Connecting Probability and Geometric Progressions

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Geometric progressions is an important concept used in many mathematical applications. Probability is a source of significant examples of this type of progression.

Consider the following situation: Greg and Joel, beginning with Greg, alternately roll a fair hexahedral die. The first one to roll a 6 wins. What is the probability that Greg wins or that Joel wins?

First consider Greg. He will win if one of these events is satisfied:

- Event 1: Greg rolls a 6 on the first try.
- Event 2: Greg and Joel both fail to roll 6s on the first try. Greg rolls a 6 on his second try.
- Event 3 Greg and Joel both fail to roll 6s on the first two tries. Greg rolls a 6 on his third try.
- Event $(n + 1)$: Greg and Joel both fail to get 6s on the first n tries. Greg rolls a 6 on his $(n + 1)$ try.

The probabilities of these distinct events are

- Event 1: 1/ 6
- Event 2: $\frac{5}{6} \cdot \frac{5}{6} \cdot \frac{1}{6} = \left(\frac{5}{6}\right)^2 \cdot \frac{1}{6}$
- Event 3. $\left(\frac{5}{6} \cdot \frac{5}{6}\right) \cdot \left(\frac{5}{6} \cdot \frac{5}{6}\right) \cdot \frac{1}{6} = \left(\frac{5}{6}\right)^4 \cdot \frac{1}{6}$
- Event $(n + 1)$: $\left(\frac{5}{6}\right)^{2n} \cdot \frac{1}{6}$

Because these events are mutually exclusive, the probability that Greg wins is the sum of the separate event probabilities.

$$P(G) = \frac{1}{6} + \frac{25}{36} + \frac{1}{6} + \left(\frac{25}{36}\right)^2 \cdot \frac{1}{6} + \ldots + \left(\frac{25}{36}\right)^n \cdot \frac{1}{6} + \ldots$$

$$= \frac{1}{6}\left(1 + \frac{25}{36} + \left(\frac{25}{36}\right)^2 + \ldots + \left(\frac{25}{36}\right)^n + \ldots\right)$$

$$= \frac{1}{6}\left(\frac{1}{1 - \frac{25}{36}}\right)$$

$$= \frac{1}{6}\left(\frac{1}{\frac{11}{36}}\right)$$

$$= \frac{1}{6} \cdot \frac{36}{11}$$

$$= \frac{6}{11} \approx 0.5\ 6$$

Recall that if

$S = 1 + r + r^2 + r^3 +$

then

$rS = r + r^2 + r^3 + r^4 +$

Consequently,

$S - Sr = 1$

$S(1 - r) = 1$

$S = \frac{1}{1 - r}.$

We will next compute the probability of Joel's winning in two ways.

Method 1

- Event 1: Greg fails on his first roll, and Joel rolls a 6 on his first try.
- Event 2: Greg fails twice and Joel fails once to roll a 6. Joel then rolls a 6 on his second try.
- Event 3 Greg fails three times, and Joel fails twice to roll a 6. Joel then rolls a 6 on his third try.

$$P(J) = \left(\frac{5}{6}\right) \cdot \frac{1}{6} + \left(\frac{5}{6}\right)^3 \cdot \frac{1}{6} + \left(\frac{5}{6}\right)^5 \cdot \frac{1}{6} + \ldots$$

$$= \frac{5}{6} \cdot \frac{1}{6}\left(1 + \left(\frac{5}{6}\right)^2 + \left(\frac{5}{6}\right)^4 + \cdots\right)$$

$$= \frac{5}{36}\left(1 + \frac{25}{36} + \left(\frac{25}{36}\right)^2 + \cdots\right)$$

$$= \frac{5}{36}\left(1 - \frac{1}{1 - \frac{25}{36}}\right)$$

$$= \frac{5}{36} \cdot \frac{36}{11}$$

$$= \frac{5}{11} \approx 0.44$$

Method 2

Because only two disjoint outcomes are ultimately possible, their probabilities must have a sum of 1. Since P(G) = 6/11, P(J) must be 5/11.

Now, remove the requirement that Greg and Joel must both perform the same activity. Let Greg roll a die, hoping for a 6, and let Joel flip a fair coin, hoping for a head. Find the probability of Greg winning or of Joel winning.

$$P(G) = \frac{1}{6} + \left(\frac{1}{6} \cdot \frac{1}{2}\right)\frac{1}{6} + \left(\frac{1}{6} \cdot \frac{1}{2}\right)^2 \cdot \frac{1}{6} + \ldots$$
$$= \frac{1}{6} + \left(\frac{1}{12}\right) \cdot \frac{1}{6} + \left(\frac{1}{12}\right)^2 \cdot \frac{1}{6} + \ldots$$
$$= \frac{1}{6}\left(\frac{1}{1 - \frac{1}{12}}\right)$$
$$= \frac{1}{6} \cdot \frac{12}{11}$$
$$= \frac{2}{11}$$

Note that Greg's probability is less than 1/2 even though he goes first. This is because his winning outcome is much less likely than Joel's.

P(J) = 1 − 2/11 = 9/11

As a further refinement of this situation, suppose that Greg's winning outcome has probability p_1, while Joel's winning outcome has probability p_2. The probabilities that they do not win in any given try are, respectively, q_1 and q_2 where $q_1 = 1 - p_1$ and $q_2 = 1 - p_2$. Then,

$$P(G) = p_1 + (q_1 q_2) p_1 + (q_1 q_2)^2 p_1 + \ldots$$
$$= p_1 [1 + (q_1 q_2) + (q_1 q_2)^2 + \ldots]$$
$$= p_1 \left(\frac{1}{1 - q_1 q_2}\right)$$
$$= \frac{p_1}{1 - q_1 q_2}$$
$$P(J) = 1 - \frac{p_1}{1 - q_1 q_2}$$
$$= \frac{1 - q_1 q_2 - p_1}{1 - q_1 q_2}.$$

What must be the relationship between p_1 and p_2 to make this overall sequence a fair game? The fair game condition would require that

$$\frac{p_1}{1 - q_1 q_2} = \frac{1 - q_1 q_2 - p_1}{1 - q_1 q_2}$$
$$p_1 = 1 - q_1 q_2 - p_1$$
$$2p_1 = 1 - (1 - p_1)(1 - p_2)$$
$$2p_1 = 1 - [1 - p_1 - p_2 + p_1 p_2]$$
$$2p_1 = p_1 + p_2 - p_1 p_2$$
$$p_1 - p_2 + p_1 p_2 = 0$$

If

$$p_1 = \frac{1}{6},$$

then

$$\frac{1}{6} - p_2 + \left(\frac{1}{6}\right) p_2 = 0; \; -\frac{5}{6} p_2 = -\frac{1}{6}; \; p_2 = \frac{1}{5}.$$

Joel's event must be equivalent to drawing a specific card from a deck of five cards.

To extend this situation one more step, let Nadine enter the game as the third player in order. On any given trial, Greg, Joel and Nadine have winning probabilities of, respectively, p_1, p_2 and p_3, whereas q_1, q_2 and q_3 are their respective probabilities of failing on any given trial. In this expanded setting,

$$P(G) = p_1 + (q_1 q_2 q_3) p_1 + (q_1 q_2 q_3)^2 p_1 + \ldots$$
$$= p_1(1 + q_1 q_2 q_3 + (q_1 q_2 q_3)^2 + \ldots)$$
$$= p_1 \left(\frac{1}{1 - q_1 q_2 q_3}\right)$$
$$= \frac{p_1}{1 - q_1 q_2 q_3}.$$
$$P(J) = q_1 p_2 + (q_1 q_2 q_3) q_1 p_2 + (q_1 q_2 q_3)^2 q_1 p_2 + \ldots$$
$$= q_1 p_2 (1 + q_1 q_2 q_3 + (q_1 q_2 q_3)^2 + \ldots)$$
$$= q_1 p_2 \left(\frac{1}{1 - q_1 q_2 q_3}\right)$$
$$= \frac{q_1 p_2}{1 - q_1 q_2 q_3}.$$
$$P(N) = q_1 q_2 p_3 + (q_1 q_2 q_3) q_1 q_2 p_3 + (q_1 q_2 q_3)^2 q_1 q_2 p_3 + \ldots$$
$$= q_1 q_2 p_3 (1 + q_1 q_2 q_3 + (q_1 q_2 q_3)^2 + \ldots)$$
$$= q_1 q_2 p_3 \left(\frac{1}{1 - q_1 q_2 q_3}\right)$$
$$= \frac{q_1 q_2 p_3}{1 - q_1 q_2 q_3}.$$

Because exactly 1 person must ultimately win, the relationship P(G) + P(J) + P(N) = 1 must hold. To verify this algebraically,

P(G) + P(J) + P(N) =

$$\frac{p_1 + q_1 p_2 + q_1 q_2 p_3}{1 - q_1 q_2 q_3} = \frac{(1 - q_1) + q_1 (1 - q_2) + q_1 q_2 (1 - q_3)}{1 - q_1 q_2 q_3} =$$
$$\frac{1 - q_1 + q_1 - q_1 q_2 + q_1 q_2 - q_1 q_2 q_3}{1 - q_1 q_2 q_3} = \frac{1 - q_1 q_2 q_3}{1 - q_1 q_2 q_3} = 1.$$

Suppose, for instance, that Greg rolls a die (hoping for a 6), Joel flips a coin (hoping for a head) and Nadine draws a card from a standard deck of 52 cards (hoping for a heart). Then

$$p_1 = \frac{1}{6}; \; p_2 = \frac{1}{2}; \; p_3 = \frac{13}{52} = \frac{1}{4}.$$
$$P(G) = \frac{\frac{1}{6}}{1 - \frac{5}{6} \cdot \frac{1}{2} \cdot \frac{3}{4}} = \frac{\frac{1}{6}}{1 - \frac{15}{48}} = \frac{\frac{1}{6}}{\frac{33}{48}} = \frac{1}{6} \cdot \frac{48}{33} = \frac{8}{33}.$$
$$P(J) = \frac{\frac{5}{6} \cdot \frac{1}{2}}{\frac{33}{48}} = \frac{5}{12} \cdot \frac{48}{33} = \frac{20}{33}.$$
$$P(N) = \frac{\frac{5}{6} \cdot \frac{1}{2} \cdot \frac{1}{4}}{\frac{33}{48}} = \frac{5}{48} \cdot \frac{48}{33} = \frac{5}{33}.$$

Challenges

1. Compute these types of probabilities using other situations.
2. Generalize this problem to n players.
3. Find other situations in which geometric progressions can be productively employed.
4. For three players, what must be the relationship between p_1, p_2 and p_3 to produce a fair game?