Cost Allocation: An Application of Fair Division

Albert Goetz

Although the subject of cost allocation has been extensively discussed in the literature of political economics, it has been generally neglected in mathematical literature. However, cost allocation affords a practical extension of fair-division techniques—one that is readily accessible to secondary school students and that gives them a simple yet powerful application of mathematics to real-world problem solving. A study of the concepts and the mathematics involved in cost allocation is most appropriate in a discrete mathematics course or a modeling course, but a case can be made for including this topic in other courses, as well. This article presents a typical cost-allocation problem with possible solutions and includes suggestions for presenting similar problems in the classroom. The basics of the problem follow closely from Young (1994).

The Sewage-Treatment-Plant Problem, Part 1

> Let us consider two towns, Amity and Bender, each of which needs to build a new sewage-treatment plant. Let us further suppose that the cost for Amity to build the sewage-treatment plant is \$15 million and that the cost for Bender to construct the plant is \$9 million. Were the two towns to pool their resources, the cost of one sewage-treatment plant, built to service both towns, would be \$19 million. Should the two towns decide to build only one plant, and if so, how should the cost be divided?

I find that having small groups work on this problem is both productive and enjoyable for students. Each group is first given one of the two towns to represent and asked to plan a negotiating strategy for the town. Each group is then paired with a group that represents the other town so that the groups can work out a solution.

One question that students frequently ask concerns the populations of the towns. I deliberately withhold this information initially, and I instruct students to devise possible solutions without knowing the populations.

Students should recognize that splitting the cost equally is an inferior solution for Bender. Students should devise two preferable kinds of solutions, either on the basis of the cost or on the basis of the savings involved. Splitting the savings equally between the two towns is an example of the latter. Since \$5 million, that is, \$24 million minus \$19 million, represents the amount saved, each town should save \$2.5 million, so that the \$19 million cost would be divided in the ratio of 12.5 to 6.5, that is, (\$15 – \$2.5) to (\$9 – \$2.5).

A possible solution on the basis of cost is to allocate costs in proportion to opportunity, that is, stand-alone, costs. In this solution,

$$\frac{9}{24} = \frac{3}{8}$$

of the cost, or \$7.125 million, should be borne by Bender; and

$$\frac{15}{24} = \frac{5}{8},$$

or \$11.875 million, by Amity. The same solution can be obtained by allocating savings in proportion to opportunity costs, so that the cost for Bender, for example, would be

$$9 - \frac{9}{24} \cdot 5,$$

or \$7.125 million. See Table 1; where necessary, numbers in tables are rounded to three decimal places.

Table 1

Payments by Town on the Basis of Costs or Savings

	Amity Share (Millions of \$)	Bender Share (Millions of \$)
Stand-alone costs	15	9
Split costs	9.5	9.5
Split savings	12.5	6.5

Students often rebel against first finding solutions without knowing the populations of the towns, and their concern is worthy of classroom discussion. But if the populations are cleverly constructed, the problem becomes more complex rather than easier. For example, if the population of Bender is 10,000 and the population of Amity is 40,000 and costs are allocated in proportion to population, then Amity should pay four-fifths of the cost, or $15.2 million. Such a solution is clearly not in Amity's best interest, just as splitting the cost equally is not in Bender's best interest. A question to ask students is, Under what circumstances does the ratio of the populations of the towns produce a solution that encourages each town to participate? However, if the savings are divided equally among the residents, then Amity pays $11 million, that is,

$$\left(15-\frac{4}{5}\bullet 5\right),$$

and Bender pays $8 million.

Three solutions appear to be in the best interests of both towns, as indicated in Table 2:

- Dividing the savings equally—A (Amity) pays $12.5 million, B (Bender) pays $6.5 million
- Dividing the savings equally among the residents—A pays $11 million and B pays $8 million on the basis of the given populations
- Dividing the costs or the savings proportionally to opportunity costs or savings—A pays $11.875 million, and B pays $7.125 million

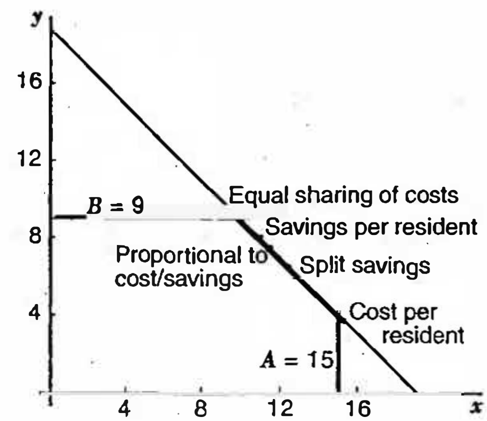
Which of the three solutions is the fairest? Young (1991) takes an interesting geometric approach to this question. *Core* is the term that game theorists and political economists give to the set of possible solutions in which neither player, or town, pays more than the opportunity costs. In Figure 1, the x-axis represents Amity's payments; the y-axis, Bender's payments.

Table 2
Three Solutions in the Best Interests of Both Towns

	Amity Share (Millions of $)	Bender Share (Millions of $)
Dividing savings equally	12.5	6.5
Dividing savings equally among residents	11	8
Dividing costs or savings in proportion to opportunity costs	11.875	7.125

The line segment joining the points (0, 19) and (19, 0) is the set of all possible allocations; the portion of that line segment between the horizontal at 9 and the vertical at 15 represents the core. Students can easily replicate this figure on a graphing calculator in a window that goes from 0 to 20 in each direction. The equation of the line segment in question is $y = -x + 19$, and the DRAW menu can be accessed from the home screen, as opposed to the graph, to obtain the desired horizontal and vertical segments. The previously discussed solutions, both those in the core and those outside it, are labeled in the figure.

Figure 1
A Diagram of Possible Solutions in a Two-Town Game



A good case can be made for choosing the midpoint of the line segment representing the core as the solution to the problem. That point corresponds to equal savings for each town. In that solution, A pays $12.5 million and B pays $6.5 million. When students try to negotiate an equitable settlement in their groups, this solution is often the most appealing.

The Sewage-Treatment-Plant Problem, Part 2

We next suppose that a third town, Cordial, is involved. The stand-alone cost for Cordial is $7 million, and the cost for a sewage-treatment plant that would service all three towns is $23 million.

Before students can break up into groups to decide how to solve this problem, costs for all possible coalitions must be assigned. One possible way follows:

- The cost for Amity and Bender together remains as before, $19 million.

- Were Amity and Cordial to participate together, the cost would be $17 million.
- Were Bender and Cordial to participate together, their cost would be $13 million.

If we use the method of proportional allocation, which gave us a solution in the core in the two-town game, then Amity contributes $15.862 million, a solution that is not in the core. Moreover, dividing savings equally among residents fails to fall within the core because Bender and Cordial can form a coalition that leaves Amity out and build the plant for roughly $2.5 million less than by joining with Amity and using that method. Table 3 summarizes results from the other methods used in the two-town game. For these results, we assume that the population of Cordial is 8,000 and that the populations of the other towns are as stated initially. Students can investigate which of these methods fall within the core and which are outside it.

In the classroom, letting students play with the problem before analyzing it in this fashion is advisable; fascinating student interactions can result. If the class is divided into three groups, each representing one of the towns, students can caucus among themselves to determine a "strategy," or method that is equitable from their point of view, to divide costs. Pairs of students from each group are then randomly assigned to negotiate a settlement; in other words, two students from A (Amity), two from B (Bender) and two from C (Cordial) work as one group; another two from A, two from B and two from C work in a second group; and so on.

Young (1991) presents a geometric analog to the line segment that denoted the core in the two-town game. We construct an equilateral triangle with its altitude numerically equal to the cost if all three towns cooperate. Each vertex of the triangle represents one town's payment of the full cost, and any point in the interior of the triangle represents the towns' splitting the $23 million in some fashion. The core in this game is the shaded area in Figure 2.

A Combinatoric Approach

L. S. Shapley, a political economist at Princeton, developed a cost-allocation method (Shapley 1981) that is similar to his approach to power indices in voting games. We consider all possible permutations of the three towns. Each permutation is treated as if the towns join the coalition sequentially and make up the difference between what has already been contributed and the total cost for the coalition. For example, in the permutation ABC, A joins first and must contribute 15. When it joins the coalition, B must contribute 4, the difference between A's 15 and the cost for AB, which is 19. When C joins, C must also contribute 4, the difference between 23 and 19. The *Shapley value* is the average of all possible contributions for a town. The values for the problem are summarized in Table 4.

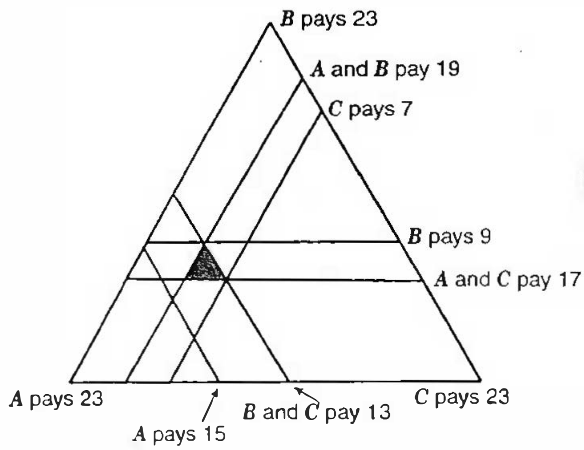
Table 3

Payments by Town for the Three-Town Game

	Payments by Town		
	Amity	Bender	Cordial
Stand-alone costs	15	9	7
Split costs	7.67	7.67	7.67
Split savings	12.33	6.33	4.33
Cost divided in proportion to stand-alone costs	11.129	6.677	5.194
Costs divided among residents	15.862	3.966	3.172
Savings divided among residents	9.483	7.621	5.897

Figure 2

A Geometric Diagram of Possible Solutions in a Three-Town Game



The Geometric Solution for Three Players

The Shapley solution obtained previously is within the core and is thus a valid solution to the problem, but we have no guarantee that the Shapley value will be in the core (Young 1991). Can we guarantee a solution that is in the core of a three-player game if a core exists? We can easily construct a situation in which the core does not exist. We consider the core in Figure 2. We try to extend the midpoint solution of the two-player game, called the *standard solution,* to three players. The core here is a triangle, although we have no guarantee that the core will be a triangle. To visualize this result, we move the line designated "A and B pay 19" parallel to itself and away from vertex C. As that line moves, the core changes from a triangle to a quadrilateral to a pentagon. The upper vertex of the core triangle represents B's paying a share of 9. This amount is B's maximum payment within the core. B's minimum payment is represented by the line designating "A and C pay 17," or 6. We average those payments at 7.5 and construct through that point the horizontal segment with endpoints on the borders of the core. See Figure 3. The left endpoint of the segment represents C's minimum cost, and therefore A's maximum cost, given that B will pay 7.5. The right endpoint represents A's minimum cost and C's maximum cost. If we simply average the maximum and minimum costs for A and C, we obtain the solution that A pays 10.75, B pays 7.5 and C pays 4.75.

A spreadsheet that neatly summarizes all these solutions in the three-town game can be constructed. Such a spreadsheet appears as Table 5. Entries in the top half of the spreadsheet represent the costs to each town or coalition of towns for each possible solution. Entries in the bottom half of the spreadsheet represent the savings for each coalition. Any negative entry in the bottom half of the table indicates that the solution does not fall within the core of the game.

Table 4

Allocation Using a Combinatoric Approach

Coalition order	A	B	C
	Individual Contributions		
ABC	15	4	4
ACB	15	6	2
BAC	10	9	4
BCA	10	9	4
CAB	10	6	7
CBA	10	6	7
Total contribution	70	40	28
Shapley value	11.67	6.67	4.67

Figure 3

The Core Triangle from Figure 2

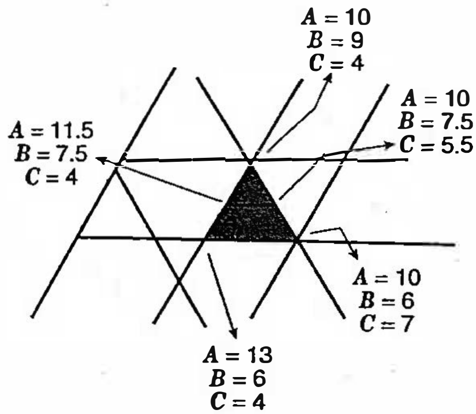


Table 5

Summary of All Solutions in the Three-Town Game

	Amity	Bender	Cordial
Costs			
Stand-alone costs	15	9	7
Split-cost solution	7.67	7.67	7.67
Split-savings solution	12.33	6.33	4.33
Costs prop. to oppty.	11.129	6.677	5.194
Prorated costs	15.862	3.966	3.172
Prorated savings	9.483	7.621	5.897
Geometric solution	10.75	7.5	4.75
Shapley solution	11.67	6.67	4.67
Savings			
Split cost	7.33	1.33	-0.67
Split savings	2.67	2.67	2.67
Costs prop. to oppty.	3.871	2.323	1.806
Prorated costs	–0.862	5.034	3.828
Prorated savings	5.517	1.379	1.103
Geometric	4.25	1.5	2.25
Shapley	3.33	2.33	3.33

Students usually need help in arriving at either the geometric solution or the Shapley value. They do have quite a bit to say about these and the other solutions that they may generate on their own, and talking through the solutions in class has always been interesting and provocative.

Problems of cost allocation are inherently interesting to students and are rich in mathematical applications. Those that come to mind most readily include graphing straight lines, geometric constructions, parallelism, combinatorics and proportions. The aspect that makes cost-allocation problems so valuable in the classroom, however, is that students are motivated to talk about mathematics with one another and to experience a real-life application of the mathematics that they know.

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A rectangle is 2 m longer in length than in width. If we add 4 m to the length and width, the area of the rectangle increases by 72 m^2. Find the length of the sides of the original rectangle.