Pigeonholes and Mathematics

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The Pigeonhole Principle, also known as the Dirichlet Principle, states the following: if $xy + 1$ pigeons are divided evenly into y holes with x pigeons in each hole, then at least one hole must hold $x + 1$ pigeons. For example, if 201 pigeons ($xy + 1$) are divided evenly into 100 pigeon holes (y), then there are 2 pigeons (x) in each hole, except one in which there must be 3 pigeons ($x + 1$). (Even if the pigeons are not divided evenly into the pigeonholes, the principle still holds true.)

Suppose there are n holes with at most x pigeons in each hole. Then, the total number of pigeons would be at most xn. However, if there are $xn + 1$ pigeons, then at least one hole must hold more than x pigeons.

Let us see how we can use the Pigeonhole Principle to solve other problems of the same nature.

Problem 1

A sack holds black marbles and white marbles, identical in all ways but color. A marble is removed without looking into the sack. How many times must this be done to be sure that two marbles of the same color will be removed?

Solution 1

In this case, the pigeons are the marbles drawn, x, and the pigeon holes are the colors, y, of the marbles. Therefore, the question becomes how many pigeons ($xy + 1$) must there be to ensure that two pigeons ($x + 1$) of the same y color end up in one of the holes? The answer is 3.

Problem 2

Given any 12 integers, show that 2 of them can be chosen whose difference is a multiple of 11.

Solution 2

We have only 11 slots or pigeonholes. In a mod 11 system, all integers fit the system in this manner:

0	1	2	3	4	5	6	7	8	9	10
0	1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20	21
22	23	24	25	26	27	28	29	30	31	32
33	34	35	36	37	38	39	40	41	42	43
.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.

The top line of the chart shows the value of the remainders if the number 11 is divided into each integer. The difference of any 2 integers in a vertical column is a multiple of 11. Let the 11 vertical columns be the pigeonholes (y) in which all integers belong. Because there are $(xy + 1) = 12$ integers, or pigeons, 2 of them ($x + 1$) must belong to the same column. Hence, there must be 2 numbers whose difference is a multiple of 11.

Problem 3

Given 8 different natural numbers, none greater than 15, show that if we take the (positive) difference between pairs of these numbers, at least 3 of these differences will be equal.

Solution 3

From the 8 natural numbers selected, there are $28 = {}_8C_2$ differences one can produce. The premise that these 8 natural numbers can be no greater than 15 determines that there are only 14 possible differences (1 through 14). These 14 possible differences serve as the pigeonholes. If the 28 differences produced by the 8 selected numbers are placed into these 14 pigeonholes, at least 3 of the 28 must fall into one of the holes because, in this case, there are pigeonholes that have special properties. For example, out of the 8 numbers selected, because the greatest number is 15, the difference, 14, can be produced in only one way (that is, $15 - 1 = 14$). This limits the pigeons that can belong to that pigeonhole to 1. Then there are 27 differences to be divided up among 13 pigeonholes, forcing one of the pigeonholes to carry at least 3 pigeons. Note that the difference 13 can be produced in only two ways (that is, $15 - 2 = 13$ and $14 - 1 = 13$), thus further limiting the pigeons that can belong to hole 13 to 2.

Application of the Pigeonhole Principle is not limited to any one field of mathematics. Regardless of the field—be it arithmetic, combinatorics or geometry—the difficulty in applying this principle lies only in determining which are the pigeons and which are the pigeonholes. Once these can be identified, the solution is arrived at through elementary reasoning.

Bibliography

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A jeweler had three clocks in his shop for repairs. In the test he always performed after repairing a clock, he found that one clock was now keeping perfect time, one was gaining 1 minute every 24 hours and one was losing 1 minute every 24 hours. At 9:00 on March 19, 1998, he set all three clocks to show the correct time. When would the three clocks again show the correct time if they were kept running at the rates they had shown in the test?

Find a positive integer t such that $s^2 - t^2$ is a prime number and $s = 14$.