ross, Craig Loewen, Steve Carlyle, John Percevault and Jolene Keogh were honored as friends of MCATA.

Friday evening, delegates were offered a choice of movies for math movie night, and about 100 delegates participated in this social event. Saturday's breakfast featured guest speaker David Pimm, University of Alberta. Delegates continued to take in sessions until the general closing session with Ted Lewis, University of Alberta. Because of the generous support of our many displayers, about 30 door prizes were given away as a conference finale.

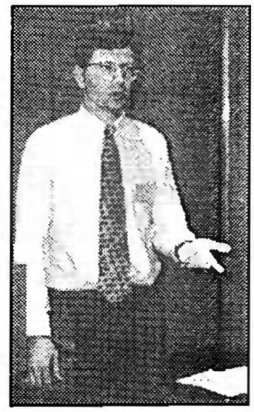
Graham Keogh

A Pictorial Potpourri of the 2000 MCATA Conference

Zalman Usiskin, keynote speaker, "Educating the Public About School Mathematics."

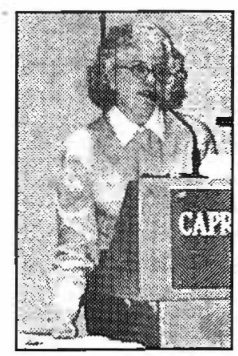
An attentive audience listening to Zalman Usiskin.

Conference Planning Committee (l-r) Elaine Manzer, Cynthia Ballheim; (middle) Patricia Chichak, Sandra Unrau, Graham Keogh; (back) Daryl Chichak in costume promoting the 2001 Conference, "A Math Odyssey," and Rick Johnson.

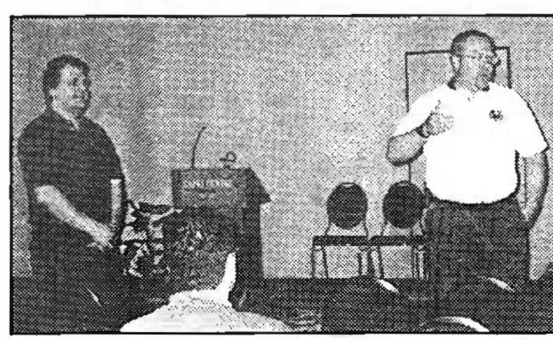


Terry Kaminski conducting a workshop on the Addison-Wesley Applied Mathematics 10.

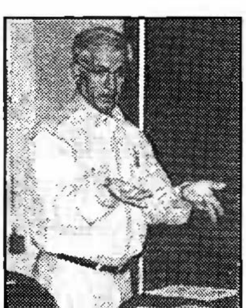
Scott Carlson and Indy Lagu (not in picture) presented the "Normal Is in the Die of the Beholder" workshop.



Cynthia Ballheim spoke about how Mathpower 12 students could approach diploma exams.



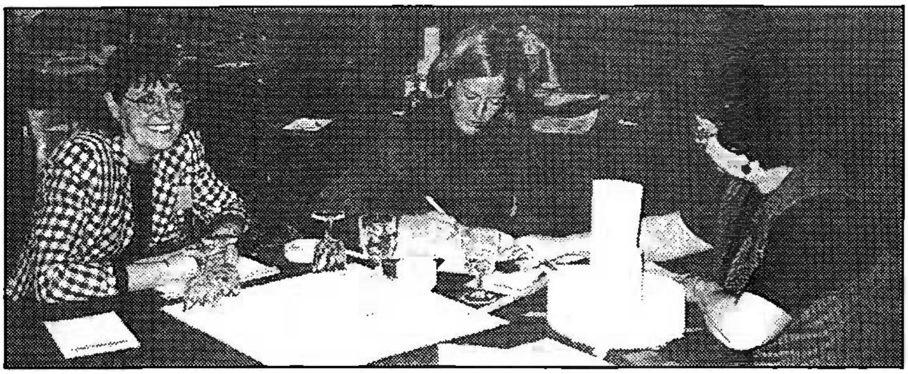
Len Arden (l) and Holt Zaugg (r) talked about portfolios for senior high math.



Enzo Timoteo led the "Math in Motion – Base 10 Blocks, Decimals" workshop.

Evelyn Sawicki revisited the Alberta diagnostic math program.

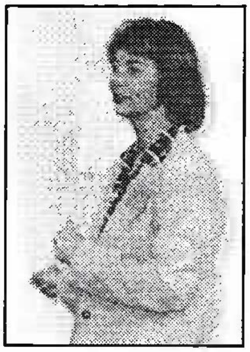
Jack LeSage working with teachers in "Linking Geoboards at the Junior High Level."



(l–r) Ellen Radomski, Jennifer Burke and Jacqueline Willette enjoyed the "Michelangelo at Work" workshop.

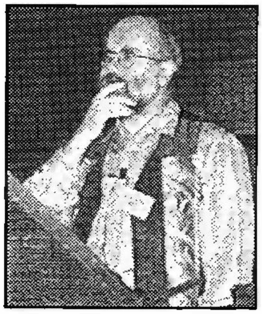
Elaine Simmt asked, "What do broccoli, Cantor's set and a math class in flu season all have in common?"

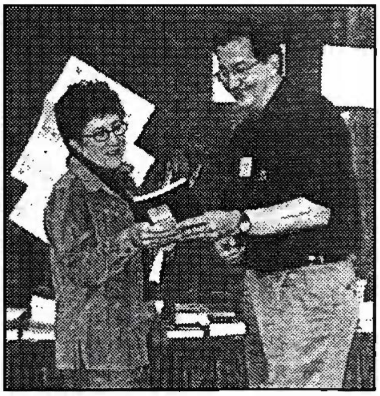
Ernie Klassen presented "Numero—Having Fun with Numbers."



Carol Hellman and Paula Bruner (not in picture) conducted the workshop, "Using Illustrative Examples to Inform Mathematics Instruction."

David Pimm, keynote speaker, "Seeing, Saying and Writing: Where does the Mathematics Lie?"





Dale Burnett, MCATA executive member, busy with customer Carol Klassen at the MCATA desk in the display area.

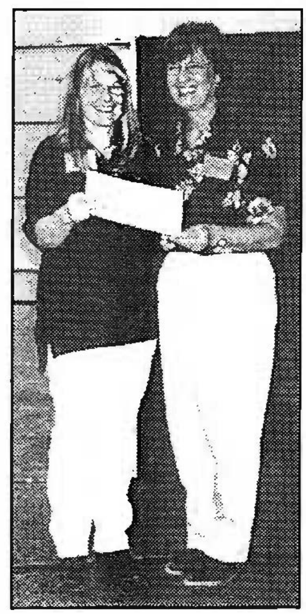
Carol Henderson (c), PEC liaison, discusses the ATA policy on provincial achievement tests with teachers.

(l-r) Janette Pethick, Susan Galloway and Pat Lore (teachers with Edmonton Public Schools) enjoyed Jane Felling's session on "Beyond" Boxcars and One-Eyed Jacks.

Len Bonifacio presented a Pure Math 30 session on statistics, perms and combinations.

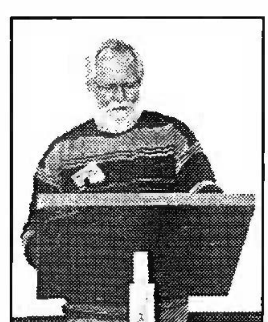
Doug Knight presented a session on the classroom assessment toolkit for ICT.

Richard Kopan worked with teachers in "Geometry Tools to Enhance the Classroom, Grades 7–9."



Sandra Unrau (l), MCATA president, presents a Friends of MCATA certificate to Jolene Keogh. Other Friends of MCATA Dale Karpluk, Steve Carlyle, Don Ross, John Percevault and Craig Loewen were not present.

Sandra Unrau (l) presents Math Educator of the Year Awards to (l–r) Evan Fleetwood, assistant principal, Ardrossan Junior/Senior High School; Shauna Boyce, Alberta Learning representative; Cathy McCabe, high school math consultant, Edmonton Public Schools; and Len Bonifacio, coordinator of mathematics and science, Holy Trinity High School, Edmonton.



Ted Lewis, closing keynote speaker, "Math Fairs—A Story of Successful Interaction Between Schools and University Mathematics and Science Departments."

Promoting our next conference: "A Math Odyssey," October 26–28, 2001, in Edmonton.

NCTM Standards in Action Content Standard: Geometry

Klaus Puhlmann

The *Principles and Standards for School Mathematics* (NCTM 2000) identifies 10 content standards that represent what should be valued in school mathematics education. The standards represent a connected body of mathematical understandings and competencies, and they specify the understanding, knowledge and skills that students should acquire from pre-Kindergarten through Grade 12. Geometry, like the other content standards, applies to all grade levels, albeit with varying emphasis both within and between grade levels.

In particular, the geometry standard identifies the following areas of emphasis that enable all students to

- analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships;
- specify locations and describe spatial relationships using coordinate geometry and other representational systems;
- apply transformations and use symmetry to analyze mathematical situations; and
- use visualization, spatial reasoning and geometric modeling to solve problems.

Geometric shapes and representations surround us, thus allowing the easy transfer of what students learn to other areas of mathematics and to real-world situations. Geometric representations can help students to make sense of area and fractions, histograms and scatterplots can give insight about data, and coordinate graphs can serve to connect geometry and algebra.

Spatial reasoning is helpful in reading maps, designing plans or creating artworks. Geometry is more than definitions—it is about relationships and reasoning, which help students to learn and see the axiomatic structure of mathematics. Therefore, this content standard has a strong focus on the development of careful reasoning and proof. Technology also plays an important role in the teaching and learning of geometry. The use of technology allows students to generate many examples as a way of forming and exploring conjectures.

When students begin the study of geometry, the initial activities consist primarily of observing and describing shapes and noticing properties. These are essential activities because they form a strong foundation for further study. As the students proceed in their study, the learning includes exploration of properties, concepts of similarity and congruence, deductive reasoning and formal proof techniques. Eventually students are able to construct their own proofs.

The standards also require students to specify locations and describe spatial relationships using coordinate geometry and other representational systems. As students focus on this standard, the concepts initially deal with relative position, such as *above, behind, near* and *between*. They then proceed to locating points in a rectangular grid system, which allows them to discover and analyze properties and shapes. In the middle grades, students find distances between points in the plane, using the Pythagorean relationship. Here, a fundamental connection between algebra and geometry is established as well. By the time students reach high school, they will have become proficient in using the Cartesian coordinate plane to solve a variety of problems.

Applying transformations and using symmetry to analyze mathematical situations begin by capitalizing on students' intuitive understanding of how shapes can be moved. In particular, students learn about motions such as slides, flips and turns through the use of mirrors, paper folding and tracing. Understanding the effects of transformations occurs in the upper elementary grades, and by the time students have entered the middle grades, they know what it means for a transformation to preserve distance. At the high school level, students learn multiple ways of expressing transformations, including using matrices to show how figures are transformed on the coordinate plane, as well as function notation. The concept of symmetry is learned at all grade levels providing insight into mathematics, the arts and esthetics.

Using visualization, spatial reasoning and geometric modeling to solve problems is an important content standard that applies across the grades. The early experiences are hands-on and involve a variety of geometric objects. Through the use of technology, students learn to turn, shrink and deform two- and three-dimensional objects. In later years, students learn to infer attributes that cannot be seen. Students are challenged to physically and mentally change the position, orientation and size of objects, as they develop their understanding about congruence, similarity and transformation. Moving between two- and three-dimensional shapes and their representations is important content acquired at the elementary level. This leads to wrapping blocks into nets. By the time students reach the middle grades, they should be able to interpret and create top or side views of objects.

This skill is further developed so that students learn to build structures given only side and front views. By the time students are at the secondary level, they are able to find the minimum number of blocks needed to build given structures. Visualizing and drawing cross-sections of structures and a range of geometric solids is a content skill required at the high school level.

Geometry gives students a different view of mathematics. As they explore patterns and relationships with models, blocks, geoboards and graph paper, students learn about the properties of shapes and sharpen their intuition and awareness of spatial concepts. One of the most important connections in all of mathematics is the one between geometry and algebra, thus making geometry an important content standard.

The three articles that follow relate to the geometry standard. The first presents four related problems that are examples of ways to include justification of interesting mathematics prior to a geometry course. The reasoning used in solving these problems involves both algebra and geometry and encourages the meaningful use of technology. The second article suggests ways that interactive geometry software, with classic problems as context, can be used to connect students to a richer experience of what mathematics is and what it means to know and do mathematics. The third article describes how mathematics teachers can structure classroom activities so that students will be intellectually challenged. It also offers techniques for developing tasks for group investigations.

Bibliography

Alberta Education. *The Common Curriculum Framework for K–12 Mathematics.* Edmonton: Author, 1985.

National Council of Teachers of Mathematics (NCTM). *Curriculum and Evaluation Standards for Teaching School Mathematics.* Reston, Va.: NCTM, 1989.

———. *Principles and Standards for School Mathematics.* Reston, Va.: NCTM, 2000.

> What is best in mathematics deserves not merely to be learned as a task but to be assimilated as a part of daily thought, and brought again and again before the mind with ever-renewed encouragement.
>
> *Bertrand Russell*

Geometry Problems Promoting Reasoning and Understanding

Alfred B. Manaster and Beth M. Schlesinger

Current mathematics curricula emphasize "truth rather than reasons for truth" (Harel 1998, 497). This observation was supported in the TIMSS videotape study of Grade 8 mathematics classes, which found few occurrences of explicit mathematical reasoning in any courses other than geometry (Manaster 1998). The four related problems presented in this article provide examples of ways to include justifications of interesting mathematics in courses taught before a geometry course. At different times during their study of quadratic functions, students can solve these problems and can fully understand their solutions. This understanding requires that the students follow chains of reasoning that furnish convincing justifications of the correctness of the general results. The reasoning involves both algebra and geometry, but all the problems can be done before the student takes a formal geometry course.

> *Problem A.* Find the dimensions of a rectangle with a perimeter of 30 inches and sides of integral length that has the largest possible area.

A straightforward solution to this problem, which is appropriate for students in middle school and above, involves constructing a table containing dimensions and areas of all rectangles with sides of integral length and a perimeter of 30 inches (see Table 1).

When they examine the complete table, students discover that a rectangle—7 inches by 8 inches or 8 inches by 7 inches—exists with maximum area of 56 square inches. Since the table lists all possible rectangles satisfying the given conditions, a brief discussion completes a proof that the problem has been solved.

Table 1

Side$_1$ (In.)	Side$_2$ (In.)	Area (In.^{2})
1	14	14
2	13	26
3	12	36
4	11	44
5	10	50
6	9	54
7	8	56

> *Problem B.* Find the dimensions of a rectangle with a perimeter of 30 inches that has the largest possible area.

When we remove the constraint that the sides have integral length, the nature of the problem changes dramatically. The student cannot make a complete table of all possible rectangles.

By extending the table to include some rectangles with fractional lengths, students can observe symmetry in the table and may suspect or believe that the solution is the square with sides of 7.5 inches. The issue that we address in this article concerns the role of proof in developing a deeper understanding that this conjecture is indeed correct.

A next step in developing a more complete table might be to find the formulas for the height and area in terms of the base of the rectangle. Since $2h + 2b = 30$,

$$h = \frac{30}{2} - b$$

and

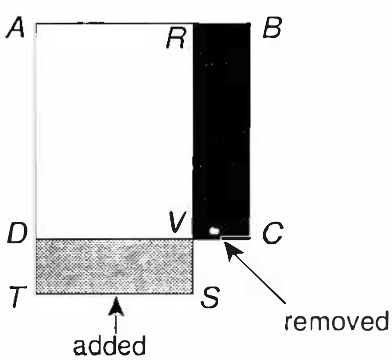
$$A = b \cdot h = b \cdot \left(\frac{30}{2} - b\right).$$

Students can use the formula and a graphing calculator to graph the area as a function of the length of the base. They can see by inspection that a maximum value of 56.25 square inches appears to exist when the base is 7.5 inches. By zooming in on the graph or by constructing tables with smaller and smaller step sizes, they can gather more evidence in support of this conjecture. Sophisticated students might use the "Maximum" function of a calculator to obtain the value 7.5. Teachers must be aware that this procedure may supply information, but it cannot lead to full understanding unless the student also knows how the maximum was found and why that algorithm works.

At some point, some students will notice that the apparent solution is a square. On the basis of that insight, it is possible to construct a beautiful geometric proof that a square always has the largest area of all rectangles with a given perimeter.

In Figure 1, rectangle $ARST$ was formed from the square $ABCD$ by first shortening the square's base an arbitrary amount equal to VC. The height of the new rectangle must be increased by the same amount, DT, to keep the perimeter constant. Rectangles $RBCV$ and $DVST$ have the same width, that is, $VC = DT$. Since RV is equal to a side of the original square and DV is shorter than a side of the original square, the area of $RBCV$ is greater than the area of $DVST$. The area that we removed is greater than the area that we added; therefore, the area of the square $ABCD$ is greater than the area of the new rectangle $ARST$. Because VC could represent any length less than AB, the new rectangle could be any rectangle with the same perimeter as square $ABCD$. Since $ABCD$ was any square, we have completed a geometric justification that the square has the largest area of all rectangles with a given perimeter.

Figure 1



Since the perimeter of *ABCD* equals that of *ARST*, the area added is less than the area removed.

Not all students are likely to have the insight that leads to the preceding argument; it is therefore worthwhile to explore other ways to understand why the square is the solution. One approach is to analyze the formula for the area as a function of the base.

Since $A = b(15 - b)$, then $A = -b^2 + 15b$. We want to find the largest possible value for A and the value of b at which it occurs. Because these values are difficult to ascertain from this formula, we use algebraic identities to rewrite it in a form that we can analyze to find these values:

$$\begin{aligned} A &= -b^2 + 15b \\ &= -(b^2 - 15b) \\ &= -\left(b^2 - 15b + \left(\tfrac{15}{2}\right)^2 - \left(\tfrac{15}{2}\right)^2\right) \\ &= -\left(\left(b - \tfrac{15}{2}\right)^2 - \left(\tfrac{15}{2}\right)^2\right) \\ &= \left(\tfrac{15}{2}\right)^2 - \left(b - \tfrac{15}{2}\right)^2 \end{aligned}$$

The final expression helps us find the largest value of A fairly easily. The first term is simply a constant. The maximum value of A occurs when we subtract the smallest possible value. Because we subtract a perfect square, its value is always greater than or equal to zero. If we make this term zero, we subtract as little as possible and make A as large as possible. Since the second term can equal zero only when $b = 15/2$, we see that the largest value of A is 56.25, which occurs only when $b = 7.5$.

Problem C. Consider all rectangles with perimeter equal to the circumference of a circle with radius 1 m. Find the dimensions of the rectangle that has the largest possible area.

The first step is to note that the circumference of the circle is 2π m. The next step depends on the student's knowledge about a solution to problem B. A student who understands the general principle that the square has the largest area can apply that result to this problem to see that the sides of the square have length $\pi/2$ m and that the area of the square is $(\pi/2)^2$ square metres. Otherwise, the student can use the same approaches that were used for problem B. A graphing calculator will help students find several slightly different values by observation, depending on the window chosen to view the graph. An algebraic approach gives

$$\begin{aligned} A &= -\left(b - \tfrac{\pi}{2}\right)^2 + \left(\tfrac{\pi}{2}\right)^2 \\ &= \left(\tfrac{\pi}{2}\right)^2 - \left(b - \tfrac{\pi}{2}\right)^2, \end{aligned}$$

so that the maximum occurs when $b = \pi/2$ and has the value $(\pi/2)^2$. It might be a good pedagogical strategy to present problem C some weeks after problem B so that students need to rethink their solution to a problem of this type. Redoing the algebra is likely to strengthen their understanding of the usefulness of underlying algebraic techniques. One advantage of seeing that both approaches give the same result is that students can observe that more than one good approach exists and that each reinforces the other.

Problem D. Is the ratio of the area of a square to the area of the circle whose circumference is equal to the perimeter of the square always the same? Why or why not?

Since the students have two examples, they might begin to find an answer by computing the requested ratios for each. For the square and circle in problem C, the area of the circle should be familiar to the students and is π square metres. The students have already found that the area of corresponding square is $(\pi/2)^2$ square metres; therefore, the ratio of the area of the square to the area of the circle is

$$\frac{\left(\frac{\pi}{2}\right)^2}{\pi} = \frac{\pi}{4}.$$

Our classroom experience indicates that high school students are more likely to use decimals to compute numerical approximations than to use formulas to compute exact values. In problem B, the area of the square was found to be 56.25 square inches. The circumference of the corresponding circle is 30 inches, so the radius is approximately 4.77 inches and the area is approximately 71.48 square inches. The ratio is approximately 0.79, which is also the two-decimal-place approximation of $\pi/4$.

We have seen that both ratios are $\pi/4$, or approximately 0.79. One question that calculators cannot answer conclusively is whether the two ratios are exactly the same. When π is used throughout the calculations, some calculators will show the difference between the two computed ratios as 0, whereas others will display a very small number. Other variations will depend on the rounding that students use in computing or estimating the areas in problem C. It might be helpful for students to look at other examples. They should eventually realize that the ratio is always about 0.79, which should lead them to ask whether the ratios are exactly the same and, even more important, why.

Fortunately, the algebraic solutions for problems B and C can lead to a proof that the ratios are all the same. Building on the result of problem C, if we call the perimeter of the square p, we see that the area of the square is

$$\left(\frac{p}{4}\right)^2 = \frac{p^2}{16}$$

and that the radius of the circle is $p/2\pi$, so the area of the circle is

$$\pi\left(\frac{p}{2\pi}\right)^2 = \frac{p^2}{4\pi}.$$

For any value of p, then, the ratio of the area of the square to the area of the circle is

$$\frac{\frac{p^2}{16}}{\frac{p^2}{4\pi}} = \frac{4\pi}{16} = \frac{\pi}{4}.$$

This result confirms our previous observations, and since no p exists in the final expression, $\pi/4$, the ratio is always the same.

Another explanation for the answer to problem D uses properties of similar figures and proportional reasoning. Let P_1 and P_2 be any two values for the perimeters of the circle and square. The ratio between any corresponding lengths in the similar figures, such as the radius of the first circle to the radius of second or the side of the first square to the side of the second. is P_1/P_2; and the ratio between any corresponding areas, for example, the semicircle of the first circle to the semicircle of the second, is

$$\left(\frac{P_1}{P_2}\right)^2.$$

Let C_1 and C_2 represent the areas of the circles with perimeters P_1 and P_2 respectively, and let S_1 and S_2 represent the areas of the squares with those perimeters. Since

$$\frac{S_1}{C_2} = \left(\frac{P_1}{P_2}\right)^2 = \frac{C_1}{C_2},$$

we see that

$$\frac{S_1}{S_2} = \frac{C_1}{C_2}$$

and

$$\frac{S_1}{C_1} = \frac{S_2}{C_2},$$

Therefore, the ratio of the area of the square to that of the circle does not depend on their common perimeter.

These four related problems and their solutions are accessible to students in courses other than geometry. They invite exploration. They encourage the meaningful use of technology. They call for writing. They lead to interesting and deep mathematics. They blend algebra and geometry. The variety of methods used to establish the mathematical results builds the students' appreciation of the power of the techniques they have developed and helps them recognize that mathematics is at least as much about "why" as about "how."

References

Harel, G. "Two Dual Assertions: The First on Learning and the Second on Teaching (or Vice Versa)." *American Mathematical Monthly* 105 (June–July 1998): 497–507.

Manaster, A. B. "Some Characteristics of Eighth Grade Mathematics Classes in the TIMSS Videotape Study." *American Mathematical Monthly* 105 (November 1998): 793–805.

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Interactive Technology and Classic Geometry Problems

Jean J. McGehee

Interactive geometry software connects visual justification and empirical thinking to higher levels of geometric thinking with logical justification in formal proof. Battista and Clements (1995, 53) recommend the use in the secondary school geometry curriculum of software that "should guide students to learn significant and interesting concepts."

Some of the more engaging concepts are embedded in the classic works of the Greek geometers, yet interest can be killed if students struggle with the emphasis on the formal aspects of the Euclidean model of mathematical inquiry (Perkins et al. 1995). The enjoyment and challenge come from making ideas that were conceived with the limited tools of the ancient Greeks come alive for our students with the help of the computer. The task for the teacher is not so much to present the constructions and theorems in a colorful display on the computer screen; instead, the challenge is to design activities that engage students in the dynamic and interactive features of the software so that they can demonstrate real understanding.

An example of a classic construction problem is the circle of Appolonius: the locus of all points, P, in a plane such that given any two points A and B in the plane, $PA : PB$ will be a constant ratio k where $k \neq 1$. Students usually do not know where to begin with this problem. Traditional algorithmic instruction and our top-down knowledge usually produce an activity with an efficient set of steps for the construction. Consider the steps for the activity in Figure 1, and ask yourself how involved your students would be in understanding the construction. Note that The Geometer's Sketchpad (Jackiw 1993) is used for all work in this article.

Figure 1
Activity Directions for the Traditional Approach

Find the locus of all points, P, in a plane such that given any two points A and B in the plane, PA : PB will be a constant ratio k where $k \neq 1$.

1. Choose any two points; label them A and B. Choose any other point in the plane, and label it P. Note that $PA \neq PB$.
2. Construct segments for ΔPAB. Select segments PA and PB and find the ratio, $PA : PB$, by using the Measure menu. This ratio will be k, the constant, $k \neq 1$.
3. Construct the angle bisector of $\angle APB$ and its intersection with side AB. Label the intersection X.
4. Construct the line through A and B. Construct the exterior-angle bisector at P and its intersection with line AB. Label the intersection Y.
5. Construct segment XY; label its midpoint O. (see Figure 2)
6. Construct the circle with center O and radius OX.
7. Select any other point on the circle; label it Q. Construct and select segments QA and QB. Find the ratio $QA : QB$. Is it the same as $PA : PB$? (see Figure 3)
8. Select Q and the circle, and under the Display menu, choose "animate" to move Q around the circle. Does the value of the ratio $QA : QB$ change? Select a point T not on the circle and find the value of $TA : TB$. Drag T to make a conjecture about points not on this circle.
9. Have you found the locus of points? Describe the locus of points.

What happens when $PA : PB = 1$? (see Figure 4)

This activity uses many of the features of Sketchpad—especially the animation feature—to convince students that the circle meets the criterion of the constant ratio. Also, the use of the interior- and exterior-angle bisectors at *P* is an essential hint in the proof that the locus of points is a circle. But do students have any insight into the connection between the bisectors and the constant ratio? Although students are asked to summarize their work with a description and to drag *P* to find out what happens with *k* equal to 1, they really have little ownership in this construction.

The steps in the traditional activity are very efficient and almost eliminate any guesswork and questions on the students' part. Students may have a shallow surface level of understanding of the problem and can fail to make connections among the important concepts behind the construction. Because they have not worked through the difficulty of the problem, they are not ready to make formal arguments that the locus is a circle.

Students often have a perception that when a mathematician sits down to work a problem or prove a conjecture, he or she very quickly produces a flawless paper. Actually, the paper is the final clarification of a lot of hard work with some dead ends, a few frustrations and the rewarding moments of "aha."

Figure 2
Student's Sketch of Steps 1–5

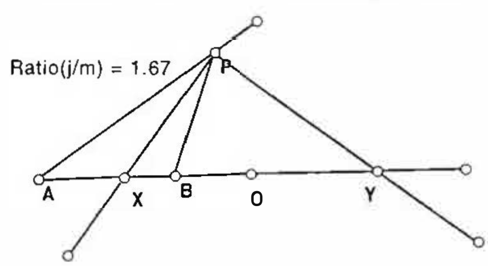
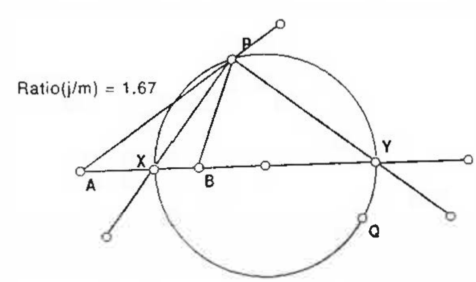


Figure 3
Student's Sketch of Steps 6 and 7

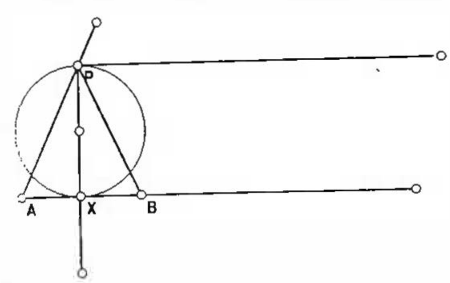


A more interactive approach to this activity would take students through the complete process of what a mathematician does: play with an idea, make a conjecture, make a formal argument. Schwartz (1995, 98) describes the software as an intellectual mirror in which "users explore their own understanding of a mathematical domain. One feature of such software environments is that while the user explores a particular question, the environment can display a logical universe of inquiries to which the user's particular inquiry belongs."

A connection is made between playing with a sketch and making a formal argument because the software requires a syntax that is similar to formal geometric thinking. For example, to draw a circle, think "Given points *O* and *X*, construct a circle with center *O* and radius *OX*" as you select *O* and *X* to activate the Construct menu. As teachers, we can capitalize on the connection between the words and the action to smooth the students' transition from lower to higher levels of geometric thinking.

Instead of using the dragging feature at the end of the activity to verify a prescribed process, use it in the beginning to foster discovery. Consider how the steps in the interactive activity in Figure 5 have evolved from the original list of steps. Unfortunately, the power of the animation cannot be demonstrated here. However, you can notice that *Q*2 is now above line *AB*. The ratio, *s/t*, corresponds to *Q2A*: *Q2B*, and it remains unchanged as *Q*2 moves around the circle.

Figure 4
Student's Sketch of Steps 6 and 7



PA: *PB* = 1.
When *PA* : *PB* = 1, the locus is not a circle.
The exterior-angle bisector is now parallel to line *AB*.
The locus of points is the perpendicular bisector of segment *AB*

Figure 5
Activity Directions for the Interactive Approach

Find the locus of all points, *P*, in a plane such that given any two points *A* and *B* in the plane, *PA* : *PB* will be a constant ratio *k* where $k \neq 1$.

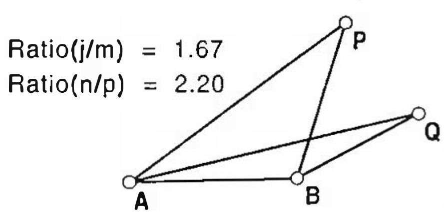
1. Choose any two points; label them *A* and *B*. Choose any other point in the plane, and label it *P*. Choose *P* so that $PA \neq PB$.
2. Construct segments *PA* and PB and line *AB*. Select $\overline{PA}$ and $\overline{PB}$, and find the ratio, *PA* : *PB*, by using the Measure menu. This ratio will be *k*, $k \neq 1$.
3. Choose another point *Q*. Construct and select $\overline{QA}$ and $\overline{QB}$. Find ratio, *QA* : *QB*. Compare this ratio with *k*. (see Figure 6)

4a. Drag *Q* as needed until *QA* : *QB* = *k*. (see Figure 7)

4b. Repeat step 4a at least five times. Can you identify a region on the plane where you would not choose *Q*? Can you choose a point above line *AB*? Below line *AB*? On segment *AB*? On line *AB* to the left of *A*? On line *AB* to right of *B*?

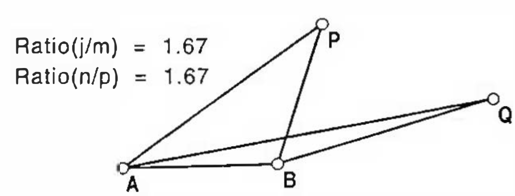
5. What figure does the locus of points suggest? If you were not guessing points, how would you describe the locus? Be very specific. Are any points other than *A*, *B* and *P* essential to your description? (see Figure 8)
6. Use your description to construct the locus, or path, of points. Does this path go through all the points you chose for step 4b? Note: the round-off error may make some of the points slightly off the locus. (see Figure 9)
7. As you animate *P* around the locus, verify that the ratio, *k*, remains constant. (see Figure 10)

Figure 6
Student's Sketch of Steps 1–3



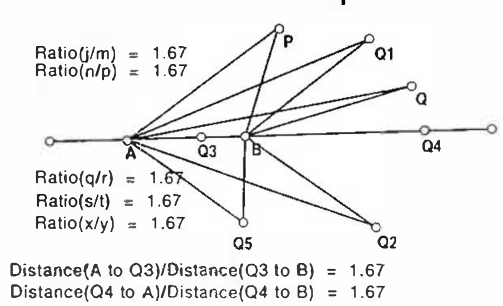
At this point, *QA* : *QB* > *PA* : *PB*.

Figure 7
Student's Sketch of Step 4a.



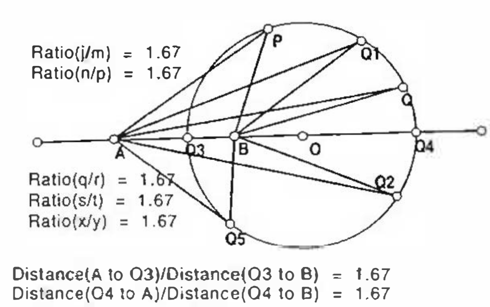
The ratios are the same.

Figure 8
Student's Sketch of Steps 4 and 5



The points suggest a circle, and *Q3* and *Q4*, the points on line *AB*, are on the diameter of this circle.

Figure 9



The locus of points is the circle with centre at the midpoint of segment *Q3Q4*.

This version of the activity does not immediately give away the fact that the locus is a circle or immediately identify the essential points X and Y for the construction. I have found that in the original activity, students lose sight of the role of the constant ratio. In this revised activity, the ratio is essential to their choice of points. However, the activity is still very empirical. The final steps connect this work to formal proof (see Figure 11).

Step 12 allows students to verify the construction of the circle of Appolonius for points A, B and P. In the more traditional activity, students could drag P and preserve the circle. In the interactive activity, P appears to be on the circle, but the computer may not recognize it as a point of the circle. All points in step 4b were fixed according to their measurement relationships to A and B, and we know that numerical measure is not a part of legitimate construction. Steps 1–11 give students the opportunity to discover the construction and the important relationships among the parts of the process. Step 12 lets students use induction to test the generalizability of the process. In writing the script, they should produce a list of commands that are essentially like the steps in the original activity. They are now ready for deductive proof.

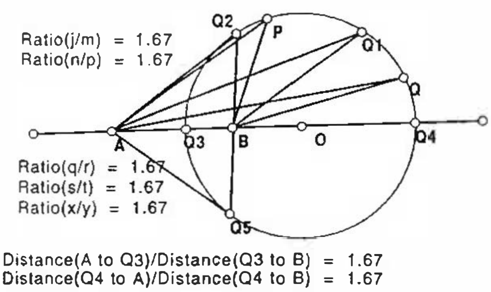
At this point, students should have made three observations that can be restated as lemmas to prove that P, X and Y determine a circle. It is important that in groups or as a class, the students discuss the restriction that $k \neq 1$. If $k = 1$, then ΔAPB is either isosceles or equilateral. Point X would be the foot of the perpendicular bisector to $\overline{AB}$ from P, and locus of points would be this perpendicular bisector. The exterior-angle bisector at P is parallel to $\overline{AB}$ (see Figure 4).

A whole-class discussion can bring the class to a consensus regarding steps 8–12. Focus students' attention on the connection between the bisectors and the constant ratio so that the following lemmas can be stated:

I. Point X is the point where the interior-angle bisector at P intersects $\overline{AB}$ in ΔABP if and only if $XA : XB = PA : PB$.
II. If $AP \neq BP$, Y is the point where the exterior-angle bisector at P intersects line AB and only if $YA : YB = PA : PB$.
III. The interior-angle bisector and the exterior-angle bisector from a vertex of a triangle are perpendicular.

Using these lemmas, students can describe and justify the construction of the circle of Appolonius. You may or may not want to go into the details of the lemmas. Both parts of the "if and only if" statements in lemmas I and II can be proved with the law of sines. Lemma III is a simple proof for geometry

Figure 10



As Q2 moves around the circle, the ratio is constant.

Figure 11
The Final Steps of the Activity Directions for the Interactive Approach

8. Relabel the point of the circle on segment AB as X. Relabel the point on the circle that is on line AB, but not on segment AB, as Y.
9. Choose on the display menu a color that has not been used. Construct segment PX. Measure $\angle APB$, $\angle APX$ and $\angle XPB$. What can you conclude?
   Measure $\angle AQX$ and $\angle XQB$. What can you conclude? Can you generalize for all the points that you chose in step 4b?
10. Extend lines to form the exterior angles at P and Q for ΔAPB and ΔAQB. Use angle measures to find how segments PY and QY are related to the exterior angles. What is your conjecture?
11. How are segments PX and PY related? How are segments QX and QY related? Can you make a generalization for the other points you chose in step 4b? (see Figure 12)
12. Using your generalization in steps 8–11, describe the locus of points. Test your description by making a new sketch of the locus in which you record a script—be sure to include measures in the script.

classes and involves the supplementary exterior and interior angles at vertex P. Then half of an interior angle and half of an exterior angle will be complementary, so that $\overline{PY} \perp \overline{PX}$.

Description. Given any two points A and B and a point, P, not on $\overline{AB}$: let $PA : PB$ be a constant ratio k where $k \neq 1$. The locus of points in the plane with constant ratio k is the circle with diameter $\overline{XY}$ where Y is the intersection of the interior-angle bisector at angle P with side AB and Y is the intersection of the exterior-angle bisector at P with line AB.

Proof. Consider the given points A and B, and let P be a point not on $\overline{AB}$ such that $PA : PB = k$ and $k \neq 1$. Construct ΔAPB.

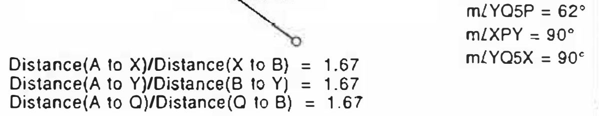
By lemma I, the intersection of the interior-angle bisector at angle P with side AB is a point X such that $XA : XB = PA : PB = k$. Then X satisfies the condition of the locus.

Since $k \neq 1$, the intersection, Y, of the exterior-angle bisector at P exists. By lemma II, $YA : YB = PA : PB = k$. Y satisfies the condition of the locus.

By lemma III, ΔXPY is a right triangle in which P is the right angle of $\overline{XY}$ is the hypotenuse. The midpoint, O, of hypotenus XY is equidistant from points X, P and Y. Therefore, for any point P such that $PA : PB = k$, P lies on the circle with center O and diameter $\overline{XY}$.

From the introduction of the problem on the computer to the final discussion, the objective is to help students focus on the conditions of the locus and to use the technology to explore and discover the essential features of the construction. Students seem easily to get to the point where they see the circle. Their work breaks down when they have to clarify how to find and describe that particular circle, but the lesson is not over until the class has communicated in logical mathematical language what has been learned. It should always be emphasized that although the Sketchpad is an effective and convincing tool for conjecture, it creates sketches that are incomplete without descriptions and justification.

Figure 12



Student's Explanation of the Relationships Between Segments and Angles and Why These Points Are on a Circle

Other classic problems that have been good topics for interactive computer activities are applications of Menelaus and Ceva's theorem, Steimer's theorem, Napoleon's theorem, problems with the nine-point circle, and re-creations of Euclid's propositions. As the students sketch to make sense of the statements of these problems or theorems, encourage them to make text notes and measures on the sketches to prepare them for any formal work that will be assigned.

A goal for these activities is to use the classic problems as a context with computer applications to connect students to a richer experience of "what mathematics is, what it means to know and do mathematics" (NCTM 1989, 16). The development of these activities will be a work in progress that will apply principles that are consistent with teaching for understanding. As you work with students, the computer-laboratory tasks will evolve from procedural activities to interactive activities, and you will begin to see the maximum benefits for students.

References

Battista, M. T., and D. H. Clements. "Connecting Research to Teaching and Geometry and Proof." *Mathematics Teacher* 88 (January 1995): 48–54.

Jackiw, N. The Geometer's Sketchpad, Ver. 3.0. Berkeley, Calif.: Key Curriculum Press, 1993. Software.

National Council of Teachers of Mathematics (NCTM). *Curriculum and Evaluation Standards for School Mathematics*. Reston, Va.: NCTM, 1989.

Perkins, D. N., et al., eds. *Software Goes to School: Teaching for Understanding with New Technologies*. New York: Oxford University Press, 1995.

Schwartz, J. L. "Shuttling Between the Particular and the General: Reflections on the Role of Conjecture and Hypothesis in the Generation of Knowledge in Science and Mathematics." In *Software Goes to School: Teaching for Understanding with New Technologies*, edited by D. N. Perkins et al. New York: Oxford University Press, 1995.

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Using ROOTine Problems for Group Work in Geometry

Regina M. Panasuk and Yvonne Greenleaf

How can mathematics teachers structure classroom activities so that students will be intellectually challenged? How can they create a learning environment that encourages students to communicate and reason mathematically, make decisions collaboratively and acquire mathematics skills and concepts that they thoroughly understand? The *Professional Standards for Teaching Mathematics* (NCTM 1991) suggests that mathematics teachers need to focus on the major components of teaching: worthwhile tasks, discourse and students' active participation and involvement. Cooperative-learning approaches offer practical classroom techniques that teachers can use to motivate all their students to learn and appreciate mathematics (Davidson 1990).

Many factors influence and affect group work, since the method of instruction is inseparable from curricular content. One crucial factor is the activity that the teacher selects or designs and offers to the students for group work. Van Hiele (1986, 39) points out that students learn not by direct teacher talking but through "a suitable choice of exercises."

In this article we introduce a technique for developing tasks suitable for group investigation. The format of the task is simple: students work from sets of numerical examples toward making generalizations. The teacher identifies the generalization—the root problem—and prepares a set of subproblems for each group. A distinctive feature of the problems, however, is that they lend themselves to different strategies for solving the same problem. This aspect promotes group discussion, problem solving and inquiry.

Problem Set 1: Angle Sum of a Triangle

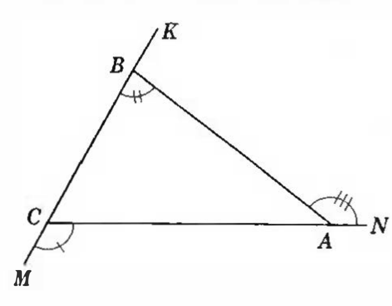
The objective of the first set of problems is to prepare students to construct the concept of the sum of the angle measures of a triangle. We assume that students are familiar with the definitions of a linear pair of angles and an exterior angle and with the property that the sum of the angles in a linear pair is equal to 180 degrees. For example, the root, or general, problem is shown in Figure 1. The subproblems for the groups are shown in Figure 2. For Figure 2, it is important that the letters on the pictures be the same for the different groups and that the depiction of the angles should correctly reflect the given magnitude of the angles.

Groups receive the "same" generic problem but with different parameters, and each group works with several of the subproblems, answering the same set of questions. Operating with different given parameters, the groups should each arrive at the same result for the second question in Figure 1, the sum of the interior angles. Then the teacher leads a class discussion in which groups share their results on each of the subproblems, so that students all observe that the result for question 2 is the same in each case.

The students are asked to explain this phenomenon. They are able to make a generalization after observing and analyzing other groups' work. Is this generalization merely a coincidence, or does it reflect conformity with a mathematical law? Depending on the developmental level and the students' ability, the teacher

Figure 1
Sum of the Angles of a Triangle: Root Problem

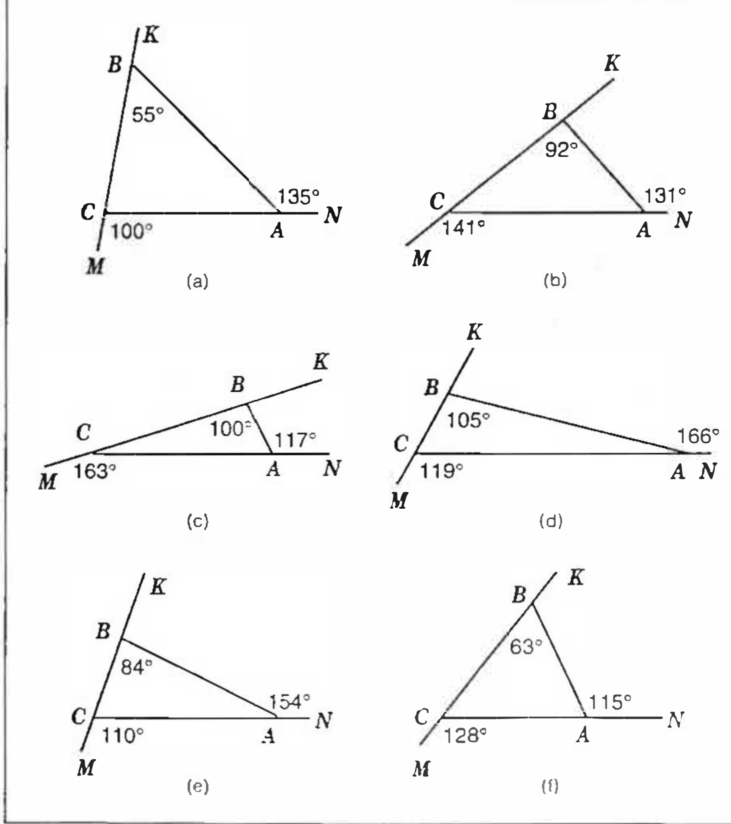
Given: $\triangle ABC$
$m\angle BAN$, $m\angle ABC$, $m\angle MCA$
Find: 1. $m\angle BCA$, $m\angle BAC$, $m\angle KBA$
2. $m\angle BAC + m\angle ABC + m\angle ACB$



may provide a rigorous proof of the theorem of the sum of the angle measures in a triangle or just state the theorem, highlighting that this theorem can be proved. This problem is accessible to students at all levels. Students who are not ready for abstract proofs can investigate it by using a paper model of a triangle, cutting the angles and then lining them up on a straight line.

This problem offers excellent exploration opportunities with The Geometer's Sketchpad (Jackiw 1990) or Cabri Géomètre (Baulac, Bellemain and Laborde 1992). The student can use one of these tools to construct a triangle and determine the individual angle measures and the sum of the three angles. Then, taking any vertex and moving it, the student observes that although the individual angle measures of a triangle may change, the sum remains constant. The software allows the investigation of many triangles, since each movement of a vertex creates a new triangle. This demonstration offers a convincing argument for most students. The experimentation should occur before any formal proof is attempted. The result of this work may be organized as in Table 1.

Figure 2
Sum of the Angles of a Triangle: Subproblems



As an additional question or as an extension of the problem. the teacher asks students to find the sum of the measures of ∠BAN, ∠KBA and ∠ACM. The students discover that 360 degrees is the sum of the measures of the three exterior angles of triangle ABC. The proof is easy if the teacher provides a generic picture as a hint (see Figure 3).

The arithmetic of the problem is simple:

$m\angle 1 + m\angle 2 = 180°$,

$m\angle 3 + m\angle 4 = 180°$,

and

$m\angle 5 + m\angle 6 = 180°$;

therefore, adding these three equations and moving some terms from the left side to the right side, we get

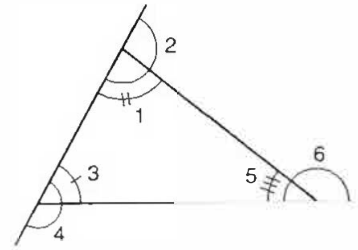
$$m\angle 2 + m\angle 4 + m\angle 6 = 540° - (m\angle 1 + m\angle 3 + m\angle 5)$$
$$= 540° - 180°$$
$$= 360°.$$

As a follow-up activity, students can then measure the exterior angle of a triangle and compare it with the sum of the measures of the two remote interior angles of the triangle. Again using technology, the students move any vertex of the triangle and observe the pattern.

The approach described gives students three angles in the diagram—one interior angle and two exterior angles. For a diagram with these angles to exist, the sum of the two given exterior angles minus the given interior angle must be 180 degrees. Thus, the information given in this approach "forces" students to the desired conclusion. The students' main task is to apply the principle that the sum of the angles of a linear pair is 180 degrees.

A more open-ended approach gives students less information and has them develop the remaining information by actual measurement. For example, in Figure 2a, one might show the same diagram

Figure 3
Sum of the Exterior Angles of a Triangle



but mark only the angles of 55 degrees and 100 degrees—and perhaps give a length for BC. Students would then draw a line and mark off a segment representing BC. They would draw rays starting from B and C to create the desired angles of 55 degrees and 100 degrees and label the point where these rays meet as A. They could then measure the remaining angles and find both the sum of the interior angles and the sum of the exterior angles.

Problem Set 2: Area of Equivalent Figures

A second set of problems connects several ideas: the Pythagorean theorem; properties of parallelograms and rectangles; and area of rectangles, triangles and parallelograms. The objective of this problem set is to strengthen the concept of area of equivalent figures. The root problem is shown in Figure 4a: ABCD is a parallelogram, and F is a point on AD. The goal is to have students compare the sum of the areas of

Figure 4
Areas of Triangles and Parallelograms: Root Problem

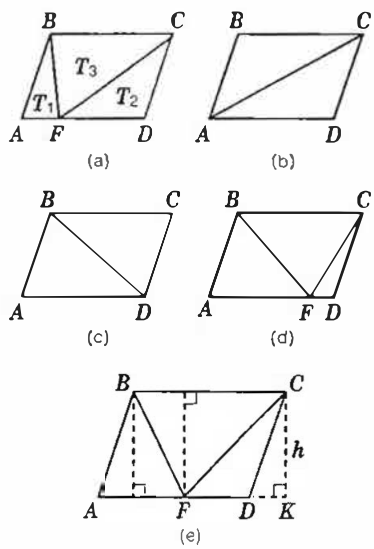


Figure 5
Areas of Triangles and Parallelogram: Subproblems

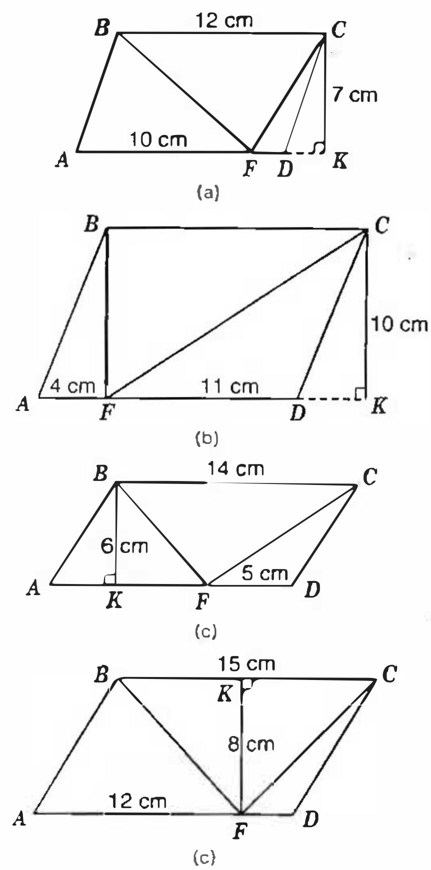
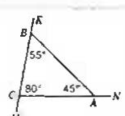
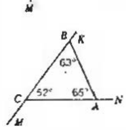


Table 1
Sum of the Angles of a Triangle

	Problem	Observation	Hypothesis	Explanation
1.		55° + 45° + 80° = 180°	The sum of the angles of any triangle is equal to 180 degrees.	
2.		52° + 63° + 65° = 180°		

triangles T_1 and T_2 with the area of triangle T_3 and with the area of parallelogram ABCD. Figures 4b and 4c show the cases in which F coincides with either A or D. Students may already have proved that the two triangles are congruent in these two instances, so their areas are equal and the sum of their areas is equal to the area of the parallelogram.

The teacher develops a set of subproblems by fixing the "moving" point and giving a numerical value to the selected segments of the parallelogram (see Figure 5).

The geoboard is a very useful tool for demonstrating the "moving" point and helping students understand the effect of the motion of the point on the shapes within the parallelogram. However, although geoboards allow the systematic study of shapes, area, perimeter and so on, they represent a discrete structure and are limited by the number of pegs. If available, software can be used to construct shapes, thus reinforcing the defining properties of geometric figures.

An analogous problem can be considered using a rectangle instead of a parallelogram. In this case, the root problem is as shown in Figure 6. The subproblems are shown in Figure 7.

Students first apply the Pythagorean theorem and obtain the lengths of segments MF and FD. They can then organize their search for areas in different ways. They may see the segment BC as the sum of the two segments, MF and FD, and find the area of triangle BFC by using the formula; or they may see the area of triangle BFC as the difference between the area of rectangle MBCD and the sum of the areas of triangle MBF and triangle FCD. Students discover that

area ΔMBF + area ΔFCD = area ΔBFC = ½ area MBCD.

In general, as shown in Figure 8,

area ΔMBF $= \frac{ax}{2}$,

area ΔFCD $= \frac{a(b-x)}{2}$,

and

$$\text{area } \Delta MBF + \text{area } \Delta FCD = \frac{ax}{2} + \frac{a(b-x)}{2} = \frac{ax + ab - ax}{2} = \frac{ab}{2}.$$

Since the area of MBCD equals ab, it then follows that the area of triangle BFC equals one-half the area of MBCD. The most important conclusion is that the foregoing inference does not depend on the location of the "moving" point F.

Figure 6
Areas of Triangles and Rectangles: Root Problem

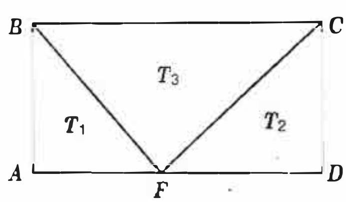


Figure 7
Areas of Triangles and Rectangles: Subproblems

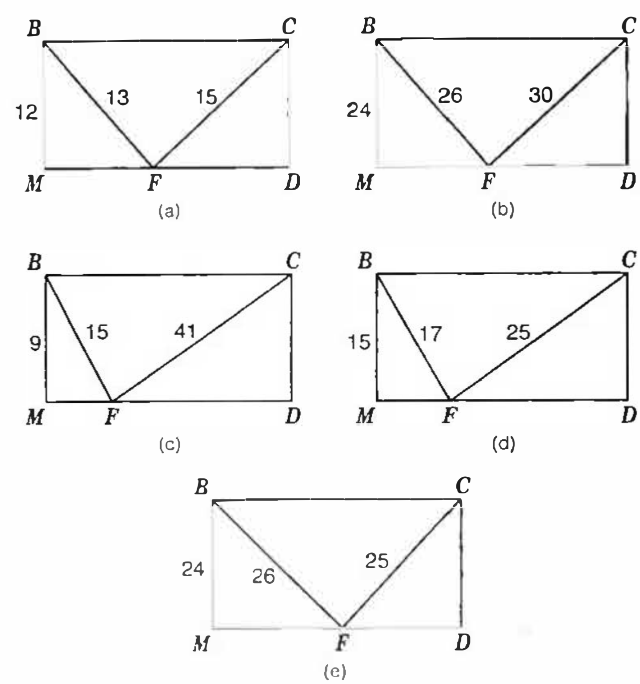
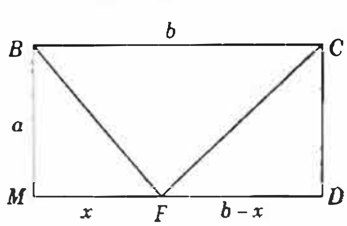


Figure 8
Areas of Triangles and Rectangles: General Case



Planning Instruction

This approach is based on van Hiele's (1986, 177) phases of learning, which can be used as a scheme when planning instruction. Do not confuse the phases with the levels of thought. According to van Hiele, the learning process leading to understanding at the next higher level has five phases, approximately, but not strictly, sequential; in other words, a student goes through various phases in proceeding from one level to the next. The transition from one phase to the following takes place "under influence of a teaching-learning program" (p. 50). Geometry activities should focus on techniques that stimulate students to move from one level of thought to the next and encourage more than one level of thought to provide learning opportunities for the range of students' abilities.

Bibliography

Baulac, I., F. Bellemain and J. M. Laborde. *Cabri: The Interactive Geometry Notebook (Cabri Géomètre).* Pacific Grove, Calif.: Brooks/Cole, 1992. Software.

Clements. D. H., and M. T. Battista. "Geometry and Spatial Reasoning." In *Handbook of Research on Mathematics Teaching and Learning*, edited by D. Grouws, 420–65. Reston. Va.: National Council of Teachers of Mathematics, 1992.

Davidson. N., ed. *Cooperative Learning in Mathematics: A Handbook for Teachers*. Reading, Mass.: Addison-Wesley, 1990.

Jackiw. N. The Geometer's Sketchpad. Berkeley, Calif.: Key Curriculum Press, 1990. Software.

National Council of Teachers of Mathematics (NCTM). *Professional Standards for Teaching Mathematics*. Reston, Va.: NCTM. 1991.

Van Hiele. P. N. *Structure and Insight*. Orlando, Fla.: Academic. 1986.

Vygotsky. L. S. *Mind in Society: The Development of Higher Psychological Processes*. Cambridge, Mass.: Harvard University Press, 1978.

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Mathematicians create by acts of insight and intuition. Logic then sanctions the conquests of intuition.

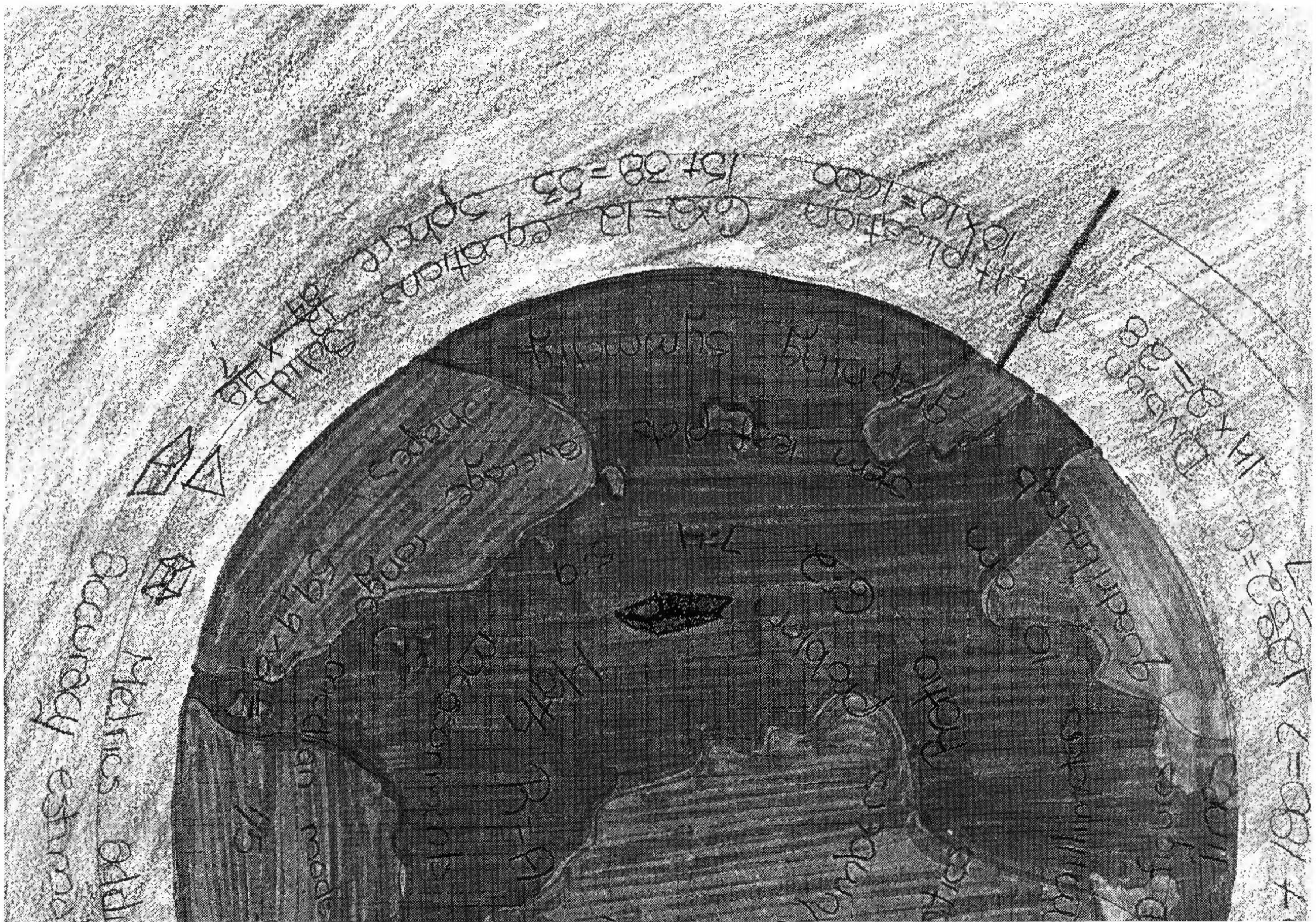
Morris Kline

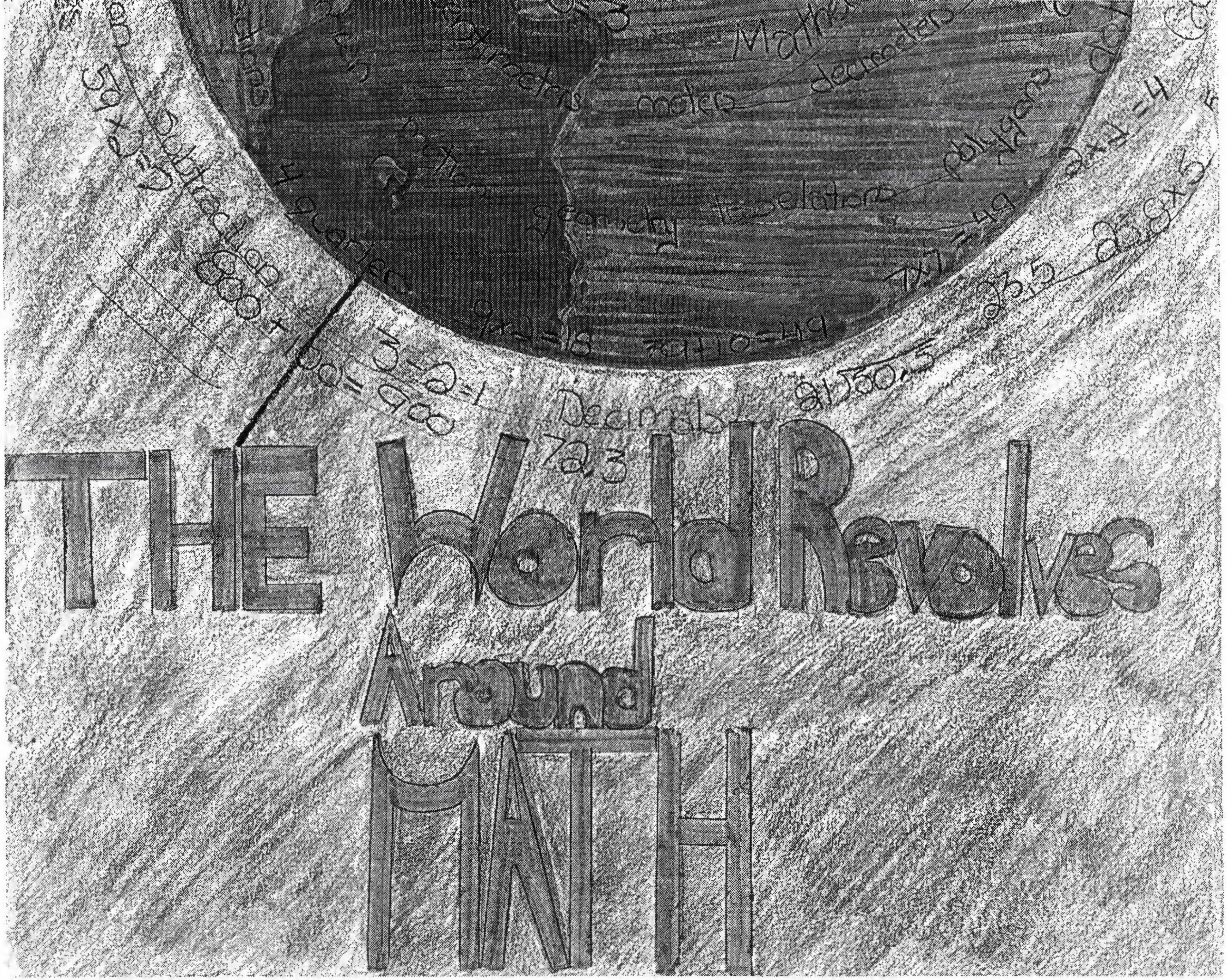
I have discovered a truly marvelous proof of this (the result known as Fermat's last theorem), which however the margin is not large enough to contain.

Pierre de Fermat

World Mathematics Year 2000 Poster

Alie Boos, Grade 6, Monsignor J. S. Smith School, Calgary




THE World Revolves Around MATH
geometry
tessellations
polygons
subtraction
4 quarters
meters
decimeters
9x2=18
7x7=49
3-2=1
800+100=900
Decimals
72.3
23.5

STUDENT CORNER

Mathematics as communication is an important curriculum standard, hence the mathematics curriculum emphasizes the continued development of language and symbolism to communicate mathematical ideas. Communication includes regular opportunities to discuss mathematical ideas and explain strategies and solutions using words, mathematical symbols, diagrams and graphs. While all students need extensive experience to express mathematical ideas orally and in writing, some students may have the desire—or should be encouraged by teachers—to publish their work in journals.

delta-K *invites students to share their work with others beyond their classroom. Such submissions could include, for example, articles on a particular mathematical topic, an elegant solution to a mathematical problem, posing interesting problems, an interesting discovery, a mathematical proof, a mathematical challenge, an alternative solution to a familiar problem, poetry about mathematics, a poster or anything that is deemed to be of mathematical interest.*

Teachers are encouraged to review students' work prior to submission. Please attach a dated statement that permission is granted to the Mathematics Council of the Alberta Teachers' Association to publish [insert title] in one of its issues of delta-K. *The student author must sign this statement (or the parents if the student is under 18 years of age), indicate the student's grade level and provide an address and telephone number.*

The following submissions were received for this issue. "Minimizing Aroma Loss," by Robert Barrington Leigh and Richard Travis Ng of Edmonton, is reprinted with permission from The College Mathematics Journal.

To promote World Mathematics Year 2000, MCATA sponsored a contest for posters that promote mathematics in K–12 classrooms. The winning poster shown on pages 28 and 29 was submitted by Alie Boos, a Grade 6 student at Monsignor J. S. Smith School in Calgary. She received a $100 prize. Special thanks to her teacher, Susan Weisenberger, who submitted the entry.

Minimizing Aroma Loss

Robert Barrington Leigh and Richard Travis Ng

Robert Barrington Leigh is a 13-year-old student at Vernon Barford Junior High School in Edmonton. He has always been interested in mathematics and has competed in several contests. In Grade 6 he won first place in the CNML, and in Grade 7, he gained the Edmonton Junior High math trophy. Robert enjoys Professor A. Liu's math club, and it is under Professor Liu's guidance that he worked on this paper.

Richard Travis Ng is 14 years old and in Grade 10 at Meadowlark Christian School. He lives in Edmonton. His favorite hobbies are reading, building websites, playing badminton and skiing. Richard also plays the violin and the piano.

Imagine that you are the owner of a small coffee shop, and you have just imported a box of the finest Colombian coffee beans. As you open it, you savor the aroma. Suddenly, your smile turns into a frown as you realize that some of the essence of the coffee has evaporated into thin air.

We use the following **mathematical model** to measure the loss. We assume that there are n kilograms of coffee beans initially, when n is a positive integer, and that you will use 1 kg each day. Each kilogram in a box loses one aroma point every time the box is opened. Fortunately, you have some empty boxes, which help in reducing future losses. Let k be the number of boxes available, including the one in which the coffee beans come. You want to minimize the total number of points lost.

Let us first work out an example with $k = 2$ and $n = 6$. After checking all cases, we find this optimal strategy. Let the boxes be numbered 1 and 2.

We now consider the general problem. Clearly, counting the number of points lost each day is not a promising approach, especially since we do not even

Day	Open Box	Points Lost	Shift to Box 2	Amount in Box 1	Amount in Box 2
1	1	6	2 kg	3 kg	2 kg
2	2	2		3 kg	1 kg
3	2	1		3 kg	
4	1	3	1 kg	1 kg	1 kg
5	2	1		1 kg	
6	1	1			
	Total =14				

know how many kilograms of coffee beans are to be transferred from which box to which and when. The main idea behind our attack of this problem is to count the number of points lost by each kilogram.

The number of points each kilogram of coffee beans loses is equal to the number of times it is exposed. We keep track of this by putting a label on each kilogram. Number the boxes 1 to k. A label is initially empty. Every time the kilogram is exposed in box i, add an i to the end of its current label. The label changes progressively until the kilogram is used up. Its length at that time is the total number of points lost.

Each label starts with a 1. By symmetry, we can arrange to have no more coffee beans in a box with a higher number than in a box with a lower one. Each day, we always open the nonempty box with the highest number. Thus we never transfer coffee beans from a box with a higher number to a box with a lower one. This means that the terms in each label are nondescending. Since exactly 1 kg of coffee beans is used each day, no 2 kg can have the same label. What we want is a set of the shortest n labels.

Let us return to our example with $k = 2$ and $n = 6$. There is only one label of length 1, namely 1. There are two labels of length 2 and three labels of length 3. They are 11, 12, 111, 112 and 122. Thus the minimum number of points lost is $1 + 2 + 2 + 3 + 3 + 3 = 14$. This justifies that our strategy is indeed optimal. In fact, it is the only one that leads to the optimal result, since the labels tell us precisely what to do.

Each kilogram is exposed in box 1 on day 1. The kilogram labeled 1 is used immediately. The kilograms labeled 12 and 122 must be shifted to box 2 then. They are used on days 2 and 3. The remaining three kilograms are all exposed in box 1 on day 4. The kilogram labeled 11 is used immediately, while the kilogram labeled 112 must be shifted to box 2. It is used on day 5, while the kilogram labeled 111 stays in box 1 throughout and is used on day 6.

The general problem is solved if we can count the number of distinct labels of length l with nondescending terms such that the first is 1 and none exceeds k. As another example, consider the case $k = 3$ and $l = 5$. There are 15 such labels:

11111	11122	11222	11333	12233
11112	11123	11223	12222	12333
11113	11133	11233	12223	13333

Counting the labels directly is no easy matter either. We now change each into a binary sequence as follows. Write down a number of 0's equal to the number of 1's in the label. Insert a 1 after this block. Then write down a number of 0's equal to the number of 2's, followed by another 1, and so on. Note that each binary sequence consists of k 1's and l 0's, starts with a 0 and ends with a 1.

As an example, consider the label 11122. We start off with three 0's followed by a 1. Then we write down two 0's followed by a 1. Finally, since the label contains no 3's, we just write down one more 1, yielding the binary sequence 00010011. Conversely, consider the binary sequence 01000101. We start off with one 1, followed by three 2's and then one 3, yielding the label 12223. It is clear that each label is matched with a unique binary sequence whose first term is 0 and last term 1, and vice versa. The corresponding binary sequences are listed after the labels in the chart below.

11111	00000111	11133	00011001	12222	01000011
11112	00001011	11222	00100011	12223	01000101
11113	00001101	11223	00100101	12233	01001001
11122	00010011	11233	00101001	12333	01010001
11123	00010101	11333	00110001	13333	01100001

It is not too difficult to count such binary sequences. As noted before, they are of length $l + k$. Since the first term is always 0 and the last term is always 1, we only need to consider the $k + l - 2$ terms in between. They consist of $l - 1$ 0's and $k - 1$ 1's, and all we have to do is count the number of ways of placing the 1's. The answer is the binomial coefficient $\binom{k+l-2}{k-1}$. When $k = 3$ and $l = 5$, $\binom{k+l-2}{k-1} = \binom{6}{2} = 15$. Hence there are indeed 15 labels of length five, as we saw earlier.

For n kilograms of coffee beans, let the longest labels have length m. This means that we use all labels of length less than m, and as many labels of length m as needed to bring the total up to n. Hence m is the largest integer such that the total number N of labels of length from 1 to $m-1$ is less than n. Clearly,

$$N=\binom{k-1}{k-1}+\binom{k}{k-1}+\binom{k+1}{k-1}+\dots+\binom{k+m-3}{k-1}=\binom{k+m-2}{k}.$$

For any positive integer n, let m be the largest positive integer such that $n > \binom{k+m-2}{k}$. Let $r = n-\binom{k+m-2}{k}$, where $1 \le r \le \binom{k+m-1}{k} - \binom{k+m-2}{k} = \binom{k+m-2}{k-1}$. Then the n labels consists of $\binom{k-1}{k-1}$ of length 1, $\binom{k}{k-1}$ of length 2, ... , $\binom{k+m-3}{k-1}$ of length $m-1$, and r of length m. It follows that the minimum number of points lost is

$$\binom{k-1}{k-1}1+\binom{k}{k-1}2+\dots+\binom{k+m-3}{k-1}(m-1)+rm,$$

and that this optimal value can be attained.

In our original example, $n = 6$ and $k = 2$. Now m is determined by $6 > \binom{3}{2}$, so that $m = 3$. Hence $r = 6-\binom{3}{2} = 3$ and the minimum number of points lost is $\binom{1}{1}1+\binom{2}{1}2+\binom{3}{1}3 = 14$, as we found. If $k = 3$, then $m = 3$, $r = 2$ and 13 points are lost. If $k = 4$, then $m = 3$, $r = 1$ and 12 points are lost. If $k = 5$, then $m = 2$, $r = 5$ and 11 points are lost. This is the best that can be done with $n = 6$, and we leave to the reader the details of how to move the kilograms of coffee around.

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READER REFLECTIONS

In this section, we will share your points of view on teaching mathematics and your responses to anything contained in this journal. We appreciate your interest and value the views of those who write.

Erratum

One of our readers pointed out that the June 2000 issue of *delta-K* (Volume 37, Number 2) contained an error on page 12. Specifically, the picture captioned "Len Arden presenting Texas Instrument Graph Links and Software" should have read "Herb Schabert presenting Exploring Technology Links with Math Power 11." I apologize to the presenters and the readers for any inconvenience that this may have caused.

Calendar Math

Arthur Jorgensen

Here are the math exercises for the month of January 2001. These problems vary significantly in difficulty. However, with minor modification most can be adapted to various grade levels. Many problems should lead to interesting discussions, which could in turn lead into many other subject areas.

1. Mary can buy three pencils or two erasers for the same amount of money. How many erasers can Mary buy if she just has enough money for 24 pencils?
2. Which pie gives you more for your money, a 30-cm diameter round pie for $16 or a 20-cm square pie for $7.50?
3. Is it cheaper to take two friends to a movie once or to take one friend twice? Support your answer if the cost of a movie is $4.
4. Sean buys a female cat for $14.50. When his cat has kittens, he plans to sell the kittens for $3.25 each. Ignoring the cost of food, how many kittens will he have to sell before he makes a profit?
5. Each of the letters has a unique numerical value. What is it?

   ABCD
   × 9
   DCBA
6. Mr. Wilson is planning to paint his deck. The deck is 4 m wide and 10 m long. If a litre of paint covers 9 m^2, what is the smallest number of litres of paint that he will have to buy if he plans to give the deck two coats of paint?
7. What will be the least cost of paint for Mr. Wilson if paint can be bought at the following prices: a 4-L pail costs $23.50, and a 1-L can sells for $7.35?
8. How long will it take Mr. Wilson to paint the deck if he can paint one square metre every 3.5 minutes?
9. If Mr. Wilson begins to paint his deck at 9 a.m., at what time will he be finished if he has to allow three hours' drying time between coats of paint?
10. If Mr. Wilson hires his friend Tom to paint the deck, how much will Mr. Wilson pay if Tom charges $0.75 per square metre to paint the deck?
11. If Mr. Wilson decides to have Tom paint the deck, what will be the total cost of the paint job, including the cost of the paint?
12. The Olsons have five sons and each son has two sisters. How many children do the Olsons have?
13. The Lauterbachs are planning to plant a row of trees along their driveway. If the driveway is 150 m long, how many trees will they have to plant if the trees are to be 10 m apart?
14. Susan wants to enclose a pasture area for her horse, Silver. She has only 157 m of rope. What is the largest area that she can enclose?
15. Cyprian thinks he should walk his dog, Buster. On the first day he walks Buster for 10 minutes. On the second day he walks for 20 minutes. On the third day he walks for 30 minutes. If he maintains the pattern for one week, how long will he walk on the seventh day?
16. What is the total time that Cyprian will have walked in the week?
17. If Cyprian walks at the rate of 3 km/h, how far will he have walked in the week?
18. If Cyprian burns 60 calories per hour while walking, how many calories will he burn during the week while walking?
19. Victor eats half his candy on Monday. On Tuesday he eats half of what is left. On Wednesday he again eats half of what is left. How much of his candy is left on Thursday?
20. If Victor continues this pattern of eating his candy, when will his candy be all gone? [*This is a good discussion question.*]

21. Mrs. Smith has her dog in a kennel that is 6 m × 4 m. She feels that it is too small. She wants a kennel that is four times as large. What will be the dimensions of the new kennel?

22. A particular insect can double its weight every hour. It reaches its maximum weight after six hours. After how many hours does it reach half its maximum weight?

23. How would you arrange the following digits to get the largest number? The smallest number? 3705. What is the difference between the largest and smallest numbers?

24. According to statistics, nine babies are born and three people die every two seconds. If this is true, how many babies are born in any one day?

25. Using the above data, how many people die in any one day?

26. According to these statistics, by how many people is the world's population increasing in any one day? [*These questions should provide the opportunity for some excellent discussion.*]

27. Alberta's population is approximately 3,000,000. If Alberta's population increases at the approximate rate of 50,000 per year, in what year will the population reach 4,000,000?

28. Of the 50,000 increase in population, 20 percent are school-aged children. How many more classrooms will be required each year, if one classroom is expected to house 25 students?

29. If it costs approximately $6,000 to educate each child per year, how much will the annual cost of educating these additional students be?

30. Jane is three years older than her brother Adolph. In five years, their combined ages will be twenty-one years. How old is Jane today?

31. If the price of a candy bar has doubled every 10 years and it costs $1 today (2001), to the nearest cent, what did it cost in 1961?

Answers

1. 16 erasers
2. The 20-cm square pie
3. It is cheaper to take two friends to the movie once because one only has to buy three tickets, versus four tickets if one takes one friend twice. If tickets cost $4 each, the difference means saving $4.
4. 5 kittens
5. A = 1, B = 0, C = 8, D = 9.
6. 9 L
7. $54.35
8. 280 minutes
9. 4:40 p.m.
10. $60
11. $114.35
12. 7 children
13. 16 trees
14. A circle will provide the largest area: 1,962.5 m^2
15. 70 minutes
16. 280 minutes
17. 14 km
18. 280 calories
19. 1/8 of his candy
20. Mathematically, the candy will never be all gone.
21. 12 m × 8 m, or any other measurement that produces 96 m^2
22. 5 hours
23. Largest: 7530, smallest: 3057, difference: 4473
24. 388,800 babies are born in 24 hours.
25. 129,600 people die in 24 hours.
26. 259,200 people
27. 20 years
28. 400 classrooms
29. $60,000,000
30. Jane is seven years old.
31. 6 cents

Discovery and learning from the very beginning of his education, the child should experience the joy of discovery.

Alfred North Whitehead

1999 Calgary Junior Mathematics Contest

Robert E. Woodrow

Part A: Short Answer Problems

No part marks.

A1. How many zero digits does the whole number $\frac{2000}{0.000\ldots01}$ have, if the denominator has 1999 zeros after the decimal point?
Solution: 2003

A2. Which number in the box makes the equation true? $\sqrt{1+\sqrt{1+\sqrt{\square}}}=2$.
Solution: 64

A3. A cube has sides of length of 1 metre. What is the largest number of corners you can choose so that none of the chosen corners are 1 metre apart?
Solution: 4

A4. How many numbers in the list 1, 2, 3, ..., 1999 are both perfect squares and perfect cubes of whole numbers?
Solution: 3

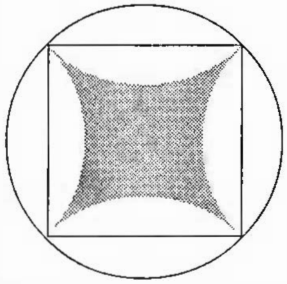
A5. Todd removes exactly 7/12 of the marbles from a jar of marbles. Then Tyler removes exactly ½ of the remaining marbles, and then Zita removes exactly 1/5 of the marbles that are left after that. What fraction of the original number of marbles is now left in the jar?
Solution: 1/6

A6. Find $(1+\frac{1}{2})\times(1+\frac{1}{3})\times(1+\frac{1}{4})\times\ldots\times(1+\frac{1}{1999})$ in simplest form.
Solution: 1000

A7. Find three different positive integers a, b, c, all less than 10, so that $1/a + 1/b = 3/c$.
Solution: $(a, b, c) = (1, 4, 2)$ or $(2, 4, 3)$ or $(2, 8, 4)$ or $(4, 8, 6)$

A8. It takes Emily 20 minutes to walk to school. It takes Fran 15 minutes to ride her bicycle to school. Fran cycles three times as quickly as Emily walks. If Emily lives 2 km from the school, how far (in km) does Fran live from the school?
Solution: $^{9}/_{2}$ or 4.5 km

A9. In the figure, the circle has radius 1 metre. A square is drawn inside it as shown and the four flaps of the circle are folded over the square. What is the area in square metres of the uncovered (shaded) region?



Solution: $4-\pi$

Part B: Long Answer Problems

Work must be shown to earn full credit. Part marks may be earned for partially correct solutions.

B1. A certain junior high school has Grades 7, 8 and 9. Last year the average grade of all the students in the school was exactly 8. Then all 99 of the Grade 9 students passed and left to go to high school, all the Grade 8 students passed into Grade 9, all the Grade 7 students passed into Grade 8 and 77 new Grade 7 students entered the school. The average grade of all the students in the school this year is still exactly 8. How many students are now in the school?
Solution: The only way the average grade of all the students in the school can be exactly 8 is if there are the same number of Grade 7 students as Grade 9 students. Since there were 99 Grade 9 students last year, there must have been 99 Grade 7 students as well. They have all become Grade 8 students this year. Since there are 77 Grade 7 students this year, and the average Grade is still 8, there must be 77 Grade 9 students this year too (these were the Grade 8 students last year). So the total number of students in the school this year is 77 + 99 + 77 = 253.

B2. Amanda and Cam are traveling to Calgary from Regina. Amanda took a plane while Cam drove. The plane's average speed was 550 km per hour; the average speed of Cam's car was 110 km per hour. Also, the road from Regina to Calgary is 100 km longer than the flight path of the plane, and Cam took 6 hours longer to make the trip than Amanda did. How far in km is it by plane from Regina to Calgary?

Solution: If we let x be the distance in km by plane from Regina to Calgary, then $x + 100$ will be the number of kilometres from Regina to Calgary by road. Since Amanda flies at 550 km/h, her trip will take $x/550$ hours. Since Cam drives at 110 km/h, his trip will take $(x + 100)/110$ hours. Thus we get the equation

$$\frac{x}{550} + 6 = \frac{x + 100}{110},$$

which simplifies to

$$x + 6(550) = (x + 100)5$$

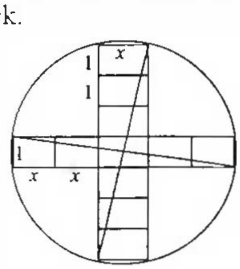
and then

$$x + 3300 = 5x + 500,$$

so $4x = 2800$ or $x = 700$ km.

Of course, this problem can also be done by "guess and test."

B3. A circular garden has two perpendicular paths across it, formed by equal-sized rectangular concrete blocks. In the east-west direction exactly five blocks fit across the garden, and in the north-south direction exactly seven blocks fit, as shown in the diagram. The shorter side of each block is exactly 1 metre. Find the length of the longer side of each block.



Solution: Let x be the length of the longer side of each block, in metres. The east-west path across the garden is a rectangle of length $5x$ metres and width 1 metre, so by the Pythagorean theorem the square of the diagonal of this rectangle will be $(5x)^2 + 1^2 = 25x^2 + 1$. The north-south path is a rectangle of length 7 and width x, so the square of its diagonal will be $49 + x^2$. Both of these diagonals are diameters of the circle, so their squares must be equal. Thus $25x^2 + 1 = x^2 + 49$, which says $24x^2 = 48$, so $x^2 = 2$. Therefore $x = \sqrt{2}$, since the distance x is positive.

B4. Yin and Zack went to a restaurant. When they got their bills, Yin added a 10 percent tip onto her bill while Zack added a 15 percent tip onto his, and the total amount they paid was \$41. If instead Yin had given a 15 percent tip and Zack had only given a 10 percent tip, the total amount they paid would have been \$40. What were the original amounts of their two bills?

Solution: If we add together the two amounts paid, the resulting total of \$81 can be thought of as the original (total) bill doubled, with a tip added on which is 15% + 10% = 25% of the undoubled bill and so 12.5% or 1/8 of the doubled bill. Thus \$81 must be 9/8 of the original doubled bill, so the original bill (doubled) must have been 8/9 of \$81 which is \$72, and the original bill must have been \$36.

On the other hand, if we subtract the two amounts paid we get \$1, and this amounts to 15% – 10% = 5% = 1/20 of the difference (Zack's bill minus Yin's bill). So the original bills must differ by exactly \$20. Therefore we want two amounts whose sum is \$36 and whose difference is \$20, and it is easy to see that they are \$28 and \$8. So Yin's bill was \$8 and Zack's was \$28.

Of course you could also do this problem by algebra or by "guess and test."

B5. Each day Len puts on his socks and shoes and laces up his shoes. Of course he must put his left sock on before he can put on his left shoe, and the same for his right sock and shoe, and he must also put each shoe on before he can lace it up. But otherwise he can put his socks and shoes on and lace them up in any order. In how many different ways can he do these (six) things?

Solution: The first thing Len must do is put on one of his socks. Let's say he puts on his left sock first; if we count the number of ways he can do the other five things after this, then the answer we want is exactly twice this, because there will be same number of ways to do the six things by starting with his right sock (just exchange left and right).

After putting on his left sock, Len has two choices: put on his right sock next or put on his left shoe next. And so forth. Here are all the ways Len can proceed:

- Left sock, right sock, left shoe, lace left shoe, right shoe, lace right shoe.
- Left sock, right sock, left shoe, right shoe, lace left shoe, lace right shoe.
- Left sock, right sock, left shoe, right shoe, lace right shoe, lace left shoe.

- Left sock, right sock, right shoe, lace right shoe, left shoe, lace left shoe.
- Left sock, right sock, right shoe, left shoe, lace right shoe, lace left shoe.
- Left sock, right sock, right shoe, left shoe, lace left shoe, lace right shoe.
- Left sock, left shoe, lace left shoe, right sock, right shoe, lace right shoe.
- Left sock, left shoe, right sock, lace left shoe, right shoe, lace right shoe.
- Left sock, left shoe, right sock, right shoe, lace left shoe, lace right shoe.
- Left sock, left shoe, right sock, right shoe, lace right shoe, lace left shoe.

These are 10 ways, so multiplying by 2, we get a total of 20 ways for Len to put on his socks and shoes.

There is a much easier way to do this problem, if you use combinations (which is not a Grade 9 topic, of course). There are six things to do, in some order, so imagine having a row of six blanks in which you write the six tasks. If you choose three of these six blanks to be where you will write in the three things Len does to his left foot, this determines the entire order, because he must do the left-foot things in a certain order, and the right-foot things must be done in a certain order in the other three blanks. So the number of orders is just the number of ways of choosing three blanks from a row of six blanks, which is

$$\binom{6}{3} = \frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1} = 5 \cdot 4 = 20.$$

B6. Karen has five cats. They each sleep 15 hours each day, though not necessarily at the same time, and not necessarily 15 consecutive hours; they could sleep several times a day for just a short period each time, as long as the total amount they sleep each day is 15 hours.

(a) Does there have to be a time each day in which all five cats are asleep at the same time? Explain your answer.

Solution: The answer to (a) is No, because, for instance, the sleeping schedules for the five cats could be this:

Cat 1: Asleep from midnight to 3 p.m. each day, awake otherwise.

Cat 2: Asleep from 4 a.m. to 7 p.m. each day, awake otherwise.

Cat 3: Asleep from 8 a.m. to 11 p.m. each day, awake otherwise.

Cat 4: Asleep from noon to 3 a.m. each day, awake otherwise.

Cat 5: Asleep from 4 p.m. to 7 a.m. each day, awake otherwise.

Then there are never more than four cats asleep at the same time.

(b) Find the least possible amount of time each day during which there are at least three cats asleep at the same time.

Solution: Suppose that for x hours in a day there are at least three cats asleep, and for the other $24 - x$ hours there are at most two cats asleep. Then the total number of sleeping hours this accounts for is at most $5x$ (which would happen if all five cats were asleep during the x hours) plus $2(24 - x)$ (if during the $24 - x$ hours there were always two cats asleep). This adds up to at most $5x + 2(24 - x) = 3x + 48$ sleeping hours. But the five cats together have a total of $5 \cdot 15 = 75$ sleeping hours, so $3x + 48$ must be at least 75. Solving this inequality we get that x is at least 9.

In fact we claim that $x = 9$ is possible. To prove this we must find a daily sleeping schedule for the cats which has only 9 hours during which at least three cats are asleep. To do this we will need that all five cats are asleep during these 9 hours, and that at other time there are never three cats asleep at once. Here is one way to do this. All cats are asleep from midnight to 9 a.m. each day, and also:

	9 a.m. to noon	noon to 3 p.m.	3 p.m. to 6 p.m.	6 p.m. to 9 p.m.	9 p.m. to midnight
Cat 1	asleep	asleep			
Cat 2		asleep	asleep		
Cat 3			asleep	asleep	
Cat 4				asleep	asleep
Cat 5	asleep				asleep

So the answer is 9 hours.

Another way to do this problem would be to find such a daily schedule that for nine hours all five cats are asleep while during the other 15 hours only two cats are asleep, and then argue that since we've packed the maximum number of sleeping cats into these 9 hours, and at all other times there are always two cats asleep, there is no "room" to have an answer of less than 9 hours.

Aspects of Numeracy in the Primary Years (K–3): Selected Challenges for Teachers

Werner Liedtke

Information about numeracy is available for teachers, as well as parents. It is referred to in the revised British Columbia Ministry of Education documents, *Supporting Learning: Understanding and Assessing the Progress of Children in the Primary Program* (2000) and *The Primary Program: A Framework for Teaching* (2000). Numeracy is the subject and title of a 1998 pamphlet prepared by the British Columbia Association of Mathematics Teachers (BCAMT). This pamphlet includes the notable statement that numeracy is as important as literacy.

What is numeracy? What are some reasons for the suggested importance of numeracy? Why do teachers play such a key role in fostering the development and growth of numeracy?

Numeracy

According to the BCAMT (1998) pamphlet, numeracy is much more than knowing about the numbers and number operations. It relates to a person's abilities to confidently apply mathematical knowledge in various, even unfamiliar, situations. These abilities include flexible thinking, willingness to take risks and connecting new ideas to what is known. The mathematical knowledge includes the important aspects of number sense, as well as spatial sense, statistical sense and sense of relationship. These components clearly illustrate and support the statement from the pamphlet that numeracy is important because people need this skill to function in everyday life, in the home, the workplace and the community.

Contributions by Teachers

Teachers and parents play key roles in fostering numeracy development (Leder 1992). Students will become flexible thinkers and acquire the willingness to take risks if teachers of mathematics value and nurture these characteristics in their classrooms. This may not be an easy task, however, because the majority of teachers likely experienced a closed or heavy-handed approach to mathematics learning. In such a setting, the focus is on a teacher who shares ways of thinking or even prescribed steps of solving different problems. Students memorize these given or prescribed procedures and strategies. Assessment is based on ability to recall what has been memorized, frequently in a timed setting, rather than on understanding and flexible thinking.

In an open-ended setting, students learn that different procedures, strategies and/or answers may exist for given problems. Creating such an awareness can, over time, contribute to the fostering of students' self-confidence, risk-taking and flexible thinking (Spungin 1996; Liedtke, Kallio and O'Brien 1998).

The following examples illustrate a possible difference between closed and open-ended approaches. In a closed setting, one answer or strategy would be considered correct during class discussions, for activity sheets or for assessment tasks such as the following:

A. Which comes next? □ ○ □ ○
   1 2 3 ____
B. Which does not belong? 2 8 9 10
C. How do you find the answer for 8 + 7?

In an open-ended approach, the questions would be

A. Which one *do you* think comes next? Why?
B. Which one *do you* think does not belong? Why?
C. What are some different ways to find the answer for 8 + 7?

After an answer is elicited, students might be encouraged to respond to the question, What is another possible answer/way? Evidence of flexible thinking becomes apparent when, for example A, students extend the pattern beyond any shape that has been chosen, that is, a square, and when they consider ways of creating repeating or growing patterns for 1, 2, 3, ___.

In one Grade 3 classroom I visited, a type of weekly challenge task consisted of examining five displayed numerals and trying to identify the one that did not belong. One numeral was considered to be the correct answer. To encourage flexible thinking, this task can easily be changed to challenging students to think of questions that would make each displayed numeral a correct answer to the request. This approach is possible for examples of the type shown in B.

An open-ended approach does not imply that students will not learn the basic facts. On the contrary, students will learn these and much more. The different strategies they learn in such settings to re-invent forgotten facts will transfer to other mathematical ideas (Isaacs and Carroll 1999). This would be true for students who know several ways, other than counting, of convincing someone that the answer for 8 + 7 is 15.

These examples illustrate that the desirable goals that are part of numeracy are unlikely to be reached without a skillful teacher who is able to create an appropriate classroom atmosphere and orchestrate discussions that provide opportunities for students to think, to think about thinking and to explain and compare thinking or thinking strategies.

Aspects of Numeracy—Goals

The BCAMT (1998) pamphlet identifies the following aspects of numeracy: number sense, spatial sense, statistical sense and sense of relationship. A detailed discussion of each aspect is beyond the scope of this article. My goal is to identify relevant outcomes from *Mathematics, K to 7: Integrated Resource Package* (British Columbia Ministry of Education 1995) for each and to indicate possible goals that might be considered characteristic of a numerate student. Because it can be argued that "a sense of number" is of prime importance, and this is reinforced by the fact that this skill appears at the top of every grade level in the integrated resource package, the greatest attention is given to this aspect of numeracy.

Prenumber and Number Sense

Table 1 identifies thinking strategies that are part of the prenumber sense. These strategies are necessary prerequisites for understanding number or acquiring a sense of number and being able to count rationally. Because activities with patterns involve strategies similar to those for ordering, the aspect of numeracy labeled sense of relationship has been included in Table 1. Although the goals included under the heading "fluent" may appear to be rather specific whenever possible, many do include reminders that flexible thinking is an integral part of fluency or being numerate.

Outcomes for the important benchmarks (to five; to ten) of early number sense are included in Table 2. The activities with numbers to five should lead students to recognize these numbers without having to count (subitizing). Finger-flash activities can be used to assist with reaching this visualization goal (Liedtke 1992–93). The part-part whole understanding of number, or being able to assign different names to a number (for example, 5 and 2 *or* 3 and 4 for 7), will transfer to the development of thinking strategies for the basic facts. Rational counting implies that students can tell why certain numbers come next in ordered sequences (for example, 2, 4, 6, _; 1, 3, 5, _; 1, 1, 2, 3, 5, _). When counting a set of objects, students should learn to realize that counting is independent of direction. Counting can be carried out in any possible way as long as names are appropriately matched with objects.

The goals of a sense of number to 999 are included in Table 3. Two key goals for two-digit numbers are related to visualization and realizing that each number can have two or more names. The visualization process can be enhanced, for example, by having students think of the fewest number of students it would take to show a given number on fingers or with base-ten blocks (that is, for 47—five students or four "tens" and seven "ones").

Activities that involve estimation can contribute to the development of number sense. Students need to know what is meant when they are asked to *guess* and when to *estimate*. Care should be taken not to use these terms carelessly or interchangeably. Care also needs to be taken when acknowledging students' responses to requests for guesses and estimates. Judging or labeling the response as good or even excellent can easily have other students consider their guesses or estimates as inappropriate. As a result they may lose some of their willingness to take risks. It is advantageous to consider all guesses and estimates as good and to let students know that that is the case. (If, by chance, criteria for judging appropriateness of estimates are used, students should know how these are determined.)

Properties of numbers can be discovered and visualized as students move counters, shade in squares on paper or use pegs and rubber bands on geoboards.

Table 1
Goals for Prenumber Sense Thinking Strategies

	Prenumber	Emerging	Early	Fluent
Sorting	• free play • response to "Which one does not belong?" cannot be explained	• sorts by color or shape or size • response to "Which one does not belong?" cannot be explained	• can sort using two characteristics • responses to "Which one does not belong?" can be explained	• is flexible, can sort beyond color, shape and size • is able to defend response to "Which one does not belong?" with several different answers • recognizes number as a common characteristic of groups of different looking objects
Ordering three objects such as toys, dolls, sticks...	• random	• selects biggest or smallest object	• can order from "smallest" to "biggest"	• can order from "biggest" to "smallest" and vice-versa
Ordering more than three objects such as toys, dolls, sticks...	• random	• can copy an ordered sequence with a set of objects of the same type	• can copy an ordered sequence with a set of objects of a different type	• can order from "biggest" to "smallest" and vice-versa and use appropriate language to describe adjacent numbers • can extend in both directions • is able to insert objects into a given sequence • is able to construct own ordered sequences for variety of characteristics • is able to apply the ordering strategies to number
Matching: find as many or show the same number (without counting)	• random, no understanding of the request "Find as many" or "Show the same number"	• begins matching but fails to complete the one-to-one correspondence and may revert to free play	• matches but counts in order to verify the result	• recognizes that matching results in equivalent sets even if the objects differ according to color, shape, or size • recognizes equivalence even after the objects are re-arranged.
Pattern	• uses term incorrectly	• able to copy a given pattern	• able to extend simple repeating patterns • selects correctly when given a choice to extend a pattern	• recognizes predictability of repeating and growing patterns • aware that patterns can be extended in more than one way • able to extend and describe hidden members of patterns and justify responses • constructs own patterns • identifies patterns in different mathematical settings • looks for patterns as a possible problem solving strategy

Table 2
Goals for a Sense of Numbers—to 5, 10 and 20

	Prenumber	Emerging	Early	Fluent
Early number (zero to five): ideas of few, many, some	• does not use this language (prefers big)	• uses terms like *many*, *lots* or *most*	• uses *few, many* and *some*	• realizes the relativity of the terms (for example), six fish is a lot in a fish bowl; not many in the pond or lake)
Early number (zero to five: **subitizing** (recognition of number)	• may not recognize one for similar objects (without generalizing about oneness)	• recognizes one and two for dissimilar objects, checks by counting	• recognizes up to three for dissimilar objects, checks by counting	• recognizes up to five similar objects without counting • recognizes number for a variety of arrangement of similar objects
Early number (zero to five): **part-part-whole** (different ways to show a number)	• no recognition that one and four is the same as five	• shows number in one different way	• can show two ways, uses counting to justify response	• can show all the possible ways and uses matching to justify
Early number (six to ten): **subitizing**	• makes guesses	• uses rote counting	• uses rote counting but sometimes counts on from five	• names all familiar arrangements for two to ten objects and can re-organize for easy recognition • for unfamiliar arrangements uses a part-part-whole strategy
Counting	• makes errors (misses)	• may make a few errors • counts up to a certain number	• error free rote counting	• error free and is aware of the patterns: one more, two more, etc.
Matching ordered numbers with numerals (such as 5 or five)	• incorrect matches	• correct matches for one and two	• occasional errors	• error free
Ordering the numerals to ten	• some correct parts of the sequence; may miss	• most of the sequence correct	• recites whole sequence	• is a rational counter: can start anywhere, count on, count back, count by twos and justify responses
Counting to twenty	• some correct parts of the sequence • may skip or miss	• major parts of the sequence are correct	• recites whole sequence using standard names	• counts rationally (see above) • can give two names for every number: standard; also knows that thirteen is one ten and three ones

Table 3
Goals for a Sense of Number—99 to 999

	Prenumber	Emerging	Early	Fluent
Two-digit numbers and numerals **to 99**	• no apparent understanding—reverses digits, misses numbers, no meaning, counts by rote to a specific number	• standard names • rote recital in one direction	• knows two different names • can count in two directions	• knows many names for a given number (37 is 30 and 7, 20 and 17, 10 and 27 and so on) • can explain how to construct a number using the least number of base ten blocks or students (showing their fingers in tens and ones) • skips counts in different directions • fills in between two given numbers
Three digit numbers **up to 999**	• considers each digit as ones • makes errors when rote counting • unable to read numbers, may read 126 as "one, two, six"	• rote recital of standard names • knows what comes after a given number • reads most numbers correctly	• can represent the number in one way • can count in two directions	• states many different names for a given number • can visualize in terms of base ten blocks or money denominations (ones, tens, hundreds) • skip counts in several ways • given numbers, knows the number before, next and in between
Estimation up to ten (about, how many?)	• guesses	• recognizes more than three	• recognizes more than five • may use the word *about* incorrectly by giving an exact answer	• uses a referent of five (one hand) or ten (two hands) to make a visual comparison • uses the word *about* correctly
Estimation up to 99	• guesses	• guesses for numbers greater than ten	• estimates but sometimes reports exact answers ("That is about seventy-nine.")	• uses ten as a referent, gives answers that end in zero • does not give exact answers when using *about*
Estimation up to 999	• guesses	• can identify extreme incorrect choices	• correctly chooses best answer from a list of choices	• can connect to experience by selecting the best answer for a given setting (Number of students in your school: 20, 200 or 900?) • combines measurement sense with number sense (height of the door: 200 cm, 500 cm or 900 cm?)
Properties of Numbers: odd even, triangular, rectangular, square	• no recognition	• given objects, inconsistent but may be able to understand even numbers to ten	• can identify even and odd for numbers up to twenty	• visualizes numbers for number names and is able to label numbers as *odd, even, triangular, rectangular, square*

Spatial and Measurement Sense

Goals for spatial sense are shown in Table 4. Some mathematics programs tend to suggest and emphasize that students learn (memorize) the names for blocks. Just as a sense of number is independent of knowing how to print numerals (or even reversing them) and number names, visualization or spatial sense is not in any way enhanced by knowing the names for blocks (my experiences in classrooms have shown that many times young students use incorrect names; for example, *square* for *cube*).

In the BCAMT (1998) pamphlet, measurement is referred to under the heading of spatial sense. Important ideas related to measurement sense are included in Table 5. One key idea for all topics of measurement is related to the fact that "our eyes may deceive us." Things that in some way appear to be different (longer; bigger in area, capacity or volume; heavier; "takes longer" and so on) may in fact be the same, and things that are similar in appearance may be quite different. Another important idea is related to appropriateness of selecting units of measurement. Students need to learn that this appropriateness depends on the type of problem to be solved and not on "speed of obtaining results" (that is, is a door to be covered with a metal frame or with newsprint?).

Statistical Sense

Goals for statistical sense are included in Table 6. Activities that involve the collecting and organizing of data are part of every grade. As students are taught how to interpret the graphs they have constructed or

Table 4
Goals for Developing Spatial Sense

	Prenumber	Emerging	Early	Fluent
Examination of 3-D figures	• free play, no connections	• fails to look at the whole block, talks about parts	• uses one label for a given block	• relates a block to many familiar objects by concluding "it depends on how you look at it"
Examining diagrams or photographs of 3-D figures ("Which block goes with this picture?")	• random matches	• selects the block that may have one or a few similar characteristics	• matches one block by placing it on top of the picture	• matches with appropriate block, describes part not visible, recognizes when more than one covered answer is possible • given a picture can hold a block in the same position
Special 2-D shapes (triangles, rectangles, circles)	• no distinction between open and closed curves	• recognizes difference between open and closed curves	• recognizes one type of triangle • recognizes most types of rectangles • recognizes circles	• correctly labels all parts of triangles (rectangles, circles) • recognizes how triangles (rectangles, circles) are the same and how they can differ
Transformation 3-D and 2-D	• not meaningful	• identifies difference, unable to explain	• identifies the results of slides • predicts the outcomes of slides and connects to experience	• predicts the outcome flips, slides, turns and connects these to events/actions from experience (printing, physical education, art)
Symmetry	• unable to verbalize ideas about symmetry	• recognizes simple symmetry and may use the word "same"	• can identify/locate one line of symmetry	• identifies/locates multiple lines of symmetry • relates to experience (printing, nature, art) • constructs symmetrical figures (3-D, 2-D)

Table 5
Goals for Developing a Measurement Sense

	Prenumber	Emerging	Early	Fluent
Length	• relies on eyesight, sees no need to measure	• considers both endpoints, but neglects what lies in between	• inconsistent use of ruler • no estimation strategies, just guesses	• able to measure to nearest unit • knows when to estimate and when to measure • knows several estimation strategies • knows that appropriateness of selecting a unit depends on the type of problem to be solved • connects to events from experience (art, science, social studies, physical education)
Area	• long means big, longer means bigger	• compares by fitting similar shapes on top of one another	• modifies dissimilar shapes to make comparisons	• realizes that eyes can "deceive me"—shapes that look different can have the same area • knows more than one estimation strategy • connects to ideas from events/experience
Capacity/ Volume	• classifies as big or small	• orders similar containers from smallest to biggest	• uses nonstandard units (handful) to check a given guess	• realizes that different-looking containers/buildings can have the same capacity/volume • has at least one estimation strategy • connects to events/actions from experience
Mass	• bigger means heavier	• realizes similar objects could differ • uses hands to make comparisons	• willing to make predictions about two objects, inconsistent	• realizes that different looking objects could be the "same"; objects of the same size and shape could be "different"; and "bigger" looking object could be "lighter" • connects to events/actions from experience
Temperature	• understands hot and cold	• understands hot, cold, warm	• relates to experience (not to the Celsius scale)	• knows that hot, cold and warm could be relative • identifies important numbers on the Celsius scale (0, 20, 37, 100)
Time	• no understanding	• understands "takes a long time" versus "takes a little time"	• rote reading of analog clock to the half hour • reads digital clock without understanding	• knows that time can be relative • uses units of time appropriately • can explain the meanings of markings and parts of a clock • connects time measurement to experience (parts of a day, days, weeks, months, years, seasons)

Table 6
Goals for Developing Statistical Sense

	Prenumber	Emerging	Early	Fluent
Examination of Collected Data	• unable to make logical interpretations of data	• detects only results obvious to visual inspections	• can make simple arithmetical interpretations (how much more, less) • can make true/false interpretations	• can interpret and create statements (talk/write) that involve *true, false, could be true, likely, unlikely, certain* and *uncertain* • uses appropriate language to talk about outcomes of events from experience • applies probabilistic thinking to data, as simple interpolations and extrapolations are made

drawn, they should be challenged to respond to many types of questions and statements. Some of these should be answerable by examination of the graphs, while others could deal with the displayed topic or idea without the data directly enabling students to prepare responses. Statements about graphs should be created that students are required to classify as *true, false* or *could be true* (but the graph does not tell us). Students should be challenged to create different types of questions and statements of their own.

As data from graphs and reports from daily events (news, weather reports, sports and so on) are discussed, ask students to make/write statements that use the terms *likely, unlikely, certain* or *uncertain*. Graphs that have been constructed to find and show the answer to a problem (Liedtke 1992)—Which color do we prefer? What is our heartbeat before and after an exercise?—could be related to other populations or events—Do you think other groups would have the same preferences? Do you think other types of exercises would show the same results? Questions such as these provide many opportunities for students to explain and justify their thinking. They can be given a chance to pursue their own ideas and problems.

Conclusion

The importance of the role of a teacher has been mentioned. Major outcomes for different aspects of numeracy have been identified. To successfully foster development and growth of numeracy, teachers must provide opportunities for students to use their own language as they justify their thinking and provide reasons for their responses. Students need to become confident as they explain their thinking. As Lappan (1999) suggests, students need to learn to persevere. A high level of confidence and a willingness to try things in mathematics are "gift[s] that will carry them forth to future mathematics and life success [and] help them learn to produce their own ideas" (p. 3).

References

British Columbia Association of Mathematics Teachers (BCAMT). *Numeracy*. Vancouver: British Columbia Teachers Federation, 1998. Pamphlet.

British Columbia Ministry of Education. *Mathematics, K to 7: Integrated Resource Package*. Victoria: Author, 1995.

——. *Supporting Learning: Understanding and Assessing the Progress of Children in the Primary Program.* Victoria: Author, 2000. Booklet.

——. *The Primary Program: A Framework for Teaching.* Victoria: Author, 2000.

Isaacs, A. C., and W. M. Carroll. "Strategies for Basic Facts Instruction." *Teaching Children Mathematics* 5, no. 9 (May 1999): 508–15.

Lappan, G. "President's Message—Fostering a Good Mathematical Disposition." *NCTM—News Bulletin* 36, no. 2 (September 1999): 3.

Leder, G. "Mathematics and Gender: Changing Perspectives." In *Handbook of Research on Mathematics Teaching and Learning*, edited by D. A. Grouws, 597–622. Toronto: Maxwell Macmillan Canada, 1992.

Liedtke, W. "Versatile Graphing." *Vector* 33, no. 2 (Winter 1992): 34–39.

——. "Promoting the Development of Number Sense in the Early Years." *Prime Areas* 35, no. 1 (1992–93): 43–45.

Liedtke, W., P. Kallio and M. O'Brien. "Confidence and Risk Taking in the Mathematics Classroom (Grade 1)." *Primary Leadership* 1, no. 2 (1998): 64–66.

Spungin. R. "First and Second Grade Students Communicate Mathematics." *Teaching Children Mathematics* 3, no. 4 (1996): 174–79.

n-Secting a Line Segment by Straightedge and Compass Methods (with an Inductive Proof Using Vector Theory)

Murray L. Lauber and Keith Harder

The motivation for this article grew out of a problem presented to me by an artist colleague at Augustana University College, Keith Harder. He had located the thirds, fourths, fifths and so on of the diagonals of a rectangle on a grid by methods in harmony with those described here and was looking for a mathematical justification of the methods. The grid illustrated a variety of methods of finding these thirds, fourths, fifths and so on. In the interests of simplicity, I have focused on just one.

The article consists of two parts: the first, a description of a method of n-secting a line segment; the second, a proof documenting why the method works using vector theory.

The Method

Construction Procedures Employed

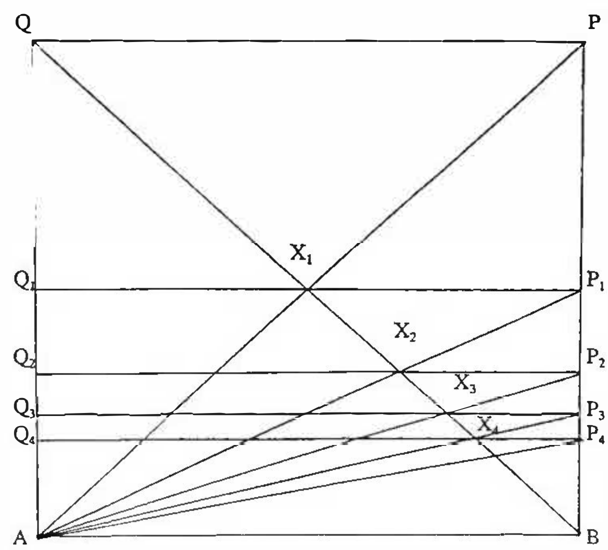
The method is based on two elementary straightedge and compass procedures:

- Constructing a rectangle with a given line segment as one of its diagonals (this can be done by bisecting the segment, constructing the circle with the segment as a diameter, then using that diameter along with a second diameter as the two diagonals of the rectangle).
- Constructing a line segment parallel to a given line segment through a given point.

Description of the Steps

The problem of n-secting a line segment is equivalent to the problem of n-secting a diagonal of a rectangle. This can be done on diagonal BQ of rectangle ABPQ as follows.

1. Let X_1 be the point of intersection of diagonals AP and BQ. Then X_1 bisects diagonal BQ.
2. Construct Q_1P_1 parallel to AB through X_1 as indicated in the diagram above. Then construct diagonal AP_1 of rectangle ABP_1Q_1. Let X_2 be the point of intersection of BQ and AP_1. Then X_2 is one-third of the way from B to Q. Call X_2 a "third" of BQ. Then the other third of diagonal BQ, the one nearest Q, can be located by an analogous process, completing the trisection of BQ.
3. Construct Q_2P_2 parallel to AB through X_2 as indicated. Then construct diagonal AP_2 of rectangle ABP_2Q_2. Let X_3 be the point of intersection of AP_2 and BQ. Then X_3 is one-fourth of the way from B to Q. Call X_3 a "fourth" of BQ. Then another fourth, the one nearest Q, can be located by an analogous process. These two fourths along with X_1 quadrasect, or 4-sect, BQ.
4. Construct Q_3P_3 parallel to AB through X_3 as indicated. Then construct diagonal AP_3 of rectangle ABP_3Q_3. Let X_4 be the point of intersection of AP_3 and BQ. Then X_4 is a "fifth" of BQ. Locate the fifth of BQ nearest Q, as well as the two-fifths of diagonal AP nearest A and P, by analagous processes. Construct the rectangle determined by these four-fifths. Then locate the thirds of the diagonal of this smaller rectangle coincident with diagonal BQ of rectangle ABPQ. These thirds, along with the fifths of BQ, 5-sect BQ.

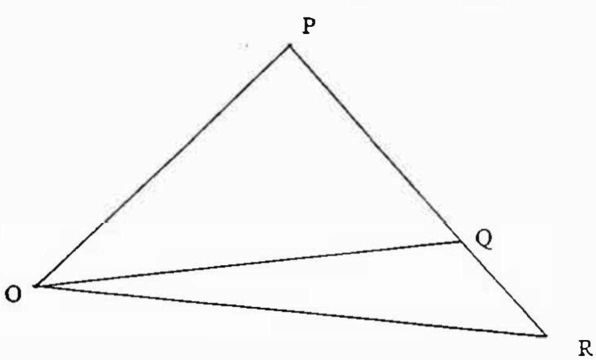


5. This process can be extended inductively, at least in theory, to n-sect BQ, for n any natural number larger than 5. For n prime, the last step in the process is exactly analogous to step 4 above. For n composite, it may be possible to reduce the number of steps in the process. For example, for $n = 6$, once the outer sixths are located, the inner sixths are the two-thirds and the bisector of BQ which have already been located.

Proof

The following is an inductive proof that a diagonal of a rectangle can be n-sected by the method described in the first section. It depends heavily on the rule that ratios of corresponding sides of similar triangles are equal and the following definition and theorem.

- **Definition**: Two vectors $\vec{u}$ and $\vec{v}$ are collinear if $\vec{u} = k\vec{v}$ where k is a nonzero real number.
- **Theorem** (Elliott et al. 1984, 302–03): If P, Q and R are collinear points such that $\vec{PQ} = k\vec{QR}$ and O is any other point, then $\vec{OQ} = \frac{k}{k+1}\vec{OR} + \frac{1}{k+1}\vec{OP}$



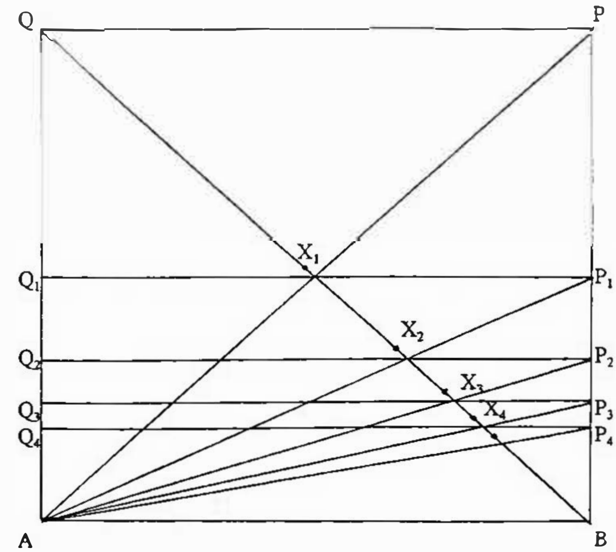
For example, if $\vec{PQ} = 2\vec{QR}$, *then* $\vec{OQ} = 2/3\ \vec{OR} + 1/3 + 1\ \vec{OP}$

The steps in the proof, based on the reference rectangle, are as follows.

1. Construct diagonals AP and BQ of rectangle ABPQ. Let X_1 be the point on diagonal BQ such that $\vec{QX_1} = X_1B$. Then, by the theorem cited above with $k = 1$,

   $\vec{AX_1} = ½\vec{AB} + ½\vec{AQ} = ½\vec{AB} + ½\vec{BP} = ½(\vec{AB}+\vec{BP}) = ½\vec{AP}$.

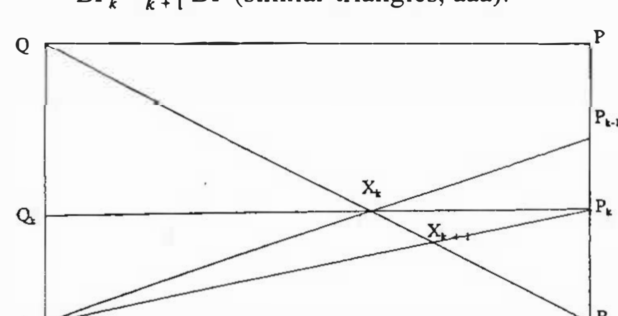
   Thus X_1 is on AP and is the point of bisection of diagonals AP and BQ.
2. Construct Q_1P_1 through X_1 parallel to AB as indicated in the diagram. Then, since $\vec{BX_1} = ½\vec{BQ}$, $\vec{BP_1} = ½\vec{BP}$ (similar triangles, aaa). Construct diagonal AP_1 of rectangle ABP_1Q_1. Let X_2 be the point on BQ such that $\vec{QX_2} = \overline{2X_2B}$. Then, by the theorem with $K = 2$, $\vec{AX_2} = {}^2/_3\vec{AB} + {}^1/_3\vec{AQ} = {}^2/_3\vec{AB} + {}^1/_3\vec{BP} = {}^2/_3(\vec{AB} + {}^1/_2\vec{BP}) = {}^2/_3(\vec{AB} + \vec{BP_1}) = {}^2/_3\vec{AP_1}$. Thus X_2 is on AP_1, that is X_2 is the point of intersection of AP_1 and BQ. Further, X_2 is a third of BQ.



3. Construct Q_2P_2 through X_2 parallel to AB as indicated. Then, since $\vec{BX_2} = {}^1/_3\vec{BQ}$ (this follows from the conclusion in 2) that $\vec{QX_2} = \overrightarrow{2X_2B}$, $\vec{BP_2} = {}^1/_3\vec{BP}$ (similar triangles, aaa). Construct diagonal AP_2 of rectangle ABP_2Q_2. Let X_3 be the point on BQ such that $\vec{Q\ X_3} = 3X_3B$. Then, by the theorem with $K = 3$,

   $\vec{AX_3} = {}^3/_4\vec{AB} + {}^1/_4\vec{AQ} = {}^3/_4\vec{AB} + {}^1/_4\vec{BP}$
   $= {}^3/_4(\vec{AB} + {}^1/_3\vec{BP}) = {}^3/_4(\vec{AB} + \vec{BP_2}) = {}^3/_4\vec{AP_2}$.

   Thus X_3 is on AP_2, is the point of intersection of AP_2 and BQ, and is a fourth of BQ.
4. Continue the above process inductively. Let X_k be the point of intersection of BQ and AP_{k-1}. Construct Q_kP_k through X_k parallel to AB and assume that $\vec{QX_k} = k\vec{X_kB}$. It follows that $\vec{BX_k} = \frac{1}{k+1}\vec{BQ}$ *and so* $\vec{BP_k} = \frac{1}{k+1}\vec{BP}$ (similar triangles, aaa).



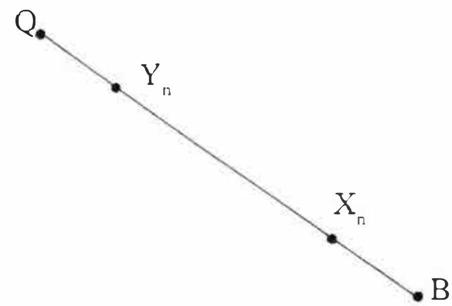
Construct diagonal AP_k of rectangle ABP_kQ_k. Let X_{k+1} be the point on BQ such that
$\vec{QX}_{k+1} = (k + 1)\ X_{k+1}\ \bar{B}$. Then

$$\vec{AX}_{k+1} = \frac{k+1}{k+2}\vec{AB} + \frac{1}{k+2}\vec{AQ} = \frac{k+1}{k+2}(\vec{AB} + \frac{1}{k+1}\vec{AQ})$$
$$= \frac{k+1}{k+2}(\vec{AB} + \frac{1}{k+1}\vec{BP}) = \frac{k+1}{k+2}(\vec{AB} + \vec{BP}_k)$$
$$= \frac{k+1}{k+1}\vec{AP}_k$$

Thus X_{k+1} is on AP_k, is the point of intersection of BQ and AP_k, and is $\frac{1}{k+2}$ of the way from B to Q.

The preceding inductive proof verifies that the methods described in this article can be used to find a point that is an *n*th of the way from B to Q along diagonal BQ of rectangle ABPQ. What remains to be shown is that the methods can be used to partition BQ into *n* equal segments. The following sketch of an inductive argument, supported by the Strong Principle of Mathematical Induction, should convince the reader that this can be done.

Suppose that X_n is an *n*th of the way from B to Q. Then a point Y_n that is an *n*th of the way from Q to B can be found by an analogous process. The completion of the n-secting of BQ can then be accomplished by (n – 2)-secting X_nY_n.



Of course the n-secting of BQ could also be accomplished by the conventional method of using a compass to duplicate segments of length BX_n along BQ, but the inductive procedure described above seems somehow more satisfying.

Bibliography

Elliott, H., et al. *Math 31*. Toronto: Holt, Rhinehart & Winston of Canada, 1984, pp. 269–312.

Nicholson, W. K. *Linear Algebra with Applications*. 3d ed. Boston: PWS Publishing, 1990, pp. 141–55.

The tantalizing and compelling pursuit of mathematical problems offers mental absorption, peace of mind amid endless challenges, repose in activity, battle without conflict, refuge from the goading urgency of contingent happenings and the sort of beauty changeless mountains present to senses tried by the present-day kaleidoscope of events.

Morris Kline

Exemplifying If-Then Principles in Logical Reasoning

David R. Duncan and Bonnie H. Litwiller

Teachers are always seeking real-world settings that exemplify symbolic logic principles. The implication $P \rightarrow Q$ (if P then Q) often causes student concern. We shall present an example that may be used to illustrate the many facets of this implication.

Consider the following two statements concerning a calendar month:

S = the month is September.
T = the month has exactly 30 days.

The implication $S \rightarrow T$ is clearly valid because every month of September has exactly 30 days; in other words, if the month is September, then it has 30 days.

Let us consider the other implications associated with $S \rightarrow T$.

1. *Converse:* $T \rightarrow S$
   This implication asserts that if the month has 30 days, then it must be September. This is clearly not valid reasoning because the month could also be April, June or November. The example supports the fundamental idea that a proposition is *not* equivalent to its converse.
2. *Inverse:* $\sim S \rightarrow \sim T$
   This implication asserts that if the month is not September, then the month cannot have 30 days. This is also not valid for the same reason cited in (1) above. As in (1), a proposition is not equivalent to its inverse.
3. *Contrapositive:* $\sim T \rightarrow \sim S$.
   Because this differs so much from the original statement, students often do not believe that they can be equivalent; however, the contrapositive reasserts the original implication. In this case, if the month does not have 30 days, then it cannot be September. This is clearly valid. A non--30-day month might be any of January, February, March, May, July, August, October or December; it cannot be September.

Another facet of the implication process involves the use of the terms *sufficient* and *necessary*. These terms are used as alternate ways of expressing $P \rightarrow Q$:

1. P is sufficient for Q.

This means that if we are trying to establish the truth of Q, then P provides sufficient information (all that is needed to do so). Although P is sufficient it may provide more information than is needed.

2. Q is necessary for P.

This means that if we are trying to establish the truth for P, then Q is essential in this process. Although Q is needed it might itself not be enough evidence to establish P.

How can the September-30 days example exemplify the sufficient and necessary settings? Recall that $S \rightarrow T$.

1. S is sufficient for T. If you had a calendar page with the last week torn off and you were trying to establish that the missing section ended with 30 days, the presence of the September label at the top of the page would be all the evidence you would need. The evidence is sufficient.
2. T is necessary for S. If you had a calendar page with the month label missing and you were trying to establish that the month was September, the first thing that you would check would be the number of days in the month. For your inquiry to continue there must be exactly 30 days. If not, admit defeat. So 30 days is necessary for September.

Can sufficient and necessary be interchanged in examples 1 and 2 just preceding?

1. Is S necessary for T?
   No, some other months also have 30 days. The absence of this piece of evidence is not critical to your case.
2. Is T sufficient for S?
   Again, the answer is no. Finding that the month has 30 days may be helpful in proving that it is September, but it is not conclusive (sufficient).

Thus, the terms *sufficient* and *necessary* are not interchangeable.

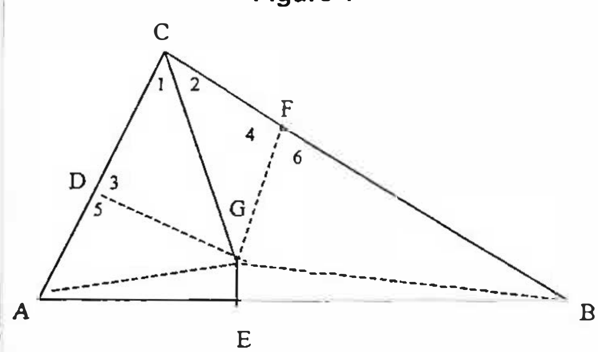
Challenge: Find other settings to illustrate the implication principles of this article.

An Interesting Mathematical Fallacy That All Triangles Are Isosceles

Sandra M. Pulver

This fallacy is a remarkable proof that all triangles are isosceles.

Figure 1



Let ABC be any triangle. Construct the bisector of $\measuredangle$ C and the perpendicular bisector of side AB. From G, their point of intersection, drop perpendiculars GD and GF to AC and BC respectively and draw AG and BG. Now in triangles CGD and CGF, $\measuredangle 1 = \measuredangle 2$ by construction and $\measuredangle 3 = \measuredangle 4$ because all right angles are equal. CG is common to the two triangles. Therefore triangles CGD and CGF are congruent. (If two angles and a side of one triangle are equal respectively to two angles and a side of another, the triangles are congruent.)

Consequently, DG = GF. (Corresponding parts of congruent triangles are equal.) Then in triangles Δ GDA and Δ GFB, $\measuredangle$ 5 and $\measuredangle$ 6 are right angles and, because G lies on the perpendicular bisector of AB, AG = GB. (Any point on the perpendicular bisector of a segment is eqidistant from the ends of the segment.) Therefore triangles Δ GDA Δ and GFB are congruent. (If the hypotenuse and another side of one right angle triangle are equal respectively to that of a second, the triangles are congruent.)

From these two sets of congruent triangles, Δ CGD and CGF and GDA and GFB, we have, respectively, CD = CF and DA = FB. By addition, we conclude that CA = CB, so that Δ ABC is isosceles by definition.

Actually, we do not know that EG and CG meet within the triangle, so we shall examine all other possibilities. The above proof is valid in the cases where G coincides with E (Figure 2) *or* where G is outside the triangle but so near to AB that D and F fall on CA and CB (Figure 3).

Figure 2

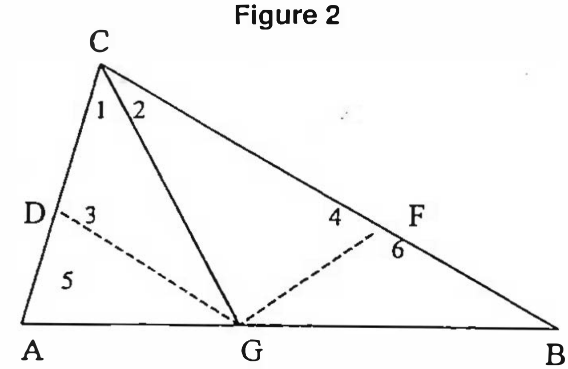
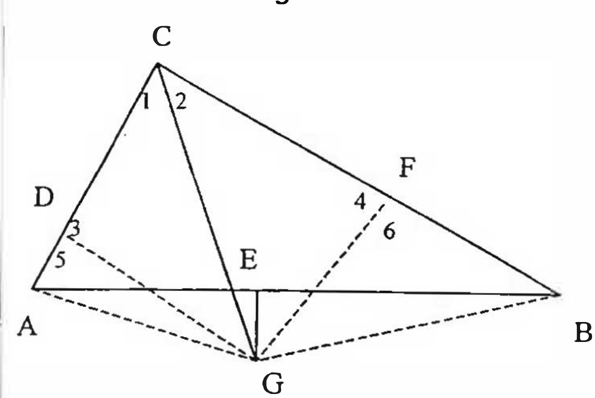
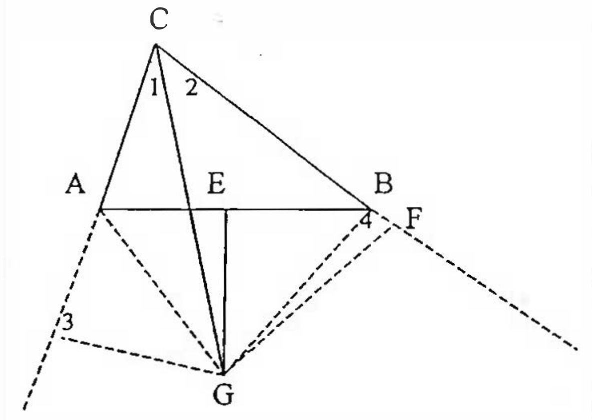


Figure 3



There is the possibility as in Figure 4 that G lies so far outside the triangle that D and F fall on CA and CB produced:

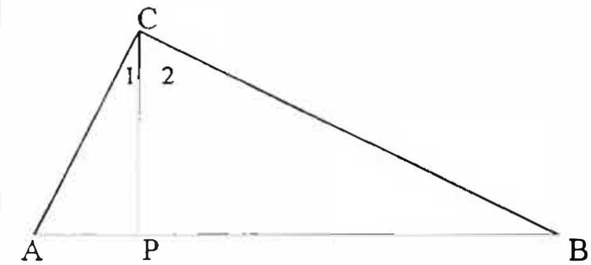
Figure 4



Again, as in the first case, Δ CGD and Δ CGF are congruent, as are Δ GDA and Δ GFB. And again, CD = CF and DA = FB. But in this case, we subtract these last two equations to get CA = CB.

Finally, it may be suggested that CG and EG do not meet in a single point G but either coincide or are parallel. Figure 5 shows that in either of these cases, the bisector CP of angle C will be perpendicular to AB, so that $\measuredangle 7 = \measuredangle 8$. Then $\measuredangle 1 = \measuredangle 2$, CP is common and Δ APC is congruent to Δ BPC. Again, CA = CB.

Figure 5



It appears that we have exhausted all possibilities and that we must accept the obviously absurd conclusion that all triangles are isosceles.

Actually, the possibility we failed to examine is the case where one of the points D and F falls outside the triangle and the other falls inside.

Bibliography

Gardner, M. *Mathematical Puzzles and Diversions.* New York: Simon & Schuster, 1959.

Meyer, J. S. *Fun with Mathematics.* New York: World, 1952.

National Council of Teachers of Mathematics (NCTM). *The Mathematics Student* 24, no. 1 (1976).

Northrup, E. *Riddles in Mathematics.* New York: Van Nostrand, 1957.

Tocquet, R. *The Magic of Numbers.* Trans. by D. Weaver. New York: A. S. Barnes, 1957.

Bees . . . by virtue of a certain geometrical forethought . . . know that the hexagon is greater than the square and the triangle and will hold more honey for the same expenditure of material.

Pappus of Alexandria

Rational Fractions: Combine and Simplify

Ron Persky

This article is motivated by the instructor's task in an algebra course to find test questions on the topic commonly labeled "adding and subtracting rational expressions." An example is $x + 8/x - 3 - x - 14/3 - x$ which reduces to 2. But the denominators are multiples of each other. The objective here is to create this type of problem in which factors are not multiples of each other and which, after combining, have a numerator/denominator common factor simplification. We show that if we restrict ourselves to problems not using polynomials of degree higher than two in any calculations, this objective is not possible without having common factors in the denominators. Then we show how to find problems when one allows common factors.

The expressions we shall examine involve just two terms. In the following list, Form 1 is a starting point. The forms progress in complexity, each one providing an opportunity for numerator/denominator cancellation that was not afforded previously. The variable is x; all other letters denote constants. Assume factors written with different constants, for example, $ax + b$ and $cx + d$, are not multiples of one another (relatively prime).

Form 1. $\frac{u}{ax+b} + \frac{v}{cx+d}$

Form 2. $\frac{ux}{ax+b} + \frac{v}{cx+d}$ Note, $\frac{ux}{ax+b} + \frac{vx}{cx+d}$ would revert this back to form 1 for our purpose.

Form 3. $\frac{ux+e}{ax+b} + \frac{vx+f}{cx+d}$

Form 4. $\frac{u}{ax+b} + \frac{v}{(cx+d)(ex+f)}$ This can be modified by replacing v by vx or by $vx + w$.

Form 5. $\frac{u}{(ax+b)(cx+d)} + \frac{v}{(ex+f)(qx+r)}$

Form 5 is as complex as will be considered. If we change, for example, u to ux, the form 5 numerator combines to a third degree polynomial. This is too much for a test question.

None of the forms above will result in numerator/denominator cancellation when combined providing that the restriction of having factors relatively prime is kept. Showing this is basically the same for all forms, as exemplified for form 4:

$$\frac{u}{ax+b} + \frac{v}{(cx+d)(ex+f)} = \frac{u(cx+d)(ex+f)+v(ax+b)}{(ax+b)(cx+d)(ex+f)}.$$

Suppose we wish to have a numerator/denominator cancellation of the factor $ax + b$. Thinking of the numerator as a polynomial, $p(x)$, it would require that $p(\frac{-b}{a}) = 0$ for $ax + b$ to be a factor of $p(x)$. For that to happen, either $c(\frac{-b}{a}) + d = 0$ or $e(\frac{-b}{a}) + f = 0$. Suppose $c(\frac{-b}{a}) + d = 0$. This is equivalent to $\frac{a}{c} = \frac{b}{d}$ which is equivalent to $ax + b = k(cx + d)$. In other words, for $ax + b$ to be a common factor in the numerator and denominator, it would not be relatively prime to either $cx + d$ or $ex + f$. Similarly for either $cx + d$ or $ex + f$ to be a factor of the combined numerator.

To achieve a numerator/denominator cancellation, we relax the requirement of having all factors relatively prime. We will look at forms 4 and 5 in this light.

Consider form 4 modified to $\frac{u}{ax+b} + \frac{v}{k(ax+b)(ex+f)}$. Write this as $\frac{1}{ax+b}[u + \frac{v}{k(ex+f)}]$. Assign any values we like for u, v, k, e and f and combine what we have inside the bracket. The resulting numerator determines the common factor $ax + b$. For example, $k = 2$, $e = 3$, $f = 1$, $u = 1$ and $v = -4$ results in $ax + b = 3x - 1$. The problem presents itself as $\frac{1}{3x-1} - \frac{4}{(6x-2)(3x+1)}$.

For this same form 4, starting from $\frac{1}{ax+b}[u + \frac{v}{k(ex+f)}]$ $= \frac{1}{ax+b}[\frac{ukex+(ukf+v)}{k(ex+f)}]$, we can predetermine $ax + b$. For example, if we want $a = 6$ and $b = 5$, set $uke = 6$ and $ukf + v = 5$. If we make $k = 2$, $e = 1$ and $f = 1$, then $u = 3$ and $v = -1$. The problem presents itself as $\frac{3}{6x+5} + \frac{1}{(12x+10)(x+1)}$.

Form 5 can be modified to

$$\frac{u}{(ax+b)(cx+d)} + \frac{v}{k(ax+b)(gx+r)} = \frac{1}{ax+b}[\frac{u}{cx+d} + \frac{v}{k(gx+r)}]$$

Assign any values to the constants inside the bracket, combine and the resulting numerator determines the common factor. $ax + b$. Example: $c = 2$, $d = 1$, $k = 1$, $g = 5$, $r = 2$, $u = 3$ and $v = -7$ results in $ax + b = x - 1$.

The problem presents itself as $\frac{3}{(x-1)(2x+1)} - \frac{7}{(x-1)(5x+2)}$.

While changing u to ux in form 5 results in a third degree polynomial, this is not the case when there is a common factor. The form becomes

$$\frac{ux}{(ax+b)(cx+d)} + \frac{v}{k(ax+b)(ex+f)}$$

$$= \frac{1}{ax+b}\left[\frac{ukex^2+(ukf+vc)x+vd}{k(cx+d)(ex+f)}\right].$$

The numerator inside the bracket can be factored as $(uex + v)(kx + d)$ providing

$ukf + vc = ued + vk$

To make this happen, we can begin, for example, by picking, somewhat arbitrarily, $u = 2$, $e = 1$, $v = 3$ and $k = 2$. This gives

$4f + 3c = 2d + 6$

Set $d = -3$ (arbitrary).

$4f + 3c = 0$.

Find a solution: $f = -3$, $c = 4$. If $4f + 3c = 0$ had no solution, we could reassign a value to d.

Now the numerator inside the bracket is $(2x + 3)(2x - 3)$. Assign either factor to $ax + b$. If we use $ax + b = 2x + 3$, the problem presents itself as

$$\frac{2x}{(2x+3)(4x-3)} - \frac{3}{(4x+6)(x-3)}$$

The moving power of mathematical invention is not reasoning but imagination.

Augustus de Morgan

The pleasure we obtain from music comes from counting, but counting unconsciously. Music is nothing but unconscious arithmetic.

Gottfried Wilhelm von Leibniz

Friday the Thirteenth

Klaus Puhlmann

One day Susanne did not show up for work. She had not called in sick nor did she take a holiday. I was quite worried that something had happened. I called her, but there was no answer. After work, I drove to her place. The lights were on in her apartment. I rang the bell, but no one answered the door. Finally, after I repeatedly rang the doorbell, Susanne shouted from within, "Who is it?" I was relieved to hear her voice. Hesitantly, she opened the door.

I asked, "What is the matter? Are you sick?"

She responded with a question. "Don't you know what day it is?"

"Yes," I replied, "it is Friday the 13th!"

Susanne backed up and sat down in the armchair. "By the time I was driving to work I had already had three bad omens today that something would happen," she said. "I decided to turn around, come back to my apartment and go to bed, where I stayed the whole day."

"I didn't know that you are so superstitious," I said.

"Even if you don't believe in it, something bad could happen to you," Susanne responded. "This year is especially bad as there are several Fridays that fall on the 13th." Her remark was a surprise for me, because I thought that such Fridays occur at most once a year. This assumption is obviously false. Here is the question:

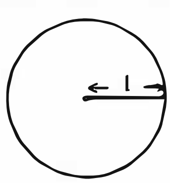
What is the maximum number of Fridays in a year that fall on the 13th day of a month?

Unsolvable Problems: Quadrature of the Circle

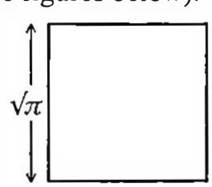
Klaus Puhlmann

The Quadrature of the Circle, often referred to as squaring the circle, involves the construction of a square equal in area to a given circle, using only straight-edge and compass. This is one of three famous problems of antiquity that cannot be solved. (The other two will be dealt with in future issues of *delta-K*.)

Although the Greeks found themselves confronted by these problems which they could not solve, at least by the use of the unmarked ruler and the compass alone, it was not until the late 19[th] century that it was proved to be impossible. Its impossibility arises from the fact that π is a *transcendental* number, and therefore a length equal to $\sqrt{\pi}$, which is also transcendental, cannot be constructed (see figures below).



$A = \pi \times 1^2 = \pi$



$A = \sqrt{\pi} \times \sqrt{\pi} = \pi$

In a circle with radius 1, the edge of the square would be $\sqrt{\pi}$.

Essentially there are three methods of attacking the problem: first, by the use of the ruler and compass only; second, by the use of higher plane curves; third, by such devices as infinite series, leading to close approximations.

Greek mathematicians seem to have found the insolubility of the first method, but they did not prove it. They were successful with the second method, but less skillful with the third.

Many noted mathematicians have attempted to square the circle. Antiphon (c. 430 BCE) tried the quadrature by inscribing a polygon and then doubling the number of sides successively until he approximately exhausted the area between the polygon and the circle.

Hippocrates of Chios (c. 460 BCE) also attempted the solution and was the first to actually square a curvilinear figure. Other noteworthy attempts at squaring the circle were made by Deinostratus (c. 350 BCE) and Archimedes, and they all focused on finding the best approximation of π.

Approximations of π were also attempted by many European mathematicians, including Fibonacci, Tycho Brahe and Vieta. Vieta (c. 1593) gave an interesting approximation of π using continued products for the purpose:

$$\frac{2}{\pi} = \sqrt{\frac{1}{2}} \times \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}} \times \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}}} \cdots$$

John Wallis (1655) gave the form:

$$\frac{4}{\pi} = \frac{3 \times 3 \times 5 \times 5 \times 7 \times 7 \times 9 \times 9 \times 11 \times 11 \cdots}{2 \times 4 \times 4 \times 6 \times 6 \times 8 \times 8 \times 10 \times 10 \times 12 \cdots}$$

Leibniz (1673) used the form:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$

While William Jones (1706), an English writer, was the first to use π to stand for the ratio of circumference to diameter, it was F. Lindemann (1882) who proved the transcendence of π, thus showing the impossibility of squaring the circle by the use of ruler and compass alone.

Bibliography

Rudio, F. "Der Bericht des Simplicius über die Quadraturen des Antiphon and des Hippokrates." *Bibl. Math.* III 3, no. 7. Leipzig: 1907.

Smith, D. E. *History of Mathematics*. Vol. 2. New York: Dover, 1953.

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