delta-k

JOURNAL OF THE MATHEMATICS COUNCIL OF THE ALBERTA TEACHERS' ASSOCIATION

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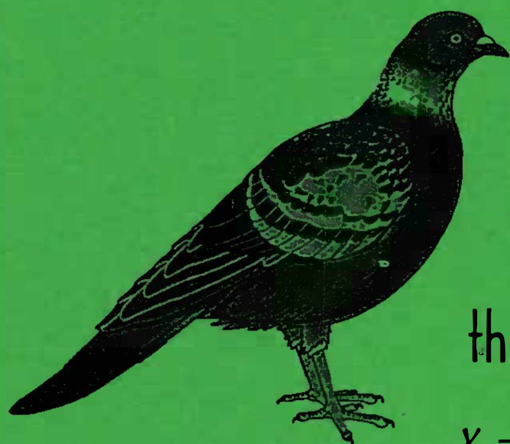
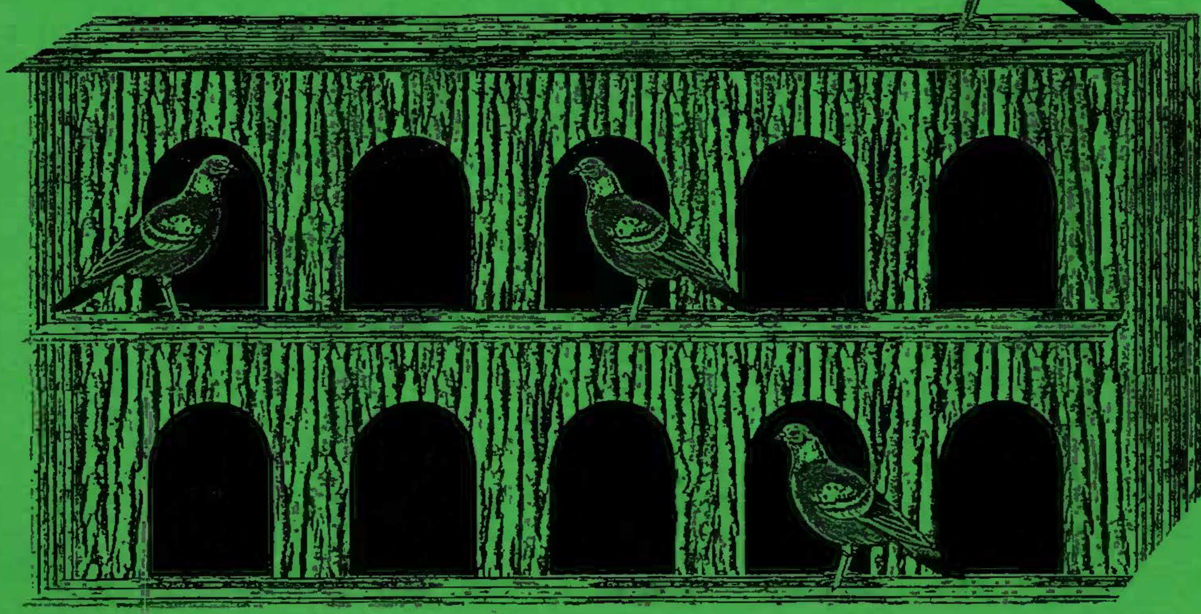
The Pigeonhole Principle . . .



If $xy + 1$ pigeons are divided evenly into y holes with x pigeons in each hole, then at least one hole must hold $x + 1$ pigeons.



GUIDELINES FOR MANUSCRIPTS

delta-K is a professional journal for mathematics teachers in Alberta. It is published to

- promote the professional development of mathematics educators, and
- stimulate thinking, explore new ideas and offer various viewpoints.

Submissions are requested that have a classroom as well as a scholarly focus. They may include

- personal explorations of significant classroom experiences;
- descriptions of innovative classroom and school practices;
- reviews or evaluations of instructional and curricular methods, programs or materials;
- discussions of trends, issues or policies;
- a specific focus on technology in the classroom; and
- a focus on the curriculum, professional and assessment standards of the NCTM.

Manuscript Guidelines

1. All manuscripts should be typewritten, double-spaced and properly referenced.
2. Preference will be given to manuscripts submitted on 3.5-inch disks using WordPerfect 5.1 or 6.0 or a generic ASCII file. Microsoft Word and AmiPro are also acceptable formats.
3. Pictures or illustrations should be clearly labeled and placed where you want them to appear in the article. A caption and photo credit should accompany each photograph.
4. If any student sample work is included, please provide a release letter from the student's parent allowing publication in the journal.
5. Limit your manuscripts to no more than eight pages double-spaced.
6. A 250–350-word abstract should accompany your manuscript for inclusion on the Mathematics Council's website.
7. Letters to the editor or reviews of curriculum materials are welcome.
8. *delta-K* is not refereed. Contributions are reviewed by the editor(s), who reserve the right to edit for clarity and space. **The editor shall have the final decision to publish any article.** Send manuscripts to Klaus Puhlmann, Editor, PO Box 6482, Edson, Alberta T7E 1T9; fax 723-2414, e-mail klaupuhl@gyrd.ab.ca.

Submission Deadlines

delta-K is published twice a year. Submissions must be received by August 31 for the fall issue and December 15 for the spring issue.

MCATA Mission Statement

Providing leadership to encourage the continuing enhancement of teaching, learning and understanding mathematics.

delta-K

Volume 38, Number 2 May 2001

CONTENTS

 delta-K is published by the ATA for the Mathematics Council (MCATA). EDITOR: Klaus Puhlmann, PO Box 6482, Edson, Alberta T7E 1T9. EDITORIAL AND PRODUCTION SERVICES: Central Word Services staff, ATA. Opinions expressed herein are not necessarily those of MCATA or of the ATA. Address correspondence regarding this publication to the editor. *delta-K* is indexed in the Canadian Education Index. ISSN 0319-8367

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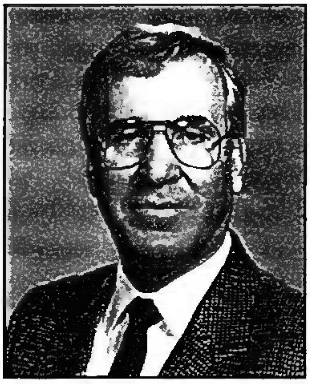
EDITORIAL



Curriculum development for mathematics and, subsequently, the teaching, learning and assessment of school mathematics have been guided over the past decade in Alberta by three important National Council of Teachers of Mathematics (NCTM) publications:

- *Curriculum and Evaluation Standards for School Mathematics* (NCTM 1989)
- *Professional Standards for Teaching Mathematics* (NCTM 1991)
- *Assessment Standards for School Mathematics* (NCTM 1995)

The first document, *Curriculum and Evaluation Standards for School Mathematics* (NCTM 1989), was an attempt by a professional organization to articulate clearly defined standards for teachers and curriculum developers. This document revolutionized how mathematics curricula were to be written. The standards documents outline clearly stated goals and content areas for the grade bands, but they were also designed to challenge the assumption that mathematics is only for the select few. These documents articulate the view that all students should gain mathematical power—that is, the ability to explore, conjecture and reason logically, as well as the ability to effectively use a variety of mathematical methods to solve nonroutine problems.

The standards documents advocate the use of manipulatives, but they do not suggest that manipulatives are the only tools used to teach concepts. They suggest that students should have adequate procedural facility and encourage calculator use but not at the expense of mental math skills or number sense.

The original standards documents intended to provide guidance and a vision of the teaching and learning of mathematics. However, with such an enormously wide range of classroom practices and interpretations in place—all claiming to follow the standards—the documents often received fairly negative comments from the public sector, which was determined to identify the culprit responsible for our poor performances in mathematics.

In April 2000, with the benefit of 10 years' hindsight and the knowledge that the original standards were not entirely understood by everyone, the NCTM released an updated document entitled *Principles and Standards for School Mathematics* (*PSSM*). *PSSM* outlines six principles and ten standards for school mathematics and organizes the standards across four grade-level bands—Grades K–2, Grades 3–5, Grades 6–8 and Grades 9–12.

The six principles for school mathematics are as follows:

1. *The Equity Principle*—Effective mathematics education requires equity—high expectations and strong support for all students.
2. *The Curriculum Principle*—A curriculum is more than a collection of activities: it must be coherent, focused on important mathematics and well articulated across the grades.
3. *The Teaching Principle*—Effective mathematics teaching requires understanding what students know and need to learn and then challenging and supporting them to learn it well.
4. *The Learning Principle*—Students must learn with understanding, actively building new knowledge from experience and prior knowledge.
5. *The Assessment Principle*—Assessment should support the learning of important mathematics and furnish useful information to both teachers and students.
6. *The Technology Principle*—Technology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances students' learning.

The 10 standards for school mathematics are as follows:

1. Number and Operation
2. Algebra
3. Geometry
4. Measurement
5. Data Analysis and Probability
6. Problem Solving
7. Reasoning and Proof
8. Communication
9. Connections
10. Representation

PSSM discusses each standard in some detail and suggests how it can be implemented across the four grade-level bands.

As teachers contemplate how to put these principles and standards into practice, they may appreciate knowing that various support structures and resources are available. For example, the electronic edition of *PSSM* (with additional examples and resources) is available at Illuminations (illuminations.nctm.org). This website provides opportunities to see how NCTM's principles and standards can work in the classroom. It is also an excellent tool for developing new teaching strategies and enhancing professional growth. Once you launch Illuminations, you also have access to the hallmark of the site, i-Math Investigations, which are Internet-based activities for all grade bands. Other features of the site include teacher-oriented Reflections on Teaching, professional development activities for teachers based on online video vignettes; Selected Web Resources, which includes links to sites that have been reviewed by an expert panel; and Internet-Based Lesson Plans, which shows how Internet links can be used to create effective standard-based mathematics lessons. Other features are constantly being added.

If the principles and standards articulated in *PSSM* are to be implemented, mathematics teachers must first become familiar with them. We also need to remember that the principles and standards are rooted in research, meaning that they reflect the best and most comprehensive view of what effective teaching and learning of mathematics are all about.

I encourage mathematics teachers to participate in professional organizations (for example, MCATA and NCTM) and in other professional growth opportunities to gain a solid understanding of the principles and standards so that they may be practised in all classrooms across the province.

References

National Council of Teachers of Mathematics (NCTM). *Curriculum and Evaluation Standards for School Mathematics.* Reston, Va.: NCTM, 1989.

——. *Professional Standards for Teaching Mathematics.* Reston, Va.: NCTM, 1991.

——. *Assessment Standards for School Mathematics.* Reston, Va.: NCTM, 1995.

——. *Principles and Standards for School Mathematics.* Reston, Va.: NCTM, 2000.

Klaus Puhlmann

Two workers need 6 2/3 days to do a particular job. How long does it take the first worker if the second worker needs 3 more days?

From the President's Pen

Earlier this school year, the MCATA executive affirmed its direction and the work it does on behalf of members. We identified the following list of activities and goals:

- Pursuing MCATA's mission: Providing leadership to encourage the continuing enhancement of teaching, learning and understanding mathematics
- Planning, organizing and holding the annual MCATA conference
- Ensuring diversity in member representation on the executive by including K–12 mathematics teachers from around the province, a Faculty of Education representative, a Mathematics Department representative, an Alberta Learning representative, a PEC liaison and an ATA staff advisor
- Conducting regular executive meetings (four per year) that focus on MCATA's business, implementing our strategic plan and promoting our mission
- Producing a variety of publications including *delta-K,* a newsletter, monographs, position papers and a website (www.mathteachers.ab.ca)
- Submitting advocacy positions to NCTM and the ATA's Annual Representative Assembly (ARA)
- Planning, organizing and holding two Math Leaders Symposia in cooperation with Alberta Learning
- Promoting the importance of mathematics in schools and communities through promotion materials and contests
- Maintaining our NCTM affiliation
- Involving MCATA executive members in NCTM leadership training courses, liaison roles with NCTM and promotion of NCTM materials
- Representing MCATA on Alberta Learning committees
- Participating in the Beginning Teachers' Conference
- Sponsoring and/or supporting math camps, special projects and contests
- Sending representatives to the ATA Summer Conference and seminars
- Recognizing teachers through the Math Educator of the Year award
- Recognizing teachers, MCATA members and/or nonteachers with the Friends of MCATA certificate
- Awarding grants to members in support of teaching and learning projects

As we examined the work necessary to maintain each of these areas, it became apparent that there was considerable overlap. We also realized how involved each executive member is in making MCATA activities work for members.

I would like to thank the executive members for their work this year. It is a huge commitment on their part—all of them volunteer their time outside of their full-time jobs.

I would also like to thank the MCATA members who call or e-mail us with concerns, issues and comments about mathematics education in the province. We represent you best when we understand your concerns and opinions.

Don't forget the annual MCATA conference! Registration will be on the evening of Thursday, October 25, 2001. Sessions will be held all day Friday, October 26, and Saturday, October 27. The conference theme is "2001—A Math Odyssey," and we already have a good lineup of speakers. The venue is the Fantasyland Hotel in the West Edmonton Mall, which is a great place for your family to enjoy a getaway, too. Visit our website at www.mathteachers.ab.ca for more information.

Sandra Unrau

The Right Angle

Shauna Boyce

Learning Technologies Branch

The Learning Technologies Branch (LTB) is responsible for providing leadership and consultation in identifying, developing, implementing and evaluating effective distance learning strategies and techniques in Alberta schools. Recently, LTB has developed many secondary and elementary mathematics resources.

Secondary Mathematics Resources

The following print resources for secondary mathematics have been developed by LTB and are available from the Learning Resources Centre (LRC):

- *Mathematics 7 Student Pack* (1996) (Product #311069)
- *Mathematics 8 Student Pack* (1997) (Product #349812)
- *Mathematics 9 Student Pack* (1997) (Product #348103)
- *Mathematics 10 Preparation Student Pack* (2000) (Product #411322)
- *Applied Mathematics 10 Student Module Pack* (2000) (Product #434978)
- *Applied Mathematics 10b (Bridging Course)* (2000) (Product #435398)
- *Pure Mathematics 10 Student Module Pack* (1999) (Product #434358)
- *Pure Mathematics 10b (Bridging Course)* (1998) (Product #407644)
- *Pure Mathematics 20 Students Package* (1999) (Product #398265)
- *Pure Mathematics 20b (Bridging Course)* (1999) (Product #407652)
- *Pure Mathematics 30 Student Module Pack* (2000) (Product #434738)
- *Mathematics 31 Student Pack* (1995) (Product #296740)

The following electronic resources have been developed by LTB and are available from LRC:

- *Pure Mathematics 30 Multimedia Segments* (CD-ROM v.1.0. Windows/Mac) (2000) (Product #430843)
- *Applied Mathematics 10 Multimedia Segments* (CD-ROM v.1.0, Windows/Mac) (2000) (Product #431164)
- *Learning Technologies Branch LXR Test & Question Banks* (CD-ROM, Mac) (1999) (Product #400416) Note: This resource can be used for Mathematics 7, 8, 9 and 31.
- *Learning Technologies Branch LXR Test & Question Banks* (CD-ROM, Windows) (1999) (Product #400424) Note: This resource can be used for Mathematics 7, 8, 9 and 31.

LTB is currently developing print resources for Applied Mathematics 20, Applied Mathematics 20b and Applied Mathematics 30. As well, LTB is developing question banks for Applied Mathematics 10, 20 and 30.

Elementary Mathematics Resources

The following print resources for elementary mathematics have been developed by LTB and are available from LRC:

- *Mathematics Grade 1 Student Pack* (2000) (Product #427311)
- *Mathematics Grade 4 Student Pack* (2000) (Product #422288)

LTB is currently developing print resources for Mathematics 2, 5 and 6.

For more information, visit our website at www.learning.gov.ab.ca/ltb/.

Learner Assessment Branch

Diploma Examinations

Diploma examinations in four math courses will be available in June. Mathematics 33, Pure Mathematics 30 and Applied Mathematics 30 will be released exams, but Mathematics 30 (old) will be secured. All four courses will have diploma examinations in August, also. Because Applied Mathematics 30 and Pure Mathematics 30 are pilot exams, they are worth 20 percent of a student's final mark. Please note that the Mathematics 30 (old) diploma exams are available only to students who are repeating the course or who have been granted special permission. A request for special status must be made in writing to Raja Panwar, Director, Curriculum Branch. Alberta Learning, 6th Floor East Tower, Devonian Building, 11160 Jasper Avenue NW, Edmonton T5K 0L2; fax (780) 422-3745, e-mail Raja.Panwar@gov.ab.ca. The projects related to the diploma examinations for Pure Mathematics 30 and Applied Mathematics 30 are available on the Alberta Learning website at www.learning.gov.ab.ca/k_12/testing/diploma/projects/default.asp. Teachers and students can also find the projects and corresponding sample solutions from the first semester at this site.

Calculator Policy

The list of approved calculators and instructions for clearing calculators can be found at www.learning.gov.ab.ca/k_12/testing/diploma/bulletins/default.asp. Students writing the Applied Mathematics 30 and the Pure Mathematics 30 diploma exams will require a graphing calculator from the list of approved calculators. Students writing Mathematics 33 or 30 exams should follow the same guidelines as students writing a science diploma examination. That is, they may use a scientific calculator or a graphing calculator approved by Alberta Learning. All information stored in programmable or para-metric memory must be cleared before writing the examination.

Diploma Examination Information Bulletins

These bulletins provide students and teachers with information about the diploma examinations scheduled for the 2000–2001 school year. They include the blueprints for the examinations, the scoring criteria for the 2000–2001 school year, suggestions for students about writing the examinations, descriptions of the standards for the courses and examinations, and examples of students' responses.

Revisions to the information bulletins for Applied Mathematics 30 and Pure Mathematics 30 are under way and will include changes to curriculum standards and the example questions. The new information bulletins for the 2001–2002 school year, which will include examples of questions that have been validated by both teachers and students, will be posted on our website soon.

Mr. Jones was philosophical about losing money. "I have as many pennies as I had dollars before, but half as many dollars as I had pennies before, and half of my money is gone. Can you tell me how much money I have now?"

In this section, we will share your points of view on teaching and learning mathematics and your responses to anything contained in this journal. We appreciate your interest and value the views of those who write.

Erratum

The previous issue of *delta-K* (Volume 38, Number 1, December 2000) failed to include one of the authors in Comments on Contributors. The following acknowledgment statement should have been included:

> Werner W. Liedtke is a professor in the Faculty of Education, Department of Elementary Education, at the University of Victoria, British Columbia. He also teaches distance education for the Knowledge Network. In addition, he supervises student teachers and delivers inservice courses to parents of preschool children and to teachers, locally and throughout the province.

My apologies for this omission and any inconvenience it may have caused for the author and the readers.

Finding Your Inner Mathematician

Keith Devlin

Many people assume that it takes a special kind of brain to be able to do mathematics—that unless you were born with some kind of "math gene," you simply are not going to be able to get math, no matter how hard you try. As someone who struggled hard with math in school until I was 15, and then got it all at once, I never believed the math-gene theory. What made the difference for me was that everything suddenly made sense—perfect, simple, elegant sense.

Having taught mathematics for 30 years, I am convinced that everyone has the capacity to do mathematics, at least through high school algebra and geometry. In fact, all you really need to do math are nine basic mental abilities that our ancestors developed thousands of years ago to survive in a hostile world:

1. *Number sense.* This is not the same as being able to count. It's much more basic than that and includes the ability to recognize the difference between one object, a collection of two objects and a collection of three objects—and to recognize that a collection of three objects has more members than a collection of two. Number sense is not something we learn. Child psychologists have demonstrated conclusively during the past 20 years that we are born with number sense.
2. *Numerical ability.* This does involve learning—both to count and to understand numbers as abstract entities. Early methods of counting, such as making notches in sticks or bones, go back at least 30,000 years. The Sumerians are the first people we know of who used abstract numbers; between 8000 and 3000 B.C., they inscribed numerical symbols on clay tablets.
3. *Spatial-reasoning ability.* This includes the ability to recognize shapes and to judge distances accurately, both of which have obvious survival value. In addition to forming the basis for geometry, this ability is important for a lot of mathematical thinking that is not, on the face of it, visual or geometric.
4. *A sense of cause and effect.* Much of mathematics depends on "if this, then that" reasoning, an abstract form of thinking about causes and their effects.
5. *The ability to construct and follow a causal chain of facts or events.* A mathematical proof of a theorem is a highly abstract version of a causal chain of facts.

6. *Algorithmic ability.* An algorithm is a step-by-step procedure for performing a certain mathematical task—the mathematician's equivalent of a recipe for baking a cake. In elementary school, we are taught algorithms for adding, subtracting, multiplying and dividing whole numbers and fractions. Secondary school algebra requires that we learn algorithms to solve equations. Algorithmic ability is an abstract version of the fifth ability on this list.
7. *The ability to understand abstraction.* Humans developed the capacity to think about abstract notions, along with acquiring language, 75,000 to 200,000 years ago.
8. *Logical-reasoning ability.* The ability to construct and follow a step-by-step logical argument is fundamental to mathematics. It is another abstract version of the fifth ability.
9. *Relational-reasoning ability.* This involves recognizing how things and people are related to each other, and being able to reason about those relationships. Much of mathematics deals with relationships between abstract objects.

The human brain acquired those nine abilities at least 75,000 years ago. They are basic mental attributes crucial to our daily lives. The question is: What does it take to put those abilities together and do math?

The key is the ability to handle abstraction—the seventh ability on the list. We can all use our brains to reason about physical objects we are familiar with, and we can carry out the same kinds of reasoning about imaginary variants of those objects—for example, the characters in a Harry Potter book or on *Star Trek*. Mathematical thinking involves one more step: reasoning about purely abstract objects. The trick is to make those abstract objects seem real—to fool the brain into thinking that it's dealing with real objects. Once you have taken that step into the world of the abstract, the rest is comparatively easy. After all, the mind is then performing tasks that it finds natural and instinctive.

Although making the abstract seem real sounds hard, we all do much the same thing whenever we read a novel or watch a movie. So am I saying that to do mathematics you have to treat it like reading a novel or watching a movie?

In fact, I'm going a step further. When you start reading a novel, or you watch a movie for the first time, you have to familiarize yourself with the characters and the situation in which they find themselves. In the case of mathematics, the characters never change, only their situations. You have to familiarize yourself with the characters just once, and from then on everything amounts to finding out new things about them.

What does that remind you of? It reminds me of a television soap opera, like the long-running *As the World Turns.* That isn't a joke. The secret to being able to do mathematics is to think of math as a soap opera.

I'm not talking about the love lives of mathematicians here—it's math itself that constitutes the soap opera. The characters are not fictitious people but mathematical objects: numbers, geometric figures, topological spaces and so on. The facts and relationships of interest are not births, deaths, marriages, love affairs and business deals, but mathematical facts and relationships like: Are objects A and B equal? What object has property P? What is the relationship between objects X and Y? Do all objects of type X have property P? How many objects of type Z are there?

Mathematicians think about mathematical objects and the relationships among them using the same mental abilities that most people use to think about physical space or about other people.

Mathematicians don't have a different kind of brain. They have learned to use a standard-issue brain in a slightly different way. What distinguishes a great mathematician from a high school student struggling in a geometry class is the degree to which each one can cope with abstraction. The mathematician learns to create and hold an abstract world in her mind, and then reason about that world as if it were real.

The importance of abstraction and the brain's difficulty in handling abstract objects have three clear implications for mathematics teaching. First, we should start with what is familiar and concrete, and move gradually into the abstract. Second, we must realize that the key—the real challenge—is for the student to come to view the abstract objects of mathematics as real. Third, we need to accept the fact that a period of repetitive training is unavoidable—because repeated use is the only way to make abstract objects seem sufficiently real for the brain to process them.

Much of the current debate about mathematics teaching is focused on whether rote learning of basic math skills is still important in an age of electronic calculators and computers. That debate misses the point. The real value of learning basic math skills today is not that you will need to use those skills per se; chances are you won't. Rather, the benefit is to make the abstract objects of mathematics become so familiar—and seem so real—that you can reason about them using the same mental capacities you use to reason about everyday things. Unless you can get to that stage, you'll never be able to master the more sophisticated kinds of mathematics that today are part of the jobs of stockbrokers, architects, scientists, builders, Olympic coaches, physicians and many other people.

Of course, not everybody will use those forms of math in their daily lives. But mastering mathematical abstraction, like learning a foreign language, is much easier when you are young. Good, effective instruction in math should be part of everyone's education, so that no one is shut out of such an important area of modern life.

Reprinted from The Chronicle of Higher Education, *Volume 47, Number 5 (September 29, 2000) with permission from the author. Minor changes have been made to spelling and punctuation to fit ATA style.*

Evaluate: $\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\ldots}}}}}$

Two Fishermen

Ron: Paul, if you gave me one of your fish, I would have twice as many as you.

Paul: But if you gave me one, we would have the same number of fish.

How many fish does each fisherman have?

Mathematics as communication is an important curriculum standard; hence, the mathematics curriculum emphasizes the continued development of language and symbolism to communicate mathematical ideas. Communication includes regular opportunities to discuss mathematical ideas and explain strategies and solutions using words, mathematical symbols, diagrams and graphs. While all students need extensive experience to express mathematical ideas orally and in writing, some students may have the desire—or should be encouraged by teachers—to publish their work in journals.

delta-K *invites students to share their work with others beyond their classroom. Submissions could include, for example, papers on a particular mathematical topic, an elegant solution to a mathematical problem, an interesting problem, an interesting discovery, a mathematical proof, a mathematical challenge, an alternative solution to a familiar problem, poetry about mathematics, a poster or anything that is deemed to be of mathematical interest.*

Teachers are encouraged to review students' work prior to submission. Please attach a dated statement that permission is granted to the Mathematics Council of the Alberta Teachers' Association to publish the work in delta-K. *The student author (or the parents if the student is under 18 years of age) must sign this statement, indicate the student's grade level, and provide an address and telephone number.*

The following article, "An Example of an Error-Correcting Code," was written by Mark Rabenstein when he was a Grade 8 student at McKernan School in Edmonton. The article was originally published in Mathematics Magazine *and is reprinted here with the kind permission of the author and the Mathematical Association of America.*

Mark earned a Ph.D. in chemistry in 1996 and is now doing pharmaceutical formulations research for Bend Research in Bend, Oregon. While in Edmonton, he attended the Saturday Mathematical Activities, Recreations & Tutorials Club (SMART Club) under the tutelage of Dr. Andy Liu, University of Alberta. He feels that the opportunity to participate in the SMART Club and work under the guidance of Dr. Liu contributed immensely to his intellectual development.

An Example of an Error-Correcting Code

Mark Rabenstein

I am a student in Grade 8. Recently, I went to an enrichment program run by Andy Liu of the University of Alberta. The topic we studied is called "error-correcting codes."

The problem goes like this. A secret agent has to send a message back to headquarters. He uses a transmitter which sends a string of 0s and 1s. Unfortunately, from time to time, a 1 gets changed into a 0 while the message is on its way, or vice versa. So he has to send some extra digits to make sure there is no misunderstanding. Fortunately, no more than k digits in the expanded or encoded message are changed at one time.

We studied many interesting schemes for encoding a message. The first one is really simple. Just repeat the message $2k + 1$ times, and the copy that appears at least $k + 1$ times is the correct one. However, this requires lots of digits, and this is not good for a secret agent.

When I thought things over, I did not see why it was necessary to repeat the message $2k + 1$ times. If there are no more than k mistakes and the message is repeated $k + 1$ times, one of the copies must be correct! The only problem is: How can we tell which is the correct one?

Well, there is a simple way to tell whether a copy has one mistake (or any odd number of mistakes). Add a 1 to the original message if it has an odd number of 1s, and add a 0 otherwise. This way, the number of 1s in the encoded message is always even.

This extra digit is called a "parity-check digit," parity meaning odd or even. If an odd number of mistakes is made in the encoded message, then the number of 1s in it will be odd and not even as it is supposed to be, and in this case we can tell something is wrong. Of course, the method fails for an even number of mistakes.

Let us go back two paragraphs and see how parity-check digits can be of help there. As I said, repeat the message $k + 1$ times. Now add a parity-check digit to each of the last k copies. I will show that this works.

Check each of the last k copies of the received message to see if anything is wrong. Suppose we find a copy with an odd number of mistakes. Well, throw it out! With each such copy goes at least one mistake. In an extreme case, everything is gone except the first copy. It must be correct because all the mistakes have been thrown out.

Suppose we are left with $l + 1$ copies (the last l copies having parity-check digits). We know that there are at most l mistakes, and they come in pairs in the last l copies.

What does this mean? This means that at least half of these l copies contain no mistakes. We should be able to tell what the correct message is, unless there is a two-way tie. In that case, we still have the first copy, which must be correct because all the mistakes have been used up.

So my scheme does work. Of course, it still needs lots of digits, unlike some of the really clever schemes I learned in the enrichment program.

Remarks by A. Liu

The code presented in this note is apparently new. The reader may supply a more formal proof. The "apology" in the last paragraph is really not necessary in that the ease of encoding and decoding for this scheme offsets its lack of sophistication. Its rate of information is asymptotically nearly twice that of the repetition codes (Alt 1948).

Codes with higher rates (the "really clever schemes") were known early on in the history of error-correcting codes (see, for example, Golay 1949, Hamming 1950 and Shannon 1948). A recent publication (Thompson 1983) gives an interesting historical account and shows the interrelationship of error-correcting codes with other areas of mathematics. A definitive treatise (MacWilliams and Sloane 1977) details the state of the art as well as listing over 1,000 references.

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NCTM Standards in Action Process Standard: Connections

Klaus Puhlmann

NCTM's *Principles and Standards for School Mathematics* (2000) identifies 10 standards that form an essential and comprehensive foundation for students from Kindergarten to Grade 12 to learn mathematics. The standards are divided into five content standards (number and operations, algebra, geometry, measurement, and data analysis and probability) and five process standards (problem solving, reasoning and proof, communication, connections, and representation). This article focuses on one process standard: connections.

The revisions to the K–12 mathematics curricula in Alberta have consciously incorporated investigations of the connections and interplay between various mathematical topics and their applications. *Principles and Standards for School Mathematics* (NCTM 2000) suggests that the mathematics instructional programs for K–12 should include the connections standard so that all students

- recognize and use connections between mathematical ideas,
- understand how mathematical ideas interconnect and build on one another to produce a coherent whole, and
- recognize and apply mathematics in contexts outside of mathematics.

The ability to connect mathematical ideas, transfer skills and concepts to other mathematical areas, and apply mathematics to areas outside of mathematics and to practical situations will lead to a deeper and more lasting understanding of mathematics. Students will also come to understand that mathematics is not a collection of separate topics or strands but, rather, an integrated whole. This view of mathematics can only be developed if teachers provide opportunities to study connections within the mathematics curriculum of a particular grade and between the grades. Knowing what has been studied in previous grades and what will be studied in the following grades is essential for teachers.

How do we teach students to recognize and use connections between mathematical ideas? The simple answer is by emphasizing mathematical connections. This implies that teachers need to link conceptual and procedural knowledge, relate various representations of concepts or procedures to one another, recognize relationships between topics in mathematics, use mathematics in other subject areas and apply mathematics to real-life situations. Such emphasis will lead students to see mathematics as an integrated whole rather than as an isolated set of topics.

As students progress through the grades, investigating connections between various mathematical topics should include recognizing equivalent representations of the same concept, using connections between mathematical topics, and using connections between mathematics and other disciplines. Teachers also need to guide the process carefully by asking appropriate questions, such as "How is our activity today related to our discussion yesterday or last week?" or "How is this problem or mathematical topic like things you have studied before?"

How do we teach students to understand how mathematical ideas interconnect and build on one another to produce a coherent whole? Students must have many opportunities to observe the interaction of mathematics with other subject areas and with everyday problems outside mathematics. As students progress and mature, their mathematical experiences with connections should provide not only different settings but also gradual increases in difficulty and complexity. Using a variety of solution processes would also lead to a deeper understanding of the connections among the various mathematical ideas. Teachers need to be mindful of the importance of linking conceptual understanding with procedures, and such linking should be central in the teaching/learning process.

It has also been suggested that an important part of the connections standard is teaching students to recognize and apply mathematics in contexts outside of mathematics. How is that best achieved? The Applied Mathematics 10-20-30 stream is a step in the right direction. This is not to suggest that applications to areas outside of mathematics cannot be included in the Pure Mathematics stream or the other grade levels. In fact, they should be included in all mathematics programs if students are to develop a view of mathematics as a connected and integrated whole.

The mathematical experiences of K–12 students should include opportunities to learn about mathematics by working on problems arising in contexts outside of mathematics and within their own experiences. The connection to the real world is particularly critical for students at the primary level. The level of difficulty and complexity related to connecting mathematics to the real world increases as students progress through the grades. At the high school level, the Applied Mathematics stream challenges students with complex applications and the use of sophisticated measuring devices. These experiences allow students to see the connection of mathematics to the worlds of engineering, architecture, commerce, social sciences, building industry and many other areas. Teachers have to take great care in selecting appropriate problems, activities and projects because students are unlikely to learn to make connections unless they are working on problems or situations that have the potential for suggesting such links. As a means of emphasizing the connections between mathematical ideas, new concepts should be introduced, when possible, as extensions of familiar mathematics and previously learned concepts and skills.

The connections standard derives its importance from the fact that students learn to see mathematics as an integrated whole, increasing their potential for retention and transfer of mathematical ideas. Furthermore, emphasizing connections helps to instill in students an understanding of and appreciation for the power and the beauty of mathematics. Connecting mathematics with other disciplines and with the real world also underscores the utility of the subject.

The three articles that follow provide examples of making mathematical connections. The first article is an example suitable for the junior high level and the other two articles demonstrate the connections standard at the high school level.

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In how many ways can the fraction ½ be written as the sum of two positive fractions with numerator equal to 1 and denominator a natural number?

Making Connections: A Thematic Mathematics Project for Grade 7

David B. Smith

Here is the teacher's dilemma: students are motivated by new experiences in mathematics, but many lack the basic skills and understanding of concepts that they need to be successful when the pace of instruction quickens. For example, when introducing the multiplication property of equality to solve $ax = b$, I find that many Grade 7 students have forgotten how to multiply fractions and decimals. They are interested in the prospect of learning something new, but their enthusiasm is dampened because they cannot perform simple computations. Even after years of drill and practice, familiar algorithms have not reached long-term memory.

Slavin (1994, 193) describes short-term memory as "a bottleneck through which information from the environment reaches long-term memory." I believe that trying to force basic skills and concepts through this bottleneck with periodic intensive drill-and-practice sessions is not effective for many students. Such practice is a temporary fix, and the redundancy of this effort holds no interest for students. During 20 years of teaching middle school, I have learned that practice is essential to success but that it is effective only when the goals are clear, meaningful and rewarding. Perhaps most important, I believe that effective practice is sustained only through self-motivation. To address these issues, I designed a year-long, career-based project to practise some of the basic skills that lay a foundation for algebra, such as measurement in both the customary and the metric system, fractions, decimals, percents and proportions. The rationale for this approach was that frequent rehearsal of these skills in a meaningful context and over an extended period would help students transfer them to long-term memory.

Project Overview

The project was divided into three units to correspond to three marking periods (fall, winter and spring). Unit 1, titled Drafting, included a basic introduction to mechanical drawing skills, design format and function. The curriculum objectives were to review operations with fractions, practise measurement skills, introduce the multiplication property of equality, and apply each of these skills and concepts to designing a floor plan for a one-story summer cottage. Unit 2, Real Estate, introduced terms and practices through an elaborate simulation. Students were asked to select a cottage design from unit 1, purchase a building site, build a cottage using current building costs and try to sell this property for a profit. The objectives for this unit were to review operations with decimals and percents and apply these skills to determine brokers' fees and closing costs. In unit 3, Investment, student teams invested the profits from their property sales in the stock market. This unit targeted the relationship between fractions and decimals and demonstrated the value of memorizing conversions between basic fractional units, such as fourths, thirds and eighths, and their corresponding decimals.

My pre-algebra class met four times a week, for three 40-minute periods and one 80-minute period. Half of the 80-minute period was allotted each week throughout the year for this thematic career-based project. The remaining time on that day and subsequent periods each week followed the traditional curriculum, emphasizing problem solving and drill and practice.

Drafting

To set the stage for success, the class reviewed basic measurement skills, multiplication with fractions and the multiplication property of equality. Using these skills, my students were able to measure distances accurately with a ruler and convert inches to feet using a scale of ¼ inch = 1 foot. Students were then introduced to drafting tools, including the T square, drawing board, right triangles and compass, and were given algorithms for four practice drawings, a rectangle, an L shape, a T shape and design symbols.

These exercises were quite challenging, and students benefited from peer coaching and frequent review sessions.

After completing these exercises, students received an overview sheet for designing their cottages according to the following guidelines.

- The maximum building size was 24 feet by 36 feet.
- The scale for the drawings was 1/4 inch = 1 foot or 1/8 inch = 1 foot.
- The maximum number of rooms was five.

The need for these parameters had become clear during the introductory exercises. Spatial sense is sometimes an innate gift, but more often it is an acquired understanding. Grade 7 students often have no concept of appropriate room dimensions and, consequently, exhibit little understanding of form and function. For example, one of my students was convinced that an indoor pool could be placed in his cottage, along with a 2' × 3' bathroom and a 5' × 3' bedroom with no windows. The game room, however, was understandably a priority for this student; its dimensions were 15' × 20'.

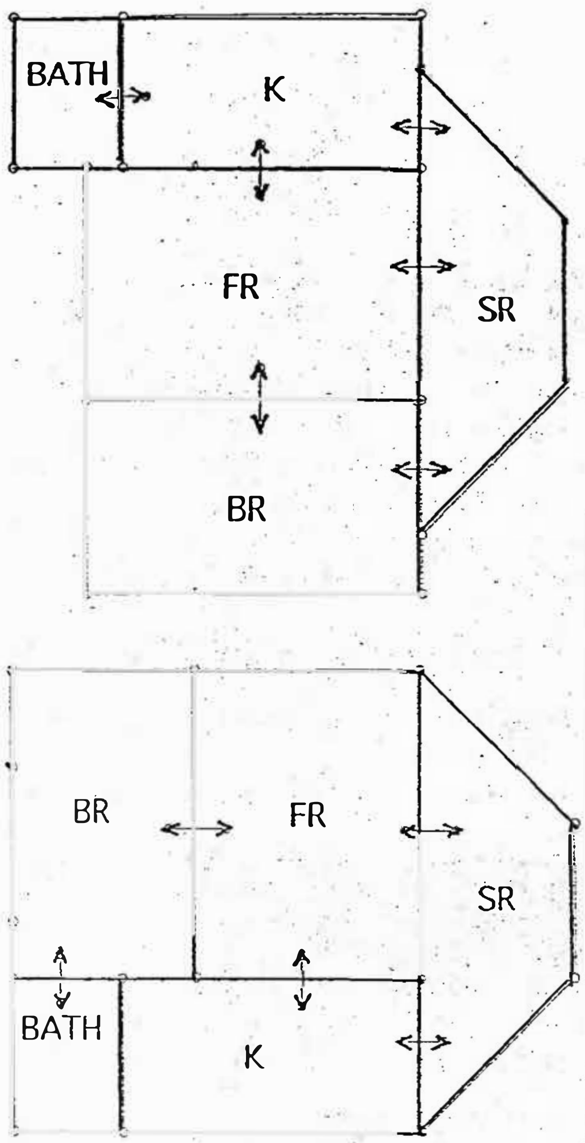
Given students' lack of spatial sense, my second objective was to model functional spaces. Students were required to measure each room in their homes and comment briefly on the advantages and disadvantages inherent in each space. One of my students commented, "My bedroom is 10' × 12' and this is big enough for my desk, bed, and dresser, but my closet is way too small, and I only have one window . . . it gets hot in the summer." I was surprised by the enthusiasm that this assignment generated; students were eager to share their findings and defend their beliefs, but they were also open to new ideas about form and function.

The third objective was to create a functional living space. First, students were asked to cut out graph-paper models of appropriately sized rooms and to piece them together in functional patterns. For example, a bedroom, sunroom, kitchen and family room can be arranged to demonstrate a variety of walking patterns, light exposures and proximities. In addition, once the pattern is selected, room dimensions can be altered slightly to lower building costs. Figure 1 shows two functional patterns created by one of my students.

When the rough drafts were complete with room assignments and symbols for doors and windows, students began their final drawings on oak tag using mechanical drawing tools. Students chose one of two scales, 1/4 inch = 1 foot or 1/8 inch = 1 foot, on the basis of the size of their design. Students were reminded to draw pale lines with a number 3 pencil to facilitate erasures. The final drawings were checked for errors and graded on accuracy and function.

When corrections were complete, the designs were given numbers and placed on tables around the room for viewing. The names of the designers were covered. Each student was given a ballot and asked to record the number of the drawing that best demonstrated the qualities of functional living space, solar efficiency and creativity. The room was full of energy for about 20 minutes. I overheard one of my students say, "There are too many doors in this den—where would you put a couch and a TV?" Another commented, "There isn't enough light in this kitchen; I would put another window on the west wall."

Figure 1
Two Functional Designs for One Student's Cottage



The winning design was taped to the chalkboard, and we reviewed what distinguished it from the others (see Figure 2). After this discussion, we displayed all the drawings in the front hall.

Real Estate

Unit 2 began in January with a review of place value, percents and multiplication with decimals. Our goal was to purchase an appropriate building site for our award-winning design. Students were asked to define a region for the search; given our proximity to the sea and several resort areas, they chose to look along the coast between Noank and Stonington, Connecticut. Ann Burgess and Kay Hill, two local real estate agents, volunteered their time to show us four waterfront properties.

After recovering from the initial shock of waterfront prices, my students listed the advantages and disadvantages of each property using the following criteria: (1) access to water, (2) zoning regulations, (3) taxes, (4) water and sewer requirements, (5) acreage, (6) total cost, (7) soil composition and (8) resale value with proposed building. This lesson was a wonderful opportunity to introduce real estate terms and practices, and the students were amazed at the number of factors to be considered when purchasing a property.

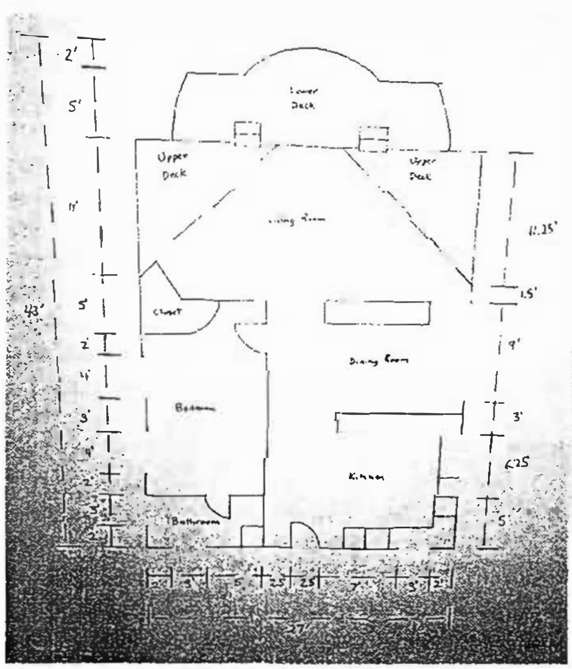
One student was concerned about Amtrak's high-speed rail service adjacent to a property in Mystic; another questioned the cost of insurance in a flood zone; and one student thought that our cottage was not big enough to increase the value of a large property with deep waterfront. She argued that the surrounding homes were considerably larger and commented, "People with a lot of money build bigger homes." This comment was particularly interesting to me because it showed the student's capacity to look beyond current needs and consider this purchase as a long-term investment.

After considerable discussion, we voted unanimously to purchase a property in Stonington with a breathtaking view of the Mystic River. We filled out a mock sales contract with an offer that was $20,000 less than the asking price, and, to our surprise and delight, the offer was accepted. We then consulted with a contractor to determine building costs per square foot and calculated closing costs. The sum of the costs to buy the lot and build our cottage was $380,000.

After we had "built" our cottage on its beautiful waterfront site, we went back to our real estate agents to conduct a market comparison and fill out a listing agreement to sell the home. The agents showed us three waterfront homes that had been recently sold or were currently on the market, were similar in style and size to our cottage, and had comparable lot sizes. Determining a sales price proved to be challenging because the market comparisons presented as many differences as similarities. Students were reminded to ask for a higher price than they thought they could get but not so high as to discourage potential buyers. This time the discussion was heated, but we finally agreed on an asking price of $445,000. Next we discussed the commission for our agent, which is usually 6 percent of the sales price. My students calculated the commission for a variety of sales prices to get a sense of this cost. After this activity, several students wanted to know how they could become real estate agents.

During the next meeting, our agents conducted a mock sale with fictional buyers from New York City, a family of four with a 36-foot sailboat. We refused their first offer of $410,000, and considerable debate ensued. Several students were concerned that we would not make enough profit from the sale after deducting a commission of $24,600, and many wanted to hold out for another buyer. Our real estate agent discussed the advantages and disadvantages of holding out and asked us whether we were prepared to rent the property to help with our monthly costs.

Figure 2
The Winning Cottage Design



My students enjoyed this role playing, and after much debate, we agreed on a price of $437,000. After closing costs and taxes, the profit on our investment was approximately $30,000.

Investment

After a review of the conversions from fractions to decimals and an overview of investment terms and principles, students were given a set of guidelines for investing their profits in the stock market. The class was divided into teams of two, and each team received the $30,000 profit from our sale to invest over an eight-week period. Their initial assignment was to survey peers, parents and neighbors about wise investment opportunities. Local companies, such as Pfizer, Dow Chemical and Stop & Shop, were attractive to my students, as were companies related to the computer industry, such as Microsoft and Yahoo. Their choices often reflected consumer trends, both because students have an excellent grasp of the youth market and because their interests are uncomplicated and highly focused. One team invested a large portion of its profits in Yahoo because personal experience convinced the two students that it is a powerful search engine. Another team selected Stop & Shop because the students had witnessed how successful a new superstore was in their community.

Each team was required to buy two blue-chip stocks and two stocks from NASDAQ. Students kept track of their investments weekly with a stock-review sheet (see Figure 3). At the end of four weeks, the teams were given the opportunity to sell unsatisfactory stock and reinvest their money. Occasional groans and shouts of "Yes!" and "Show me the money!" were testaments to students' interest in the project. Students began to follow the market daily and make conjectures about why stock values rose and fell. Students memorized decimal conversions and used calculators almost exclusively to determine the total value of their stock. See Figure 4 for two examples of student work.

At the end of eight weeks, students calculated their profits and losses and submitted their worksheets. The teams were graded on the following criteria: (1) completion of worksheets, (2) efforts to show work clearly in well-organized steps, (3) accuracy of calculations and (4) cooperation and focus during work periods. As promised at the beginning of the term, I took the top two money-making teams out for a business lunch to celebrate their wise investments and hard work.

Figure 3
Activity Sheet for Stock-Market Project

Stock Market Project Work Sheet

Stock	# of Shares	Original Invest.	Closing Price per share	Change as Decimal	Total Value of Shares Today (closing price × # of shares)	Profit (today's value – original investment)

Work:

Date:

Figure 4

Student Work for the Investment Phase of the Project

Stock Market Project Work Sheet

Stock	# of Shares	Original Invest.	Closing Price per share	Change as Decimal	Total Value of Shares Today (closing price × # of shares)	Profit (today's value – original investment)
AmOnline	100	7293.75	89 3/4	+1.75	8975	1681.25
4Health	918	7917.75	8.625	+2.5	7917.75	0
Yahoo's	75	7335.9375	115 3/16	+.125	8639.0625	1303.125
Pfizers	60	6922.5	106 7/8	−1 15/16	6412.5	−510
						2474.375

Work:

1875

8975
−72

Date: 5/8/98

Stock Market Project Work Sheet

Stock	# of Shares	Original Invest.	Closing Price per share	Change as Decimal	Total Value of Shares Today (closing price × # of shares)	Profit (today's value – original investment)
Amonline	100	7293.75	87.75	−2.25	8775	1481.25
4Health	918	7917.75	7 1/16	−.625	6483.375	−1434.375
Yahoos	75	7335.9375	120.25	−3.75	9018.75	1682.8125
Pfizer	60	6922.5	108	−2.1875	6480	−442.5

Work:

29469.937
10530.063
0625

11059.75
+ 29469.937
40529.687

Date: 5/15/98

Reflections

Two factors were important to the success of this project. First, frequent review of skills during the introduction of each unit was essential for enabling each student to achieve some level of success. For example, a few students did not master basic drafting skills and proportions before the design phase of unit 1, and their motivation dwindled because they depended on the teacher or their peers to perform simple tasks. To overcome this problem, teachers must hold frequent review sessions that are targeted toward specific difficulties.

Second, clear guidelines should be set to ensure cooperative learning for team activities. These guidelines do not require each team to follow the same process for completing the assignment, but the students must understand that each member of the team is responsible for part of the work load. For example, during the Investment unit, two students agreed to review the business section of the newspaper and underline their holdings together, then work through the stock-review sheet separately to verify gains and losses. Another team decided to work through the whole process together. The students alternated responsibilities for locating their investments, reading pertinent business news out loud, recording the data, calculating gains and losses, and checking their work with a calculator. Each approach was effective, and for each team the responsibilities and the expectations for peer interactions were clear.

The central purpose of this project was to design a format for reviewing basic skills that would hold students' interest and ensure long-term retention of these skills. I think that my students were motivated by the format and content, but their retention must be assessed next year. I look forward to discussing this assessment with the students' algebra teacher.

Additional benefits surfaced during the course of the project. First, the project served as a wonderful opportunity to link school and community. My students worked with caring professionals who helped them see the connections between school-based learning objectives and career responsibilities. Second, the components of this project targeted several of Howard Gardner's multiple intelligences, including logical/mathematical intelligence through abstract reasoning, computation and problem solving; spatial and bodily/kinesthetic intelligence through design applications and site planning; and personal intelligence through cooperative learning activities (Gardner 1985). I believe that I was able to meet the individual needs of each of my students while creating a pleasurable and productive experience.

The drafting and real estate portions of this project were designed for a suburban population of middle-level income. Many students will not have the same background, and for them real estate concepts may be unfamiliar. Therefore, more time might be required to define real estate and zoning regulations or to incorporate alternative design projects, such as remodeling apartments or designing a retail space. Regardless of focus, this project should reinforce basic skills, promote student learning and connect fundamental life skills to career opportunities. One of my students underscored this point during a conversation with a peer: "Designing a house isn't so hard," the student said, "not when you know the multiplication property of equality."

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Cost Allocation: An Application of Fair Division

Albert Goetz

Although the subject of cost allocation has been extensively discussed in the literature of political economics, it has been generally neglected in mathematical literature. However, cost allocation affords a practical extension of fair-division techniques—one that is readily accessible to secondary school students and that gives them a simple yet powerful application of mathematics to real-world problem solving. A study of the concepts and the mathematics involved in cost allocation is most appropriate in a discrete mathematics course or a modeling course, but a case can be made for including this topic in other courses, as well. This article presents a typical cost-allocation problem with possible solutions and includes suggestions for presenting similar problems in the classroom. The basics of the problem follow closely from Young (1994).

The Sewage-Treatment-Plant Problem, Part 1

Let us consider two towns, Amity and Bender, each of which needs to build a new sewage-treatment plant. Let us further suppose that the cost for Amity to build the sewage-treatment plant is $15 million and that the cost for Bender to construct the plant is $9 million. Were the two towns to pool their resources, the cost of one sewage-treatment plant, built to service both towns, would be $19 million. Should the two towns decide to build only one plant, and if so, how should the cost be divided?

I find that having small groups work on this problem is both productive and enjoyable for students. Each group is first given one of the two towns to represent and asked to plan a negotiating strategy for the town. Each group is then paired with a group that represents the other town so that the groups can work out a solution.

One question that students frequently ask concerns the populations of the towns. I deliberately withhold this information initially, and I instruct students to devise possible solutions without knowing the populations.

Students should recognize that splitting the cost equally is an inferior solution for Bender. Students should devise two preferable kinds of solutions, either on the basis of the cost or on the basis of the savings involved. Splitting the savings equally between the two towns is an example of the latter. Since $5 million, that is, $24 million minus $19 million, represents the amount saved, each town should save $2.5 million, so that the $19 million cost would be divided in the ratio of 12.5 to 6.5, that is, ($15 – $2.5) to ($9 – $2.5).

A possible solution on the basis of cost is to allocate costs in proportion to opportunity, that is, stand-alone, costs. In this solution,

$$\frac{9}{24} = \frac{3}{8}$$

of the cost, or $7.125 million, should be borne by Bender; and

$$\frac{15}{24} = \frac{5}{8},$$

or $11.875 million, by Amity. The same solution can be obtained by allocating savings in proportion to opportunity costs, so that the cost for Bender, for example, would be

$$9 - \frac{9}{24} \bullet 5,$$

or $7.125 million. See Table 1; where necessary, numbers in tables are rounded to three decimal places.

Table 1

Payments by Town on the Basis of Costs or Savings

	Amity Share (Millions of $)	Bender Share (Millions of $)
Stand-alone costs	15	9
Split costs	9.5	9.5
Split savings	12.5	6.5

Students often rebel against first finding solutions without knowing the populations of the towns, and their concern is worthy of classroom discussion. But if the populations are cleverly constructed, the problem becomes more complex rather than easier. For example, if the population of Bender is 10,000 and the population of Amity is 40,000 and costs are allocated in proportion to population, then Amity should pay four-fifths of the cost, or \$15.2 million. Such a solution is clearly not in Amity's best interest, just as splitting the cost equally is not in Bender's best interest. A question to ask students is, Under what circumstances does the ratio of the populations of the towns produce a solution that encourages each town to participate? However, if the savings are divided equally among the residents, then Amity pays \$11 million, that is,

$\left(15 - \frac{4}{5} \cdot 5\right)$,

and Bender pays \$8 million.

Three solutions appear to be in the best interests of both towns, as indicated in Table 2:

- Dividing the savings equally—A (Amity) pays \$12.5 million, B (Bender) pays \$6.5 million
- Dividing the savings equally among the residents—A pays \$11 million and B pays \$8 million on the basis of the given populations
- Dividing the costs or the savings proportionally to opportunity costs or savings—A pays \$11.875 million, and B pays \$7.125 million

Which of the three solutions is the fairest? Young (1991) takes an interesting geometric approach to this question. *Core* is the term that game theorists and political economists give to the set of possible solutions in which neither player, or town, pays more than the opportunity costs. In Figure 1, the x-axis represents Amity's payments; the y-axis, Bender's payments.

Table 2

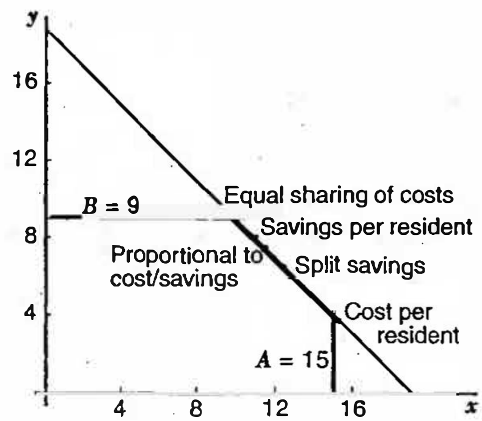
Three Solutions in the Best Interests of Both Towns

	Amity Share (Millions of \$)	Bender Share (Millions of \$)
Dividing savings equally	12.5	6.5
Dividing savings equally among residents	11	8
Dividing costs or savings in proportion to opportunity costs	11.875	7.125

The line segment joining the points (0, 19) and (19, 0) is the set of all possible allocations; the portion of that line segment between the horizontal at 9 and the vertical at 15 represents the core. Students can easily replicate this figure on a graphing calculator in a window that goes from 0 to 20 in each direction. The equation of the line segment in question is $y = -x + 19$, and the DRAW menu can be accessed from the home screen, as opposed to the graph, to obtain the desired horizontal and vertical segments. The previously discussed solutions, both those in the core and those outside it, are labeled in the figure.

Figure 1

A Diagram of Possible Solutions in a Two-Town Game



A good case can be made for choosing the midpoint of the line segment representing the core as the solution to the problem. That point corresponds to equal savings for each town. In that solution, A pays \$12.5 million and B pays \$6.5 million. When students try to negotiate an equitable settlement in their groups, this solution is often the most appealing.

The Sewage-Treatment-Plant Problem, Part 2

We next suppose that a third town, Cordial, is involved. The stand-alone cost for Cordial is \$7 million, and the cost for a sewage-treatment plant that would service all three towns is \$23 million.

Before students can break up into groups to decide how to solve this problem, costs for all possible coalitions must be assigned. One possible way follows:

- The cost for Amity and Bender together remains as before, \$19 million.

- Were Amity and Cordial to participate together, the cost would be $17 million.
- Were Bender and Cordial to participate together, their cost would be $13 million.

If we use the method of proportional allocation, which gave us a solution in the core in the two-town game, then Amity contributes $15.862 million, a solution that is not in the core. Moreover, dividing savings equally among residents fails to fall within the core because Bender and Cordial can form a coalition that leaves Amity out and build the plant for roughly $2.5 million less than by joining with Amity and using that method. Table 3 summarizes results from the other methods used in the two-town game. For these results, we assume that the population of Cordial is 8,000 and that the populations of the other towns are as stated initially. Students can investigate which of these methods fall within the core and which are outside it.

In the classroom, letting students play with the problem before analyzing it in this fashion is advisable; fascinating student interactions can result. If the class is divided into three groups, each representing one of the towns, students can caucus among themselves to determine a "strategy," or method that is equitable from their point of view, to divide costs. Pairs of students from each group are then randomly assigned to negotiate a settlement; in other words, two students from A (Amity), two from B (Bender) and two from C (Cordial) work as one group; another two from A, two from B and two from C work in a second group; and so on.

Young (1991) presents a geometric analog to the line segment that denoted the core in the two-town game. We construct an equilateral triangle with its altitude numerically equal to the cost if all three towns cooperate. Each vertex of the triangle represents one town's payment of the full cost, and any point in the interior of the triangle represents the towns' splitting the $23 million in some fashion. The core in this game is the shaded area in Figure 2.

A Combinatoric Approach

L. S. Shapley, a political economist at Princeton, developed a cost-allocation method (Shapley 1981) that is similar to his approach to power indices in voting games. We consider all possible permutations of the three towns. Each permutation is treated as if the towns join the coalition sequentially and make up the difference between what has already been contributed and the total cost for the coalition. For example, in the permutation ABC, A joins first and must contribute 15. When it joins the coalition, B must contribute 4, the difference between A's 15 and the cost for AB, which is 19. When C joins, C must also contribute 4, the difference between 23 and 19. The *Shapley value* is the average of all possible contributions for a town. The values for the problem are summarized in Table 4.

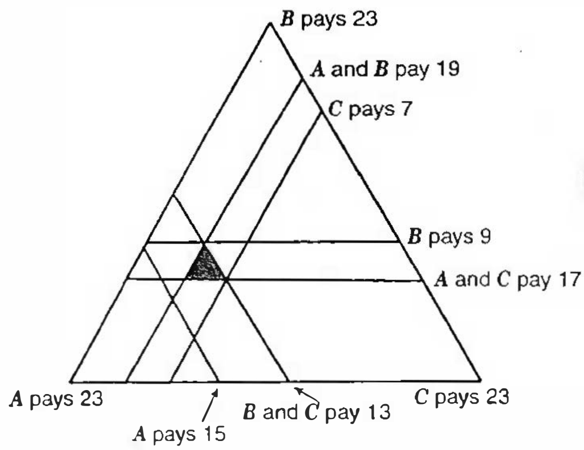
Table 3

Payments by Town for the Three-Town Game

	Payments by Town		
	Amity	Bender	Cordial
Stand-alone costs	15	9	7
Split costs	7.67	7.67	7.67
Split savings	12.33	6.33	4.33
Cost divided in proportion to stand-alone costs	11.129	6.677	5.194
Costs divided among residents	15.862	3.966	3.172
Savings divided among residents	9.483	7.621	5.897

Figure 2

A Geometric Diagram of Possible Solutions in a Three-Town Game



The Geometric Solution for Three Players

The Shapley solution obtained previously is within the core and is thus a valid solution to the problem, but we have no guarantee that the Shapley value will be in the core (Young 1991). Can we guarantee a solution that is in the core of a three-player game if a core exists? We can easily construct a situation in which the core does not exist. We consider the core in Figure 2. We try to extend the midpoint solution of the two-player game, called the *standard solution,* to three players. The core here is a triangle, although we have no guarantee that the core will be a triangle. To visualize this result, we move the line designated "A and B pay 19" parallel to itself and away from vertex C. As that line moves, the core changes from a triangle to a quadrilateral to a pentagon. The upper vertex of the core triangle represents B's paying a share of 9. This amount is B's maximum payment within the core. B's minimum payment is represented by the line designating "A and C pay 17," or 6. We average those payments at 7.5 and construct through that point the horizontal segment with endpoints on the borders of the core. See Figure 3. The left endpoint of the segment represents C's minimum cost, and therefore A's maximum cost, given that B will pay 7.5. The right endpoint represents A's minimum cost and C's maximum cost. If we simply average the maximum and minimum costs for A and C, we obtain the solution that A pays 10.75, B pays 7.5 and C pays 4.75.

A spreadsheet that neatly summarizes all these solutions in the three-town game can be constructed.

Table 4
Allocation Using a Combinatoric Approach

	Individual Contributions		
Coalition order	A	B	C
ABC	15	4	4
ACB	15	6	2
BAC	10	9	4
BCA	10	9	4
CAB	10	6	7
CBA	10	6	7
Total contribution	70	40	28
Shapley value	11.67	6.67	4.67

Such a spreadsheet appears as Table 5. Entries in the top half of the spreadsheet represent the costs to each town or coalition of towns for each possible solution. Entries in the bottom half of the spreadsheet represent the savings for each coalition. Any negative entry in the bottom half of the table indicates that the solution does not fall within the core of the game.

Figure 3
The Core Triangle from Figure 2

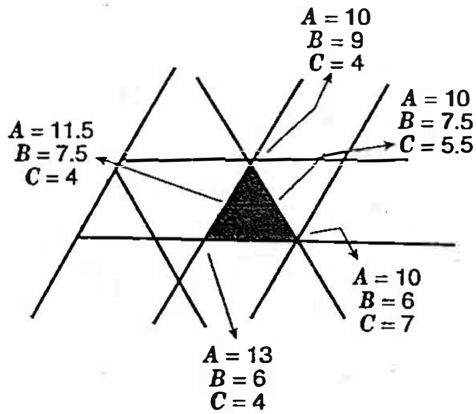


Table 5
Summary of All Solutions in the Three-Town Game

	Amity	Bender	Cordial
Costs			
Stand-alone costs	15	9	7
Split-cost solution	7.67	7.67	7.67
Split-savings solution	12.33	6.33	4.33
Costs prop. to oppty.	11.129	6.677	5.194
Prorated costs	15.862	3.966	3.172
Prorated savings	9.483	7.621	5.897
Geometric solution	10.75	7.5	4.75
Shapley solution	11.67	6.67	4.67
Savings			
Split cost	7.33	1.33	-0.67
Split savings	2.67	2.67	2.67
Costs prop. to oppty.	3.871	2.323	1.806
Prorated costs	–0.862	5.034	3.828
Prorated savings	5.517	1.379	1.103
Geometric	4.25	1.5	2.25
Shapley	3.33	2.33	3.33

Students usually need help in arriving at either the geometric solution or the Shapley value. They do have quite a bit to say about these and the other solutions that they may generate on their own, and talking through the solutions in class has always been interesting and provocative.

Problems of cost allocation are inherently interesting to students and are rich in mathematical applications. Those that come to mind most readily include graphing straight lines, geometric constructions, parallelism, combinatorics and proportions. The aspect that makes cost-allocation problems so valuable in the classroom, however, is that students are motivated to talk about mathematics with one another and to experience a real-life application of the mathematics that they know.

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A rectangle is 2 m longer in length than in width. If we add 4 m to the length and width, the area of the rectangle increases by 72 m^2. Find the length of the sides of the original rectangle.

Random Variables: Simulations and Surprising Connections

Robert J. Quinn and Stephen Tomlinson

Many traditional ideas about the content of the secondary mathematics curriculum are being challenged. The assumption that mathematics can be taught as a series of unrelated skills and algorithms is giving way to an approach that emphasizes the connections among the many branches of study. Such topics as probability and statistics must be given greater prominence as important and practical areas of mathematical knowledge. Students need opportunities to collect and interpret data, simulate probabilistic situations, consider the relationship between theoretical and experimental probabilities, and gain an understanding of random variables (NCTM 1989). Integrating probability and statistics within the curriculum provides a number of interesting and elegant connections that help students develop an appreciation for the inherent beauty of mathematics. The following lesson on random variables incorporates class discussion and experimental activities in the practical and theoretical exploration of one such connection.

This lesson is designed for, and has been used with, advanced second-year-algebra students in Grades 11 and 12. Coins, regular dice, decahedral dice and calculators are used. The lesson involves introducing three random variables followed by considering an empirical and theoretical probability for each. Approximately one and one-half hours are required for these activities. The activities can be scheduled in a single extended time block or can be split over two 50-minute class periods.

Introducing Three Random Variables

The lesson begins with a class discussion, during which the students develop a working definition of a random variable. The exact wording of this definition may vary, but it should include the sense that the value of a random variable is determined by the result of a chance experiment. For example, the number occurring on the toss of a regular die is a random variable with possible values of 1, 2, 3, 4, 5 or 6, each having a probability of 1/6.

Next, the three random variables are introduced. The first, R_x, is the number of tosses needed to get a head in a series of coin flips. R_x has a value of 1 if a head occurs on the first flip, 2 if the first occurrence of a head is on the second flip and so on. The second random variable, R_y, is defined as the number of times a die is rolled to get a 1. For example, if a 1 comes up on the first roll, the value of R_y is 1, whereas if the first three rolls of the die are 2, 4 and 1, the value of R_y is 3. Finally, the third random variable, R_z, is defined as the number of times a decahedral, or ten-faced, die is rolled to get a 1.

Before the empirical phase of the lesson begins, the students are asked to guess the most likely value of each of these random variables. Many assume that the most likely value for each can be found by dividing the number of possible outcomes of a single trial of the coin or die by 2. Thus, they believe that the most likely values of R_x, R_y and R_z will be 1, 3 and 5, respectively. Other students begin with a similar idea but decide that the initial numbers should not be divided by 2, leading them to suggest the most likely values of 2 for R_x, 6 for R_y and 10 for R_z. These guesses are recorded so that they can be compared with the empirical and theoretical results obtained later in the lesson.

The Empirical and Theoretical Investigation of R_x

The students form pairs to investigate these random variables empirically. They begin by conducting the coin-flipping experiment and recording the values of R_x for 10 trials. Each pair's data are written on the chalkboard and compiled into a single table. This larger sample space provides a better basis for deriving the estimate of theoretical probabilities. Figure 1 shows an example of a completed worksheet on which students tallied their data and copied class totals for R_x and R_y.

Figure 1

Sample Results of Student Worksheet for Coin (R_x) and Die (R_y)

Outcome	Coin		Die	
	Tally	Class Total	Tally	Class Total
1	卌	46	III	29
2	III	25	I	26
3	I	14	II	21
4		9	II	13
5	I	4		10
6		1	I	7
7		1	I	8
8		0	II	6
9		0		6
10		0	II	3
11		0		1
12		0		2
13				2
14				3
15				1
16				2
17				4
18				0
19			I	3
20				0
21				0
22				0
23				1
>23				2

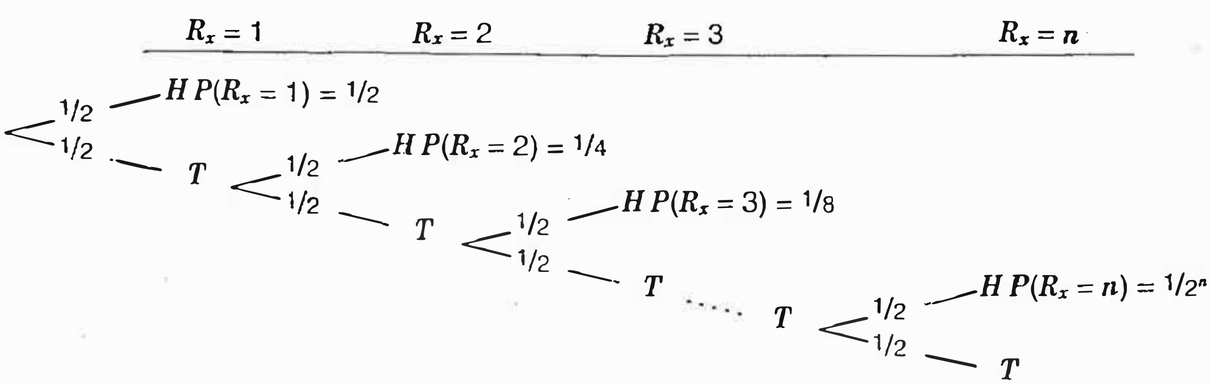
The class is encouraged to consider theoretical probabilities by predicting the likelihood of event H: getting a head on one toss of a fair coin. Most students realize that this probability is ½, that is, $P(\text{H}) = ½$. I explain that since the probability of heads on the first toss is ½, the probability that the random variable, R_x, equals 1 is also ½. The probability that the random variable takes on a value of 2 can then be found by considering the sequence of coin flips necessary for this outcome.

A tree diagram, as shown in Figure 2, demonstrates that the only way the random variable can equal 2 is if the first toss is tails and the second is heads. Since each toss is an independent event, the probability is ½ times ½, or 1/4. Similarly, the probability that $R_x = 3$ is equal to the probability of the event (tails, tails, heads) (TTH), or (½) (½) • (½) = 1/8. At this point, a pattern becomes clear to the students: the probability of any particular value of this random variable is ½ raised to the power of that value, $P(R_x = n) = (½)^n$. Figure 3 shows a worksheet on which students computed the empirical and theoretical probabilities of R_x.

After gathering the empirical probabilities and calculating the theoretical probabilities, students see that the most likely value of this random variable is 1. They also notice that the higher the value of the random variable, the less likely it is for that value to occur. Returning to Figure 2, it is interesting to observe that the probabilities of consecutive values of R_x form a geometric sequence: $P(R_x = 1) = ½$, $P(R_x = 2) = 1/4$, $P(R_x = 3) = 1/8$ and

Figure 2

Tree Diagram for R_x (Coin)



so on. Consequently, the corresponding infinite geometric series represents the probability that the value of the random variable will be some element of the set of positive integers. Since a head must occur eventually, the laws of probability provide an unusual justification showing why the sum of this infinite series must be 1:

$$P(R_x = 1 \text{ or } R_x = 2 \text{ or } R_x = 3 \text{ or} \ldots)$$
$$= P(R_x = 1) + P(R_x = 2) + P(R_x = 3) + \cdots = 1.$$

At this point, the students reconsider their original guesses concerning the most likely outcomes for R_y and R_z. Most students who had guessed that 1 was the most likely outcome for R_x do not change their original guesses for the other random variables. Surprisingly, even those who had guessed 2 for R_x tend not to change their guesses of 6 and 10 for R_y and R_z. They are either too stubborn to change or think that those who guessed 1 just "got lucky" in the coin-flipping experiment. Students are encouraged to think about the geometric sequence that corresponds to the probabilities of successive values of R_x. Occasionally, an insightful student will suggest that 1 will be the most likely outcome for R_y and R_z and that the probabilities of successive values of these random variables will decrease.

Exploring the Roll of the Die

Next, the students consider the second random variable, R_y, the number of rolls needed to obtain 1 when a die is tossed repeatedly. The students carry out this experiment, but more time is allotted for gathering data. This extra time is necessary because more time is needed for the "average" trial and because the decreased probabilities for the values of R_y imply that more trials will be needed to get a reasonably accurate picture of the distribution.

After each pair of students has carried out the experiment 15 times, the results are written on the chalkboard and compiled into a single table. To examine the theoretical probabilities of the different values of R_y, students are asked the probability of rolling a 1 on the first roll of the die. Most students realize that this probability is 1/6; therefore, $P(R_y = 1) = 1/6$. The probability that this random variable takes on a value of 2 can be found by determining the probability that the first roll of the die is "not 1" and the second roll is 1. Since $P(\text{not } 1) = 5/6$ and $P(1) = 1/6$, $P(R_y = 2) = P(\text{not } 1) * P(1) = (5/6)(1/6) = 5/36$. Similarly, $P(R_y = 3) = P(\text{not } 1) * P(\text{not } 1) * P(1) = (5/6)(5/6)(1/6) = 25/216$. Figure 4 shows a sample of student work in determining the empirical and theoretical probabilities of R_y.

Figure 3

Sample Results of Student Worksheet for Probability Calculations (Coin)

Outcome	Pair		Class			
	Freq.	Empirical Probability	Freq.	Empirical Probability	Theoretical Probability	Nearest .001
1	5	$\frac{5}{10} = .5$	46	$\frac{46}{100} = .46$	$\frac{1}{2}$	.500
2	3	$\frac{3}{10} = .3$	25	$\frac{25}{100} = .25$	$(\frac{1}{2})^2$	.250
3	1	$\frac{1}{10} = .1$	14	$\frac{14}{100} = .14$	$(\frac{1}{2})^3$	.125
4	0	0	9	$\frac{9}{100} = .09$	$(\frac{1}{2})^4$	.063
5	1	$\frac{1}{10} = .1$	4	$\frac{4}{100} = .04$	$(\frac{1}{2})^5$	.031
6	0	0	1	$\frac{1}{100} = .01$	$(\frac{1}{2})^6$	.016
7	0	0	1	$\frac{1}{100} = .01$	$(\frac{1}{2})^7$	.008
8	0	0	0	0	$(\frac{1}{2})^8$	.004
9	0	0	0	0	$(\frac{1}{2})^9$	.002
10	0	0	0	0	$(\frac{1}{2})^{10}$	.001
11	0	0	0	0	$(\frac{1}{2})^{11}$	.000
>11	0	0	0	0	$1-(1-(\frac{1}{2})^{11})$	.000

Figure 4

Sample Results of Student Worksheet for Probability Calculations (Die)

Outcome	Pair		Class			
	Freq	Empirical Probability	Freq	Empirical Probability	Theoretical Probability	Nearest .001
1	3	$\frac{3}{15} = .2$	29	$\frac{29}{150} = .193$	$\frac{1}{6}$	.167
2	1	$\frac{1}{15} = .067$	26	$\frac{26}{150} = .173$	$(\frac{1}{6})(\frac{5}{6})$	.139
3	2	$\frac{2}{15} = .133$	21	$\frac{21}{150} = .140$	$(\frac{1}{6})(\frac{5}{6})^2$	.116
4	2	$\frac{2}{15} = .133$	13	$\frac{13}{150} = .087$	$(\frac{1}{6})(\frac{5}{6})^3$	.096
5	0	0	10	$\frac{10}{150} = .067$	$(\frac{1}{6})(\frac{5}{6})^4$	.080
6	1	$\frac{1}{15} = .067$	7	$\frac{7}{150} = .047$	$(\frac{1}{6})(\frac{5}{6})^5$	.067
7	1	$\frac{1}{15} = .067$	8	$\frac{8}{150} = .053$	$(\frac{1}{6})(\frac{5}{6})^6$	.056
8	2	$\frac{2}{15} = .133$	6	$\frac{6}{150} = .040$	$(\frac{1}{6})(\frac{5}{6})^7$	.047
9	0	0	6	$\frac{6}{150} = .040$	$(\frac{1}{6})(\frac{5}{6})^8$	.039
10	2	$\frac{2}{15} = .133$	3	$\frac{3}{150} = .020$	$(\frac{1}{6})(\frac{5}{6})^9$	.032
11	0	0	1	$\frac{1}{150} = .007$	$(\frac{1}{6})(\frac{5}{6})^{10}$	.027
12	0	0	2	$\frac{2}{150} = .013$	$(\frac{1}{6})(\frac{5}{6})^{11}$	.022
13	0	0	2	$\frac{2}{150} = .013$	$(\frac{1}{6})(\frac{5}{6})^{12}$	.019
14	0	0	3	$\frac{3}{150} = .020$	$(\frac{1}{6})(\frac{5}{6})^{13}$	.016
15	0	0	1	$\frac{1}{150} = .007$	$(\frac{1}{6})(\frac{5}{6})^{14}$	.013
16	0	0	2	$\frac{2}{150} = .013$	$(\frac{1}{6})(\frac{5}{6})^{15}$	.011
17	0	0	4	$\frac{4}{150} = .027$	$(\frac{1}{6})(\frac{5}{6})^{16}$	.009
18	0	0	0	0	$(\frac{1}{6})(\frac{5}{6})^{17}$	.008
19	1	$\frac{1}{15} = .067$	3	$\frac{3}{150} = .020$	$(\frac{1}{6})(\frac{5}{6})^{18}$	.006
20	0	0	0	0	$(\frac{1}{6})(\frac{5}{6})^{19}$	.005
21	0	0	0	0	$(\frac{1}{6})(\frac{5}{6})^{20}$	.004
22	0	0	0	0	$(\frac{1}{6})(\frac{5}{6})^{21}$	.004
23	0	0	1	$\frac{1}{150} = .007$	$(\frac{1}{6})(\frac{5}{6})^{22}$	.003
>23	0	0	2	$\frac{2}{150} = .013$	$1-\frac{\frac{1}{6}-\frac{1}{6}(\frac{5}{6})^{23}}{\frac{1}{6}}$	.015

Continuing this argument produces the following:

$P(R_y = 1) = (1/6)$
$= 1/6 \quad \approx 0.167$

$P(R_y = 2) = (5/6)(1/6)$
$= 5/36 \quad \approx 0.139$

$P(R_y = 3) = (5/6)(5/6)(1/6)$
$= 25/216 \quad \approx 0.116$

$P(R_y = 4) = (5/6)(5/6)(5/6)(1/6)$
$= 125/1296 \approx 0.096$

$P(R_y = 5) = (5/6)(5/6)(5/6)(5/6)(1/6)$
$= 625/7776 \approx 0.080$

Generalizing from this pattern, students determine that the nth term of this sequence includes $n - 1$ factors of 5/6 and one factor of 1/6, as shown by the tree diagram in Figure 5.

Consequently, it becomes clear that the most likely value of this random variable is also 1. Further, the formula for successive terms is the same as that for a geometric sequence with a first term of 1/6 and a common ratio of 5/6, that is, $a_n = a_1 r^{n-1}$. Applying the traditional formula for the sum of an infinite series,

$$S = \frac{a_1}{1 - r},$$

yields

$$S = \frac{\frac{1}{6}}{1 - \frac{5}{6}} = \frac{\frac{1}{6}}{\frac{1}{6}} = 1.$$

This result is not surprising because each potential value of R_y is an element of the set of positive integers and

Figure 5
Tree Diagram for R_y (Die)

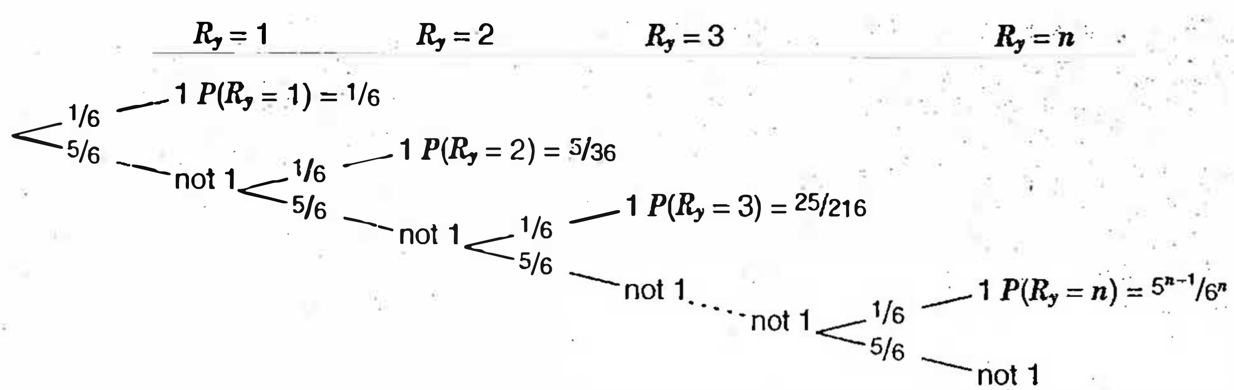
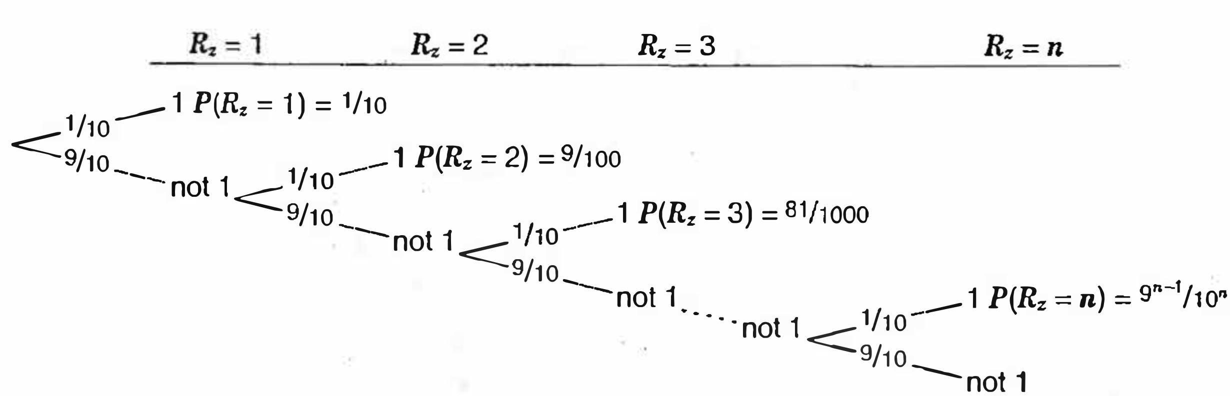


Figure 6
Tree Diagram for R_z (Die with 10 Faces)



$P(R_y = 1 \text{ or } R_y = 2 \text{ or } R_y = 3 \text{ or } \ldots) = P(R_y = 1) + P(R_y = 2) + P(R_y = 3) + \ldots = 1/6 + 5/36 + 25/216 + \ldots = 1.$

At this point many students are ready to change their guess concerning the most likely value of R_z. Their experiences with R_x and R_y enable them to generalize that 1 will be the most likely value of any random variable defined similarly.

R_z, The Decahedral Die

A consideration of R_z, the number of rolls needed to roll a 1 in a series of tosses of a decahedral die, provides students with practice using decimals. Students can collect and compile data in the same manner as in the previous two cases. If, however, most students have already concluded that 1 is the most likely outcome for R_z, the empirical aspects of this example can be omitted. Students complete this phase of the lesson by developing the subsequent probability table and the tree diagram shown in Figure 6.

$P(R_z = 1) = (.1)$
$P(R_z = 2) = (.9)(.1) = .09$
$P(R_z = 3) = (.9)(.9)(.1) = .081$
$P(R_z = 4) = (.9)(.9)(.9)(.1) = .0729$
$P(R_z = 5) = (.9)(.9)(.9)(.9)(.1) = .06561$

The probabilities of successive values of the random variable in this sequence are clearly decreasing, but students may wonder whether the sum of these terms is 1. "Maybe the values don't get small fast enough," one student hypothesized. If the Texas Instruments Explorer calculator is available, it can be used to help students investigate this result. By following the keystroke sequence .1, × .9, =, =, =, =, . . . , students notice that each time = is pressed, the previous product is multiplied by .9, yielding the consecutive values of this geometric sequence. When = has been entered many times, the value of the term will be extremely small. This process should help convince students that, although the terms decrease more slowly than the other two random variables considered, the terms do become small enough fast enough to allow the sum to remain finite. More powerful calculators should not be used. Although they will eventually produce a result of 0, the time required to achieve this answer would be extremely frustrating.

Returning to a theoretical consideration of the sequence of partial sums, S_z, the students see that the addition of each successive term brings the sum 1/10 of the way from its current value to 1. Thus, as in the previous two cases, the formula for the sum of an infinite number of these terms yields

$$S = \frac{.1}{1 - .9} = \frac{.1}{.1} = 1.$$

Figure 7

Expected Values Using Partial Sums

$$\text{Coin: } \sum_{n=1}^{t} n\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{n-1}$$

$S(1) = 0.5$
$S(2) = 1.0$
$S(3) = 1.375$
$S(4) = 1.625$
$S(5) = 1.78125$
$S(10) = 1.98828125$
$S(50) = 1.999999999999954$
$S(100) = 2$
$S(500) = 2$
$\vdots$
$S(t)$

$$\text{Six-sided die: } \sum_{n=1}^{t} n\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{n-1}$$

$S(1) = 0.1666666666666667$
$S(2) = 0.4444444444444444$
$S(3) = 0.7916666666666666$
$S(4) = 1.177469135802469$
$S(5) = 1.57034670781893$
$S(10) = 3.415910673762469$
$S(50) = 5.993846450129437$
$S(100) = 5.999998720084609$
$S(500) = 5.999999999999996$
$\vdots$
$S(t)$

$$\text{Ten-sided die: } \sum_{n=1}^{t} n\left(\frac{1}{10}\right)\left(\frac{9}{10}\right)^{n-1}$$

$S(1) = 0.1$
$S(2) = 0.28$
$S(3) = 0.523$
$S(4) = 0.8146$
$S(5) = 1.14265$
$S(10) = 3.026431198$
$S(50) = 9.690773487560795$
$S(100) = 9.997078246122364$
$S(500) = 9.999999999999984$
$\vdots$
$S(t)$

Extensions

These experiences can be enriched by having students study the expected value of each of the three random variables. I ask the students, "On average, how many times must a coin be tossed to get a head?" Similar questions can be phrased for the two types of dice. The expected value, E, of a random variable can be defined as the summation of the product of each successive value of the random variable with its probability, that is,

$$E = \sum_{n=1}^{\infty} nP(R = n).$$

When a random variable can assume an infinite number of values, as is the case with R_x, R_y and R_z, this sum can be approximated by adding these products for a relatively large number of terms. The partial sums shown in Figure 7 approach the values of E_x, E_y and E_z.

In each case, the expected value seems to converge on the number of possible outcomes for a single toss of the coin, roll of the ordinary die or roll of the decahedral die. That is, $E_x = 2$, $E_y = 6$ and $E_z = 10$. For any random variable defined similarly to R_x, R_y and R_z with q equally likely outcomes, it appears that

$$\sum_{n=1}^{\infty} n\left(\frac{1}{q}\right)\left(\frac{q-1}{q}\right)^{n-1} = q.$$

Figure 8

Proof that $\sum_{n=1}^{\infty}(n)\left(\frac{1}{q}\right)\left(\frac{q-1}{q}\right)^{n-1} = q$

$$E_q = \sum_{n=1}^{\infty} n\left(\frac{1}{q}\right)\left(\frac{q-1}{q}\right)^{n-1}$$

$$E_q = \left(\frac{1}{q}\right)\sum_{n=1}^{\infty} n\left(\frac{q-1}{q}\right)^{n-1}$$

Let

$$\frac{q-1}{q} = K.$$

Then

$$\left(\frac{1}{q}\right)\sum_{n=1}^{\infty} n(K)^{n-1}$$

converges by the root test, since

$$\lim_{n\to\infty}\left|\frac{A_n+1}{A_n}\right| = \lim_{n\to\infty}\left|\left(\frac{n+1}{n}\right)\frac{K^n}{K^{n-1}}\right|$$
$$= K\lim_{n\to\infty}\left|\frac{n+1}{n}\right|$$
$$= K \text{ and } K < 1.$$

Thus, it will be sufficient to complete the proof by showing that

$$\sum_{n=1}^{\infty} nK^{n-1} = q^2.$$

Let

$$A_n = \sum_{n=1}^{\infty} nK^{n-1}.$$

Then

$$A_n = 1 + 2K + 3K^2 + 4K^3 + \cdots$$
$$= 1 + K + K^2 + K^3 + K^4 + \cdots$$
$$+ K + 2K^2 + 3K^3 + \cdots$$
$$= \frac{1}{1-K} + K(1 + 2K + 3K^2 + \cdots)$$
$$= \frac{1}{1-K} + K(A_n);$$
$$A_n(1-K) = \frac{1}{1-K};$$
$$A_n = \frac{1}{(1-K)^2}$$
$$= \frac{1}{\left(1-\frac{q-1}{q}\right)^2}$$
$$= \frac{1}{\left(\frac{q-q+1}{q}\right)^2}$$
$$= \frac{1}{\left(\frac{1}{q}\right)^2}$$
$$= q^2.$$

Hence

$$\frac{1}{q}\sum_{n=1}^{\infty} n\left(\frac{q-1}{q}\right)^{n-1} = q.$$

This empirical evidence led us to consider proving the assertion formally. Many advanced high school students can understand this proof (see Figure 8). Further, the progression of inductive to deductive reasoning illustrated is indicative of the way that mathematicians often work. It is important for students to attempt to emulate this path in their own mathematical reasoning.

Conclusion

The students enjoyed participating in this lesson. The opportunity to guess the most likely value of each of the three random variables hooked them, piquing their competitive spirit. During the collection of the empirical data, the students vocally rooted for outcomes that would show their guess to be correct. This motivation continued during the discussion phases of the lesson as students tried to convince one another of the accuracy of their conjectures. Finally, many students commented that it was pretty cool to find a geometric sequence in a situation in which they never would have expected it.

Reference

National Council of Teachers of Mathematics (NCTM). *Curriculum and Evaluation Standards for School Mathematics.* Reston, Va.: NCTM, 1989.

The authors would like to thank Laura Worrall for implementing this lesson in her classroom and Ted Cook for taking photographs.

Reprinted with permission from The Mathematics Teacher, *Volume 92, Number 1 (January 1999), pages 4–9. Minor changes have been made to spelling and punctuation to fit ATA style.*

Hose A can fill a basin in 40 minutes. Hose B can fill the basin in 30 minutes and hose C in 20 minutes. How long would it take to fill the basin if all three hoses were used at the same time?

Calendar Math

Art Jorgensen

Below are 31 scrambled words, one for each day of May. Each of these words is often heard in the mathematics classroom. Teachers can use these scrambled words to introduce a lesson or have students figure out what the unscrambled word is. An extension to this activity might be to have students bring scrambled words for classmates to unscramble.

1. unmreb
2. qsraeu
3. etmi
4. wot
5. videdi
6. errul
7. scitamehtam
8. lytmpuil
9. rilcec
10. wasnre
11. hctreea
12. rfyto
13. gtaierln
14. msu
15. eicternmet
16. eaar
17. rshrtoe
18. melouv
19. kcolc
20. atncgelre
21. nogylop
22. ttyhir
23. dadnde
24. onpexnet
25. mobhurs
26. buscartt
27. itsivocp
28. eexntsi
29. ealoagrlrapml
30. lahf
31. zdneo

Answers

1. number
2. square
3. time
4. two
5. divide
6. ruler
7. mathematics
8. multiply
9. circle
10. answer
11. teacher
12. forty
13. triangle
14. sum
15. centimetre
16. area
17. shorter
18. volume
19. clock
20. rectangle
21. polygon
22. thirty
23. addend
24. exponent
25. rhombus
26. subtract
27. positive
28. sixteen
29. parallelogram
30. half
31. dozen

Is There a Worldwide Mathematics Curriculum?

Zalman Usiskin

We in mathematics have many names for numbers, among them square numbers, prime numbers, rational numbers, transfinite numbers, Fibonacci numbers, complex numbers, amicable numbers, and on and on. We are number people. We have many words for numbers just as Eskimos have many words for ice and Arabs have many words for camels.

As we analyze curriculum, we have also developed many names for curriculum. We have become curriculum people. In A Study of Schooling, John Goodlad identified five different curricula: the ideal curriculum (beliefs of scholars), the formal curriculum (expectations of what should be done in the class as seen in syllabi, guidelines, textbooks and so on), the instructional curriculum (what teachers report they do), the operational curriculum (what actually goes on in the classroom) and the experiential curriculum (what students report learning and what they actually learn) (Klein, Tye and Wright 1979; Goodlad 1979). Three of these were chosen, though with different names, to constitute one of the main organizing structures in the design of the Second International Mathematics Study (SIMS): the intended (ideal) curriculum, the implemented (operational) curriculum and the attained (experiential) curriculum. In the Third International Mathematics and Science Study (TIMSS), a fourth curriculum was added: the potentially implemented curriculum, a name chosen to represent the curriculum of textbooks and other available materials.

Distinguishing these various types of curricula was important in those international studies, for the various categories are used in curricular analyses that occupied a volume apiece. Yet, these curricular analyses would be purely academic exercises—and, in fact, the lack of media attention given to them suggests that they are purely academic exercises—were it not for the natural interest in comparing not *what* but *how much* is learned by students in different countries.

The existence of TIMSS and other international comparisons of performance in mathematics is founded on the premise that there exists enough of a commonality in the mathematics curriculum worldwide that a test over that commonality represents some sort of fair test of the entire curriculum. And so the question of the title of this presentation is already seen to require some clarification. If we ask, "Is there a worldwide mathematics curriculum?", to which of these curricula are we referring?

At a conference like this one, we can be a little more relaxed: Does there exist enough commonality in the curricula of different countries that when we use such content descriptors as geometry or algebra or functions or linear equations or statistics, or when we speak of the use of calculator or computer technology, we are talking about the same things? I find it useful to examine this with a type of analysis of curriculum different from the intended, implemented or attained curriculum. It is an analysis using *sizes of curriculum.*

The Sizes of Curriculum

There are at least six sizes of the mathematics curriculum, each differing from the previous by roughly one order of magnitude: (1) the individual problem or episode, (2) the problem set or lesson, (3) the unit or chapter, (4) the semester or year-long course, (5) the mathematics curriculum as a whole and (6) the entire school experience. Proceeding from the smallest to the largest, we see that the ratios of sizes are quite appropriate for a difference in orders of magnitude. There are perhaps 5–20 episodes or problems in a typical day in a mathematics classroom, 10–20 days in a typical unit, 7–15 units in a school year, 13 years of schooling from K–12, and perhaps 6–8 other subjects vying with mathematics for space in the curriculum. The fundamental property of differences in order of magnitude asserts that a strategy, practice or policy that is appropriate for one of these sizes of curriculum may not be appropriate for another.

We often see people oversimplifying educational policy by taking something that is appropriate for a small size of curriculum and then recommending it for a larger size. A pretty concept, appropriate for a unit, may be taken as the main idea behind an entire course. The major recommendation of the National Council of Teachers of Mathematics' *Agenda for Action* report issued in 1980 was that "problem solving be the focus of school mathematics in the 1980s" (p. 1), by which it was meant that the curriculum should be centred around problem solving. Here the recommenders were taking something that was hard to disagree with at the individual problem level or lesson level, namely the presentation and solving of interesting problems, and recommending that the idea be carried out three or four orders of magnitude higher.

At the time of the *Agenda for Action* recommendation, there did exist many examples of good problems and good problem-solving lessons, and a few problem-centred units, but to my knowledge there did not exist one example of a problem-centred course, and certainly there was no example of an entire curriculum of this type. What would be the place of skill work in such a curriculum? Where would mathematical systems and structure be discussed? A full curriculum requires balancing a variety of priorities, whereas a lesson, unit or even course does not require the same sort of balance, and balancing an individual problem is like balancing an individual person on a seesaw.

For the most part, a student's experience with curriculum is the union of his experiences with individual tasks, problems or episodes. The curriculum developer tries to find interesting tasks and sequence them in a way that is clear to the student and teacher. A particular problem may be there to motivate the student, or to emphasize a particular idea, or to review an idea or to set the stage for another problem that will come later. Episodes in teaching serve similar purposes. The items that are selected for testing reflect the priorities of the teacher, and when tests are analyzed by performance on individual items, one obtains a picture at this size of curriculum.

The next larger size in the order of magnitude hierarchy—the lesson—should be more than a collection of episodes or a set of problems. A good lesson is built around a concept, which for understanding requires a variety of activities. In a lesson there is always a fundamental decision to be made regarding the balance between what is explained to the student and what is expected to be learned by the student himself. For all these reasons, a good lesson needs coherence, and the best lessons have particular ideas that they emphasize.

Similarly, a good unit is more than a set of lessons. It has a sequence of related concepts that carry it from its beginning to its end. A good unit brings together these concepts in an attempt to show their power. The student, too, is asked to demonstrate power of a different sort, for one of the fundamental properties of most units in school mathematics is that they end with a performance test.

The course is normally the largest chunk of curriculum that the student encounters with a single teacher, and it is usually the only size of curriculum for which there is a grade on record. Because the course is associated with a teacher, a course has a personality. Its personality is interwoven with that of the teacher, and it is difficult to separate student opinions about a course from student views of the teacher. Problems, lessons and units tend not to be of long enough duration to develop a personality. Only in a course is there time to develop a mathematical system of any complexity; only in a course is there time to cultivate a method of thinking.

The mathematics curriculum as a whole is the sum of courses. It has properties different from those of a single course. We might not want every course to deal with mathematical proof, but the curriculum as a whole should. The study of "curriculum coverage" found in the TIMSS analysis (Schmidt et al. 1996, 52) and the earlier analysis of review in U.S. elementary textbooks by Jim Flanders (1987), each of which involves multiyear looks at the curriculum, provide pictures that no one course could provide. And seldom are tests over the entire curriculum created by individual teachers; we need teams of writers for such tests.

Some ideas work at a variety of sizes. For instance, it is often desirable and sometimes obligatory that consecutive problems, lessons, units and courses incorporate a sense of flow, of connectivity, of growth.

Analyzing the Question by Size of Curriculum

Returning to the question, "Is there a worldwide mathematics curriculum?", I would like now to interpret this question for each of the various sizes of curriculum. I will start from the largest size.

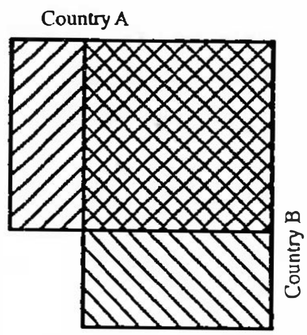
Entire Curriculum

If a student takes mathematics through secondary school in different countries, will that student cover the same mathematics? If not, then it is rather silly to speak about comparing performance in different countries, for we are comparing apples to oranges.

Obviously, we are not looking for 100 percent agreement. But it is not clear how much agreement is sufficient. The situation comparing two countries, A and B, can be represented by a diagram somewhat like a Venn diagram. In the case pictured in Figure 1, 3/4 of the topics taught in Country A are also taught in Country B, and 2/3 of the topics taught in Country B are also taught in Country A.

Figure 1

A Hypothetical Example of Overlapping Curricula



We think that there is a great deal of commonality, but in this made-up situation a full 45 percent of the total number of topics in the two countries are not common topics; that is, almost as many topics are not common as are common! Obviously, commonalities are less frequent if there are more countries.

The current trends in the mathematics curriculum that we have made themes of this conference serve to *decrease* the overlap in curricula. The movement toward mathematics for all has generally led to curricula with greater numbers of applications and data. Appropriate applications for one country may be inappropriate for another, and familiar data in one part of the world may be quite abstract in another. The use of technology in some places and not in others also creates obvious differences in what is expected of students even when the problems may be the same. For all these reasons, it is my guess that, in our quest to make it possible for mathematics to be tested worldwide, we have deemphasized the differences in total curricula.

For instance, I don't think it has been publicized that the TIMSS Grade 12 advanced mathematics test contains more geometry than algebra (see Table 1). To any person in the United States, that would seem odd, because far more time in Grades 9–12 is spent on algebra than on geometry.

Table 1

Distribution of Advanced Mathematics Items by Content Category, from the Third International Mathematics and Science Study

(taken from Mullis et al. 1998, p. B-9, Table B-2)

Category	% of items*	Number of points
Numbers and Equations	26	22
Calculus	23	19
Geometry	35	29
Probability and Statistics	11	8
Validation and Structure	5	4
Totals	100	82

*There were a total of 65 items.

Individual Courses

From the standpoint of individual courses, I think it has been demonstrated rather clearly by the TIMSS researchers that there is no worldwide curriculum. Examining Table 2, which summarizes four mainstream topics in six countries, we find that the course treatment of all four topics varies from country to country. In fact, no two of these countries treats any of these topics in the same ways over the years! The significance of this is that no one can expect to export even one or two years of a curriculum from one country to another. Individual courses simply differ by too much.

Table 3 shows the numbers of years of coverage and the numbers of years of emphasis for these four topics in these six countries.

In these tables the "mile wide, inch deep" characterization of the U.S. curriculum does not appear particularly valid, and I could not find any relation between the years of coverage or emphasis and student performance. The maximum years of coverage for the topics is shared among three countries, and the maximum years of emphasis for the topics is shared among four, and they have vastly different performance profiles.

Table 2

Curriculum Coverage for Selected Mathematics Topics Across Student Ages

(taken from Schmidt et al. 1996, p. 52, Figure 2-7)

Example 1: Properties of Whole Number Operations

	Student Age												
Country	6	7	8	9	10	11	12	13	14	15	16	17	18
France	–	–	–	–	–	–	–	–	–	–	–		
Japan		•	•	•	–								
Norway			–	–	–	–	–	–	–	–			
Spain			–	–	–	–	–						
Switzerland		–	•	•	•	–	–	•	•				
USA	–	–	–	•	•	–							

Example 2: Relation of Common and Decimal Fractions

	Student Age												
Country	6	7	8	9	10	11	12	13	14	15	16	17	18
France				–	–	–	–	•					
Japan			–	–	•	•							
Norway					–	•	–	–	–	–			
Spain					•	•	–	•	–				
Switzerland							•	–	–	–			
USA			–	–	–	•	•	–					

Example 3: Exponents, Roots and Radicals

	Student Age												
Country	6	7	8	9	10	11	12	13	14	15	16	17	18
France						–	–	•	–	–	–	•	
Japan							•		•	•	–		
Norway								–	–	–	•	–	–
Spain						–	–	–	•				
Switzerland								–	•	•	–	–	
USA						–	–	•	•	–	–	•	

Example 4: Properties of Whole Number Operations

	Student Age													
Country	6	7	8	9	10	11	12	13	14	15	16	17	18	
France									–	–	–	•	–	
Japan				•	•	•	•	•	•	•	•	•	•	
Norway			–	–	–	–	–	–	–	–	–	–	–	–
Spain								–	–	–	–	–	–	
Switzerland			–	–	–	–	–	–	•	•	–	•	•	
USA			–	–	–	–	–	•	•	•	–	•	–	

Note: Ages 9 and 13 are TIMSS Student Populations 1 and 2
– *topic covered in curriculum* • *topic emphasized in curriculum*

In analyzing curricula in the United States, the difficulty for us is the diversity that exists within our country. The most recent report we have links our own National Assessment Grade 8 scores to those of TIMSS (Mullis et al. 1998). The differences in the 41 reporting states are striking. Examine Table 4. Compared to Mississippi, 19 of the 21 countries with samples that met the international guidelines, including the U.S. as a whole, score significantly higher, and none score significantly lower. In contrast, only 6 countries score significantly higher than North Dakota, and 8, including the U.S. as a whole, score significantly lower. We must conclude that the taught curriculum is not the same in these states. But every report coming out of Washington treats our entire country as if the curriculum were the same everywhere. Why not do the obvious: find out what is done in North Dakota, Iowa, Maine and other high-performing states, and emulate it. Find out what is done in Mississippi, Louisiana and the District of Columbia and work hard to change.

It is true that the United States, despite the lack of a national curriculum, does have a common algebra curriculum, if one looks at textbooks. Here are what I believe to have been the five most used first-year algebra textbooks (counting all editions as one) in the last school year in the United States, though together they only constitute, at most, 60 percent of the first-year algebra texts in use:

Merrill Algebra 1, by Foster et al.
Algebra, by Brown, Dolciani et al.
Heath Algebra 1, by Larson, Kanold et al.
Prentice Hall Algebra, by Bellman et al.
UCSMP Algebra, by McConnell et al.

In major ways, all five books are very much alike. They have 10–13 chapters. They all begin with algebraic expressions. They have 2–5 chapters on linear equations and inequalities (here UCSMP [University of Chicago School Mathematics Project] spends more time than the others). They graph lines and then they graph and solve systems. There is work with the laws of exponents and one or two chapters on polynomials. There is a chapter on quadratics, and thus some work with radicals. All solve quadratics by the Quadratic Formula and by factoring, and all but the Brown, Dolciani et al. do this graphically. All but UCSMP have a chapter dealing with rational expressions and rational equations. All have some geometry, including area formulas and the Pythagorean Theorem. In this sense, there is very much an algebra curriculum in the United States.

There are many other algebra texts in use in the United States: the texts of Smith, Charles et al. and of Foerster published by Addison-Wesley before the merger with Scott Foresman; of Saxon published by Grassdale; of Benson et al. published by McDougal Littell; of Coxford et al. published by Harcourt Brace; and so on. These books cover the same content as the five most used books and, except for Saxon, do it in pretty much the same way.

And there are the project algebras, none used very much at this point in time: the CORD algebra, the CMP algebra out of the University of California at Davis, the computer-intensive algebra of Fey and Heid published under the title *Concepts in Algebra—A Technological Approach.*

But there are also major differences even among the books in most use. The more recent copyrights give strong attention to graphing calculators. The more recent texts have large numbers of applications and real data. The data differ significantly from book to book so that students learn different things from one book than from another. UCSMP and the recent Prentice Hall give more attention to geometry. All give some attention to functions, but some of the recent texts use function language from the beginning, whereas others do it toward the end, where most students would not even see it. The picture one receives from these books is of an algebra curriculum that is reasonably fixed, but in flux.

Table 3

Years of Coverage and Years of Emphasis of Certain Topics

(from Schmidt et al. 1996, p. 52)

Years of Coverage

Country	Whole Numbers	Fractions, Decimals	Exponents, Roots	Equations, Formulas	Average
France	11	5	7	5	7
Japan	4	4	4	10	5.5
Norway	8	6	6	12	8
Spain	5	5	4	6	5
Switzerland	8	4	5	11	7
United States	6	6	7	11	7.5

Years of Emphasis

Country	Whole Numbers	Fractions, Decimals	Exponents, Roots	Equations, Formulas	Average
France	0	1	2	1	1
Japan	3	2	3	10	4.5
Norway	0	1	1	0	0.5
Spain	0	3	1	0	1
Switzerland	5	1	2	4	3
United States	2	2	3	4	3.75

And I have not mentioned the NSF projects that are exhibiting here, in which the traditional first-year algebra topics mentioned above are dispersed over two or three years. These integrated curricula are quite different from those mentioned above, and also quite different from each other. If we gave all available curricula equal weight, then we would have to conclude that there is no standard U.S. curriculum. What percent of students need to be enrolled in similar curricula in order for there to be considered to be a standard curriculum for the entire country? It is the same question we ask for the world, but in an individual country the more appropriate size of curriculum for the question is not the entire curriculum, but the course level.

Units

At the unit level, the mathematical approach taken to a topic becomes important. How are the various ideas related? So we ask: Are the approaches taken to large chunks of content the same worldwide?

We do not have a standard way for measuring different approaches to topics. In fact, except for broad approaches to geometry, with names such as "vector approach," "transformation approach" or "synthetic approach," different approaches to mathematics have seldom been discussed. There is no universal way to decide when two approaches differ.

But I will give some examples to indicate that there are differences. Consider the approach to systems of linear equations taken in the Japanese books UCSMP translated some years ago. In the chapter entitled "Simultaneous Equations" in the Grade 8 book (Kodaira 1984, 1992), there is not one graph. The reason is that students have not yet graphed lines with equations of the form $y = ax + b$. Yet in every algebra book in the United States, the study of systems begins with graphical solutions.

The Japanese text defines slope as the number a in $y = ax + b$. All the U.S. books define slope as $\frac{y_2 - y_1}{x_2 - x_1}$. UCSMP texts and Japanese books discuss rate of change *before* they discuss slope, an approach which we have found to be very successful in enhancing student understanding of the idea of slope. Yet we use applications and the Japanese do not. Is this enough to be different? UCSMP texts and the Japanese text describe the slope as the increase in y when x increases by 1. Is this enough to be different from other texts?

The Japanese text defines figures to be congruent if one figure can be laid on top of the other by combining translations, rotations and reflections. We do the same in UCSMP texts and spend some time over a period of years developing competence in these transformations. This is not done in most United States texts. Freudenthal (1983) pointed out that the way in which a term is defined automatically constrains it for future discussion. I believe these differences in the way congruence is approached cause differences in the ways in which students think about figures and their relationships to each other, and later, in the study of functions, in the ways in which students think about their graphs. I think it's a significant difference.

However, in general, the unit level is a difficult level at which to analyze curriculum. Over the years, we have developed very little language to describe different approaches to systems of equations or quadratics or congruence or similarity. A comprehensive study of curriculum at the unit level might prove quite enlightening.

Lessons

Turning now to the lesson level, the TIMSS videotape work of Stigler suggests that there are great differences in the ways that lessons are taught in Japan, Germany and the United States. A Japanese algebra class is shown spending 27 minutes on one problem, 15 minutes on another. A Japanese geometry class is shown spending 22 minutes on one problem and then 27 minutes on an extension of the same problem. In contrast, the U.S. algebra class has students working on all sorts of problems at once—in a cooperative learning situation—and the teacher spends no more than 2 minutes discussing any one problem in front of the entire class. The U.S. geometry class is more traditional in its setup but again there are a large number of questions with not much time spent on any one of them (Seago 1997).

In reports on these lessons, Stigler criticizes the ways in which United States teachers conduct their lessons (Beatty 1997, 11–12). There is an underlying assumption that lessons in Japan, Germany and the United States could be transported from one country to another. In fact, when one looks at the classrooms and at the content, it seems that the lessons could easily be transported. But one U.S. teacher, upon viewing these lessons, said to me that there is no way that her students would tolerate spending the amount of time on one problem that the Japanese do.

Are our societies enough different to make lessons that are viable in one country not viable in another? There are people who think so even for different groups within the United States. A call for "culturally relevant" pedagogy has been made by members of groups historically underrepresented in mathematics (Ladson-Billings 1995). This call rests

Table 4

Performance of NAEP Jurisdictions Compared to 20 TIMSS Countries at Grade 8

(from Mullis et al. 1998)

Jurisdiction	# of Countries Higher	# of Countries Lower	% of Students in Top 10%	% of Students in Top 50%
North Dakota	6	8	5	64
Iowa	6	7	4	63
Maine	6	7	6	62
Minnesota (est.)	6	7	6	62
Montana	6	7	6	62
Nebraska	6	7	5	61
Wisconsin	6	7	6	61
Minnesota (actual)	6	7	7	57
Vermont	7	5	4	57
Connecticut	7	5	5	56
Massachusetts	7	5	5	55
Alaska	7	4	7	55
Michigan	7	4	5	54
Utah	9	5	3	54
Oregon	9	4	4	53
Washington	9	4	4	53
Colorado	9	4	4	52
Indiana	9	4	3	52
Wyoming	10	3	3	52
Missouri	10	3	3	49
Texas	12	3	3	46
New York	12	2	3	47
Maryland	12	2	6	45
Virginia	12	2	3	45
Rhode Island	13	2	3	46
Arizona	13	2	2	43
North Carolina	13	2	3	42
Delaware	13	2	3	41
Florida	13	2	2	40
Kentucky	13	2	2	40
West Virginia	13	2	2	38
Tennessee	15	2	2	38
Hawaii	15	2	3	37
New Mexico	15	2	2	36
California	15	1	3	38
Georgia	15	1	2	38
Arkansas	15	1	2	37
South Carolina	15	1	2	34
Alabama	16	1	1	32
Louisiana	18	0	1	25
Mississippi	19	0	1	23
Dist. of Columbia	21	0	1	13

on the assumption that mathematics is taught in the U.S. from a Eurocentric lens that works against the performance of Hispanic, African-American and Native American students. It is closely related to the ethnomathematics movement to recognize not only the contributions to mathematics of non-European cultures but also the unique ways in which mathematics is informally used every day in one's own native culture.

And yet, in viewing middle school classrooms in Shanghai some years ago, I was struck by the similarities in the mathematics far more than the differences. In more than one class studying geometry proofs, doing problems exactly like those found in Japanese or American texts, I saw, in the midst of Chinese characters, the abbreviation SAS for the Side-Angle-Side congruence theorem. I mentioned my surprise to my hosts, who reminded me that Chinese characters do not represent sounds in the way that Latin characters do, so there is no Chinese character for the first letters of words. I was still astonished that the English first letters would be used. But it definitely seems to indicate that, at least with certain content, some lessons are quite exportable from one country to another.

Problems

The smallest size of curriculum—the individual problem or task—is not the least important size. The TIMSS and other international tests of comparison are based on the premise that there is a commonality of problems or other short tasks that can be used worldwide.

For example, the publication *What Students Abroad Are Expected to Know About Mathematics* (American Federation of Teachers 1997) displays examinations that top students in France, Germany and Japan have taken. and compares these with the SAT and Advanced Placement BC Calculus exams in the U.S. The published *Baccalauréat* Exam in Mathematics from the Aix province of France, taken by students at the end of their lycée experience in Grade 12, is an exam in vector analytic geometry, calculus and algebraic descriptions of geometric transformations. The *Abitur* Exam in Mathematics from the state of Baden-Württemberg in Germany is evenly split among calculus, solid analytic geometry and stochastics. The Tokyo University Entrance Exam in Mathematics resembles one of the American Invitational Mathematics Exams we use to select students for the U.S. Olympiad team. And of course our BC Calculus exam is all calculus. If these exams cover the curriculum in their respective locales, it is rather clear that there is no worldwide mathematics curriculum for the best students. The content differs markedly from country to country.

Recognizing these differences in content, in the TIMSS Grade 12 Advanced Mathematics study, a Test-Curriculum Matching Analysis was done. An expert in each country determined whether the items were in the intended curriculum of at least half of the students in the population. The idea was "to show how student performance in individual countries varied when based only on the test questions that were judged to be relevant to their own curriculum" (Mullis et al. 1998, C-1). The expert for the United States judged that 100 percent of the items were in the intended curriculum for the U.S. students. I have never seen the entire test, but 19 of the 82 items (see Table 1) and 2 of the 6 released items required calculus, and the highest estimates are that 6 percent of U.S. students take calculus. Since 14 percent of U.S. students were in this population, these items were in the intended curriculum of less than half of the U.S. students. They should not have been considered as relevant. Curiously, the analysis of only those items identified as appropriate had no major effect on the relationships among countries on either the mathematics or the physics tests. I have no logical explanation for this. Perhaps all of the experts tried to be as ecumenical as possible in including items.

As I mentioned earlier, selecting what is appropriate is only one part of the picture, however. One must ask whether there are things the students have learned that are not being tested. At all levels, would students in other countries perform as well as U.S. on items requiring measurements in feet and inches, or in pounds and ounces? I doubt it.

A quarter-century ago, I wrote a course called "Algebra Through Applications with Probability and Statistics." At the time, we had a student in a master's program from Colombia in South America. She was very impressed by the materials and took upon herself the task of translating them into Spanish. But she said she had to change a few problems. Not the data on baseball—they play baseball in Colombia. She needed to alter those problems in which we had people going on diets and losing weight at some constant rate. She said, "In Colombia, it's not considered advantageous to be thin. People don't diet."

Answering the Question

Now, for the last time, let me state the question that I have been trying to answer with these remarks: Is there a worldwide mathematics curriculum? I did not have an answer when I first thought of the talk. For most of the time that I prepared the talk, my feeling was the usual professor's response to such a question, Yes and No. But after working through the analysis, I have a different answer.

Mathematics is a worldwide language. Beyond the writing of numerals, schools, colleges and universities use virtually the same written language for algebra, geometry, analysis and statistics. Computers worldwide use the same programming languages. Multinational companies can hire mathematicians from virtually any country in the world. The problems tackled by mathematics are universal not only in place but also in time. We are able to hold conferences like this one because we recognize those characteristics of our subject.

But in our natural desire to show off the universality of our subject, I think we may have gone too far to think that *mathematics education* is the same worldwide. From arithmetic to beyond calculus, mathematics is vast. In our different cultures, different choices are made from all the mathematics available, and different aspects of this vastness are emphasized. In France, the mathematics is more theoretical, still reflecting the influence of Bourbaki. In the U.S., the mathematics is becoming more applied. In most countries, advanced mathematics students are using calculators, but this is not true in all countries. At the broadest level, we are all teaching very much the same ideas, reflecting the commonalities of mathematics. But we do so in different course structures, with the subject matter organized sometimes in quite different units, with lessons that may be appropriate for one country but not another, and often with problems that do not transfer from one site to another.

Thus beyond the teaching of arithmetic, there are common goals but there is not today a worldwide mathematics curriculum, and let us not delude ourselves into thinking that there is. But let us not be disappointed by this. We are able to have much richer conversations because of the differences.

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Patterns and Patterning: Myths and Not Myths

Jerry Ameis

You may be having a few troublesome thoughts about the patterns and relations strand in the early and middle years curriculum documents. Because it is a new strand, there are likely to be good reasons for those thoughts. The strand covers two related processes: sorting and patterning. This article discusses both, with the focus on patterning.

In the current era of reform in mathematics education, it is important to be honest about what is mathematics and what is not. The rhetoric and pressures of reform can too easily lead us to be charmed by such clichés as "Mathematics is in everything" and "Problem solving is mathematics." Such clichés are suspect in designing instruction that involves worthwhile mathematics.

What does this have to do with sorting and patterning? Unfortunately, myths are attached to the two processes. Sorting and patterning are not mathematical processes in and of themselves. Mathematics is a cultural invention—an organized set of concepts, symbols, relationships and procedures created by people. Sorting and patterning are fundamental thinking processes that we are born with; the processes are hard-wired in us, if you will. We use them in our daily lives: when parenting, when reading a book, when shopping, when learning a language and so on. As a parent, I detected a pattern when my baby son was hungry: he cried. Is detecting that regularity doing mathematics? My wife sorts the laundry according to color; I sort it according to aroma. Are we doing mathematics when we sort the laundry?

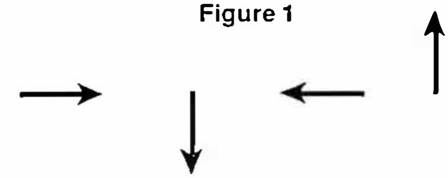
When we sort and look for patterns in ways that involve mathematical objects (for example, numbers or shapes) and/or attributes (for example, length or thickness), we are working in the domain of mathematics with the help of the processes of sorting and patterning. When we sort and look for patterns in ways that involve science concepts such as atoms and plant species, we are working in the domain of science with the help of these processes.

Patterning involves a kind of thinking generally referred to as inductive reasoning—searching for a consistent feature in something and having faith that it will continue. We are able to think in that way from birth. And patterning is not unique to humans: animals look for patterns as well, as anyone who has a pet can attest to. The ability to identify patterns is one of our mental survival tools, but it is not necessarily mathematical thinking.

What is a pattern from the perspective of mathematics? First, a mathematical pattern is not something that must repeat three times. Nor is it equivalent to a pretty visual design or a template such as a dress pattern. Mathematically speaking, a pattern involves something that remains constant about a collection of numbers, shapes or mathematical symbols, concepts or attributes. The critical matter is it *remains constant.*

Consider the series of arrows shown in Figure 1. What is the pattern? There is a regularity in the way the arrows point. The direction of each subsequent arrow involves a constant rotation or turn. One way to describe the pattern is the change in direction is always 1/4 of a turn clockwise.

Figure 1



Patterns do not shrink or grow. The elements of a pattern may shrink or grow, but the pattern does not. Consider this series of numbers: 23, 19, 15, 11, 7. The numbers (the elements) in the series decrease (shrink), but the pattern does not. One way to describe the pattern is "Each successive number is 4 less than the number before it." This pattern does not shrink for the series of numbers. It remains the same.

Consider the arrangements of dots in Figure 2. The number of dots increases or grows, but the pattern remains constant. It can be described in a variety of ways. From a geometry perspective, the pattern can be described as "The number of rows and columns of successive diagrams increases by 1." This increase of 1 in each dimension remains constant.

We could create a data table (see Table 1) from the dot diagrams and describe number patterns in it.

The table contains more than one type of numerical pattern. There is a vertical pattern that can be expressed as "The increase in the number of dots increases by 2 each time." There is a horizontal pattern that can be expressed as "The number of dots equals the square of the diagram number." Both of these patterns remain constant as the number of dots in the diagrams increases.

Figure 2

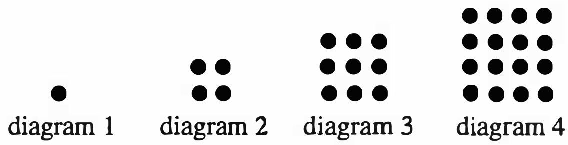


Table 1

Diagram Number	Number of Dots in Each Diagram
1	1
2	4
3	9
4	16

Teachers should not assume that, just because students are looking for and identifying patterns, this necessarily means that they are doing significant mathematics or mathematics at all. For example, determining what comes next in the color sequence Red Yellow Red Yellow . . . involves, at best, trivial mathematics (counting to one). At worst, it does not involve mathematics at all. A child does not need to count or use any other mathematical skill to identify the pattern of alternating colors.

With respect to early years curricula, identifying pattern types should not be an important goal in teaching patterning. For example, consider the following sequences: Red Yellow Yellow Red Red Yellow Yellow Red Red Yellow Yellow Red . . ., and 1331133113311331 . . . . An underlying pattern can be identified in the two sequences; some call it the ABBA pattern (not to be confused with the 1970s Swedish pop group). The underlying pattern can certainly be viewed in an ABBA way, but that is not really the point of doing patterning activities. Being able to recognize the ABBA pattern is of dubious benefit to the student or society. What can be of benefit is the student becoming comfortable with the processes of making and testing hypotheses when searching for mathematical patterns.

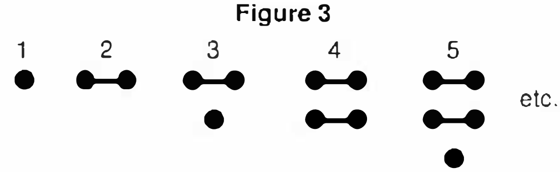
Having said this, nevertheless, some identification of underlying patterns should be pursued to assist students in generalizing. For example, it is useful to a point for students to recognize that the sequences Red Yellow Yellow Red Red Yellow Yellow Red . . . and 133113311331 . . . have something in common—even though one involves color and the other involves number. One way to describe what they have in common is to call it the ABBA pattern.

Although the mental process of searching for and identifying patterns is hard-wired in us, the language associated with that process is not. Students need to learn language descriptors for a process they can do naturally. For this reason, it is good pedagogy for teachers to make use of things that are familiar to students to help them understand the language descriptors. Nonmathematical items such as shoes and teddy bears are appropriate contexts for developing Kindergarten and Grade 1 students' understanding of language. However, once they understand the meaning of such words as *pattern*, patterning activities should involve reasonable to significant mathematics and serve two important purposes: (1) helping students learn mathematics and (2) helping stimulate the fundamental thinking process of patterning so that it can grow in depth and scope.

Patterning activities should be integrated with other strands of the mathematics curriculum. This can be done in two ways: (1) using patterns to learn concepts and skills from other mathematics strands and (2) solving patterning problems that involve concepts and skills from other strands. Using patterns to help students learn concepts and skills from other strands of the curriculum can be considered an authentic use of patterning. Three examples follow.

Early years children learn about odd and even numbers. Patterning can be used to help them understand these kinds of numbers in an activity that involves using actual objects and the symbols for numbers. For each represented number, children could be asked to connect all of the objects two at a time (make pairs) in some way (line segments are used in the diagram).

Figure 3



Children then could be asked to look for a pattern. With discussion, they should come to the conclusion that the numbers 2, 4, 6, . . . always can be paired while the numbers 1, 3, 5, . . . always have an unpaired object left over. All that remains is for the teacher to attach the word *even* to the numbers 2, 4, 6, . . . and the word *odd* to the numbers 1, 3, 5, ...

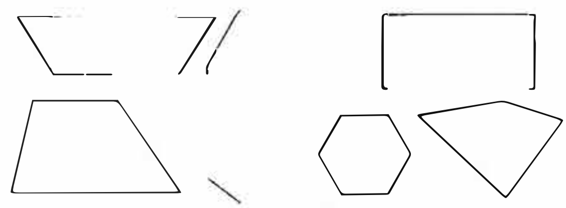
Middle years students learn about integer multiplication. Patterning can be used to help them understand that a negative number multiplied by a negative number equals a positive number. To use patterning for this goal, students must first understand that a positive number multiplied by a negative number equals a negative number. This understanding can be developed by viewing the matter in terms of "I owe . . ." (for example, 2×-3 can be interpreted as "I owe each of 2 friends 3 dollars. How much do I owe in all?"). This approach is inappropriate in the case of a negative number multiplied by a negative number because there is no such thing as a negative group in mathematics (though there can be groups of people who are negative toward something, but that is an entirely different thing). Once students understand that a positive number multiplied by a negative number equals a negative number, a situation such as the one shown in Figure 4 can be used for the case of a negative number multiplied by a negative number. Students would need to look for a pattern in the results (for this example, the answer to the multiplication gets bigger by 3 each time).

Figure 4

$3 \times -3 = -9$
$2 \times -3 = -6$
$1 \times -3 = -3$
$0 \times -3 = 0$
$-1 \times 3 = ?$

Patterning can be used to help students understand geometry terminology. For example, a teacher can display two collections of shapes (as shown in Figure 5), ask students to look for a pattern and then provide a definition of a trapezoid. One pattern is that trapezoids have four sides, one pair of opposite sides is parallel and the other pair is not parallel.

Figure 5



These are trapezoids — These are not trapezoids

Having students solve problems that involve patterning and concepts and skills from other strands is the other way to integrate patterning within the mathematics curriculum. For some of these problems, teachers can pay attention to additional matters such as constructing tables and using systematic ways to look for patterns in numbers. Many problems are possible. Mathematics curriculum documents contain good examples of such problems.

The problem presented below is different from the typical ones. The mathematics content involved in the pattern is around the Kindergarten/Grade 1 level, but the problem is likely to challenge the reader. The answer is not included here. The reader is invited to e-mail the author at j.ameis@uwinnipeg.ca after thinking about the problem for a while.

The numbers in each row are derived from the numbers in the row above in the same way. What is the next row of numbers (the eighth row)?

Figure 6

1
1 1
2 1
1 2 1 1
1 1 1 2 2 1
3 1 2 2 1 1
1 3 1 1 2 2 2 1

Connecting Probability and Geometric Progressions

David R. Duncan and Bonnie H. Litwiller

Geometric progressions is an important concept used in many mathematical applications. Probability is a source of significant examples of this type of progression.

Consider the following situation: Greg and Joel, beginning with Greg, alternately roll a fair hexahedral die. The first one to roll a 6 wins. What is the probability that Greg wins or that Joel wins?

First consider Greg. He will win if one of these events is satisfied:

- Event 1: Greg rolls a 6 on the first try.
- Event 2: Greg and Joel both fail to roll 6s on the first try. Greg rolls a 6 on his second try.
- Event 3 Greg and Joel both fail to roll 6s on the first two tries. Greg rolls a 6 on his third try.
- Event $(n+1)$: Greg and Joel both fail to get 6s on the first n tries. Greg rolls a 6 on his $(n+1)$ try.

The probabilities of these distinct events are

- Event 1: 1/ 6
- Event 2: $\frac{5}{6} \cdot \frac{5}{6} \cdot \frac{1}{6} = \left(\frac{5}{6}\right)^2 \cdot \frac{1}{6}$
- Event 3. $\left(\frac{5}{6} \cdot \frac{5}{6}\right) \cdot \left(\frac{5}{6} \cdot \frac{5}{6}\right) \cdot \frac{1}{6} = \left(\frac{5}{6}\right)^4 \cdot \frac{1}{6}$
- Event $(n+1)$: $\left(\frac{5}{6}\right)^{2n} \cdot \frac{1}{6}$

Because these events are mutually exclusive, the probability that Greg wins is the sum of the separate event probabilities.

$$P(G) = \frac{1}{6} + \frac{25}{36} + \frac{1}{6} + \left(\frac{25}{36}\right)^2 \cdot \frac{1}{6} + \ldots + \left(\frac{25}{36}\right)^n \cdot \frac{1}{6} + \ldots$$

$$= \frac{1}{6}\left(1 + \frac{25}{36} + \left(\frac{25}{36}\right)^2 + \ldots + \left(\frac{25}{36}\right)^n + \ldots\right)$$

$$= \frac{1}{6}\left(\frac{1}{1 - \frac{25}{36}}\right)$$

$$= \frac{1}{6}\left(\frac{1}{\frac{11}{36}}\right)$$

$$= \frac{1}{6} \cdot \frac{36}{11}$$

$$= \frac{6}{11} \approx 0.56$$

Recall that if

$S = 1 + r + r^2 + r^3 +$

then

$rS = r + r^2 + r^3 + r^4 +$

Consequently,

$S - Sr = 1$

$S(1 - r) = 1$

$S = \frac{1}{1-r}.$

We will next compute the probability of Joel's winning in two ways.

Method 1

- Event 1: Greg fails on his first roll, and Joel rolls a 6 on his first try.
- Event 2: Greg fails twice and Joel fails once to roll a 6. Joel then rolls a 6 on his second try.
- Event 3 Greg fails three times, and Joel fails twice to roll a 6. Joel then rolls a 6 on his third try.

$$P(J) = \left(\frac{5}{6}\right) \cdot \frac{1}{6} + \left(\frac{5}{6}\right)^3 \cdot \frac{1}{6} + \left(\frac{5}{6}\right)^5 \cdot \frac{1}{6} + \ldots$$

$$= \frac{5}{6} \cdot \frac{1}{6}\left(1 + \left(\frac{5}{6}\right)^2 + \left(\frac{5}{6}\right)^4 + \cdots\right)$$

$$= \frac{5}{36}\left(1 + \frac{25}{36} + \left(\frac{25}{36}\right)^2 + \cdots\right)$$

$$= \frac{5}{36}\left(1 - \frac{1}{1 - \frac{25}{36}}\right)$$

$$= \frac{5}{36} \cdot \frac{36}{11}$$

$$= \frac{5}{11} \approx 0.44$$

Method 2

Because only two disjoint outcomes are ultimately possible, their probabilities must have a sum of 1. Since P(G) = 6/11, P(J) must be 5/11.

Now, remove the requirement that Greg and Joel must both perform the same activity. Let Greg roll a die, hoping for a 6, and let Joel flip a fair coin, hoping for a head. Find the probability of Greg winning or of Joel winning.

$$P(G) = \frac{1}{6} + \left(\frac{1}{6}\cdot\frac{1}{2}\right)\frac{1}{6} + \left(\frac{1}{6}\cdot\frac{1}{2}\right)^2\cdot\frac{1}{6} + \ldots$$
$$= \frac{1}{6} + \left(\frac{1}{12}\right)\cdot\frac{1}{6} + \left(\frac{1}{12}\right)^2\cdot\frac{1}{6} + \ldots$$
$$= \frac{1}{6}\left(\frac{1}{1-\frac{1}{12}}\right)$$
$$= \frac{1}{6}\cdot\frac{12}{11}$$
$$= \frac{2}{11}$$

Note that Greg's probability is less than 1/2 even though he goes first. This is because his winning outcome is much less likely than Joel's.

P(J) = 1 – 2/11 = 9/11

As a further refinement of this situation, suppose that Greg's winning outcome has probability p_1, while Joel's winning outcome has probability p_2. The probabilities that they do not win in any given try are, respectively, q_1 and q_2 where $q_1 = 1 - p_1$ and $q_2 = 1 - p_2$. Then,

$$P(G) = p_1 + (q_1\, q_2)\, p_1 + (q_1\, q_2)^2\, p_1 + \ldots$$
$$= p_1\,[1 + (q_1\, q_2) + (q_1\, q_2)^2 + \ldots]$$
$$= p_1\left(\frac{1}{1-q_1\, q_2}\right)$$
$$= \frac{p_1}{1-q_1\, q_2}$$
$$P(J) = 1 - \frac{p_1}{1-q_1\, q_2}$$
$$= \frac{1-q_1\, q_2 - p_1}{1-q_1\, q_2}.$$

What must be the relationship between p_1 and p_2 to make this overall sequence a fair game? The fair game condition would require that

$$\frac{p_1}{1-q_1\, q_2} = \frac{1-q_1\, q_2-p_1}{1-q_1\, q_2}$$
$$p_1 = 1 - q_1\, q_2 - p_1$$
$$2p_1 = 1 - (1-p_1)(1-p_2)$$
$$2p_1 = 1 - [1 - p_1 - p_2 + p_1\, p_2]$$
$$2p_1 = p_1 + p_2 - p_1\, p_2$$
$$p_1 - p_2 + p_1\, p_2 = 0$$

If

$$p_1 = \frac{1}{6},$$

then

$$\frac{1}{6} - p_2 + \left(\frac{1}{6}\right) p_2 = 0;\ -\frac{5}{6}p_2 = -\frac{1}{6};\ p_2 = \frac{1}{5}.$$

Joel's event must be equivalent to drawing a specific card from a deck of five cards.

To extend this situation one more step, let Nadine enter the game as the third player in order. On any given trial, Greg, Joel and Nadine have winning probabilities of, respectively, p_1, p_2 and p_3, whereas q_1, q_2 and q_3 are their respective probabilities of failing on any given trial. In this expanded setting,

$$P(G) = p_1 + (q_1\, q_2\, q_3)\, p_1 + (q_1\, q_2\, q_3)^2\, p_1 + \ldots$$
$$= p_1(1 + q_1\, q_2\, q_3 + (q_1\, q_2\, q_3)^2 + \ldots)$$
$$= p_1\left(\frac{1}{1-q_1\, q_2\, q_3}\right)$$
$$= \frac{p_1}{1-q_1\, q_2\, q_3}.$$
$$P(J) = q_1 p_2 + (q_1\, q_2\, q_3) q_1 p_2 + (q_1\, q_2\, q_3)^2\, q_1 p_2 + \ldots$$
$$= q_1 p_2\,(1 + q_1\, q_2\, q_3 + (q_1\, q_2\, q_3)^2 + \ldots)$$
$$= q_1 p_2\left(\frac{1}{1-q_1\, q_2\, q_3}\right)$$
$$= \frac{q_1 p_2}{1-q_1\, q_2\, q_3}.$$
$$P(N) = q_1\, q_2 p_3 + (q_1\, q_2\, q_3)\, q_1\, q_2 p_3 + (q_1\, q_2\, q_3)^2 q_1\, q_2 p_3 + \ldots$$
$$= q_1\, q_2\, p_3\,(1 + q_1\, q_2\, q_3 + (q_1\, q_2\, q_3)^2 + \ldots)$$
$$= q_1\, q_2\, p_3\left(\frac{1}{1-q_1\, q_2\, q_3}\right)$$
$$= \frac{q_1\, q_2\, p_3}{1-q_1\, q_2\, q_3}.$$

Because exactly 1 person must ultimately win, the relationship P(G) + P(J) + P(N) = 1 must hold. To verify this algebraically,

P(G) + P(J) + P(N) =

$$\frac{p_1 + q_1 p_2 + q_1 q_2 p_3}{1-q_1\, q_2\, q_3} = \frac{(1-q_1) + q_1(1-q_2) + q_1\, q_2\,(1-q_3)}{1-q_1\, q_2\, q_3} =$$
$$\frac{1-q_1+q_1-q_1\, q_2 + q_1\, q_2 - q_1\, q_2\, q_3}{1-q_1\, q_2\, q_3} = \frac{1-q_1 q_2 q_3}{1-q_1\, q_2\, q_3} = 1.$$

Suppose, for instance, that Greg rolls a die (hoping for a 6), Joel flips a coin (hoping for a head) and Nadine draws a card from a standard deck of 52 cards (hoping for a heart). Then

$$p_1 = \tfrac{1}{6};\ p_2 = \tfrac{1}{2};\ p_3 = \tfrac{13}{52} = \tfrac{1}{4}.$$
$$P(G) = \frac{\frac{1}{6}}{1-\frac{5}{6}\cdot\frac{1}{2}\cdot\frac{3}{4}} = \frac{\frac{1}{6}}{1-\frac{15}{48}} = \frac{\frac{1}{6}}{\frac{33}{48}} = \frac{1}{6}\cdot\frac{48}{33} = \frac{8}{33}.$$
$$P(J) = \frac{\frac{5}{6}\cdot\frac{1}{2}}{\frac{33}{48}} = \frac{5}{12}\cdot\frac{48}{33} = \frac{20}{33}.$$
$$P(N) = \frac{\frac{5}{6}\cdot\frac{1}{2}\cdot\frac{1}{4}}{\frac{33}{48}} = \frac{5}{48}\cdot\frac{48}{33} = \frac{5}{33}.$$

Challenges

1. Compute these types of probabilities using other situations.
2. Generalize this problem to n players.
3. Find other situations in which geometric progressions can be productively employed.
4. For three players, what must be the relationship between p_1, p_2 and p_3 to produce a fair game?

Pigeonholes and Mathematics

Sandra M. Pulver

The Pigeonhole Principle, also known as the Dirichlet Principle, states the following: if $xy + 1$ pigeons are divided evenly into y holes with x pigeons in each hole, then at least one hole must hold $x + 1$ pigeons. For example, if 201 pigeons ($xy + 1$) are divided evenly into 100 pigeon holes (y), then there are 2 pigeons (x) in each hole, except one in which there must be 3 pigeons ($x + 1$). (Even if the pigeons are not divided evenly into the pigeonholes, the principle still holds true.)

Suppose there are n holes with at most x pigeons in each hole. Then, the total number of pigeons would be at most xn. However, if there are $xn + 1$ pigeons, then at least one hole must hold more than x pigeons.

Let us see how we can use the Pigeonhole Principle to solve other problems of the same nature.

Problem 1

A sack holds black marbles and white marbles, identical in all ways but color. A marble is removed without looking into the sack. How many times must this be done to be sure that two marbles of the same color will be removed?

Solution 1

In this case, the pigeons are the marbles drawn, x, and the pigeon holes are the colors, y, of the marbles. Therefore, the question becomes how many pigeons ($xy + 1$) must there be to ensure that two pigeons ($x + 1$) of the same y color end up in one of the holes? The answer is 3.

Problem 2

Given any 12 integers, show that 2 of them can be chosen whose difference is a multiple of 11.

Solution 2

We have only 11 slots or pigeonholes. In a mod 11 system, all integers fit the system in this manner:

0	1	2	3	4	5	6	7	8	9	10
0	1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20	21
22	23	24	25	26	27	28	29	30	31	32
33	34	35	36	37	38	39	40	41	42	43
.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.

The top line of the chart shows the value of the remainders if the number 11 is divided into each integer. The difference of any 2 integers in a vertical column is a multiple of 11. Let the 11 vertical columns be the pigeonholes (y) in which all integers belong. Because there are $(xy + 1) = 12$ integers, or pigeons, 2 of them ($x + 1$) must belong to the same column. Hence, there must be 2 numbers whose difference is a multiple of 11.

Problem 3

Given 8 different natural numbers, none greater than 15, show that if we take the (positive) difference between pairs of these numbers, at least 3 of these differences will be equal.

Solution 3

From the 8 natural numbers selected, there are $28 = {}_8C_2$ differences one can produce. The premise that these 8 natural numbers can be no greater than 15 determines that there are only 14 possible differences (1 through 14). These 14 possible differences serve as the pigeonholes. If the 28 differences produced by the 8 selected numbers are placed into these 14 pigeonholes, at least 3 of the 28 must fall into one of the holes because, in this case, there are pigeonholes that have special properties. For example, out of the 8 numbers selected, because the greatest number is 15, the difference, 14, can be produced in only one way (that is, $15 - 1 = 14$). This limits the pigeons that can belong to that pigeonhole to 1. Then there are 27 differences to be divided up among 13 pigeonholes, forcing one of the pigeonholes to carry at least 3 pigeons. Note that the difference 13 can be produced in only two ways (that is, $15 - 2 = 13$ and $14 - 1 = 13$), thus further limiting the pigeons that can belong to hole 13 to 2.

Application of the Pigeonhole Principle is not limited to any one field of mathematics. Regardless of the field—be it arithmetic, combinatorics or geometry—the difficulty in applying this principle lies only in determining which are the pigeons and which are the pigeonholes. Once these can be identified, the solution is arrived at through elementary reasoning.

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A jeweler had three clocks in his shop for repairs. In the test he always performed after repairing a clock, he found that one clock was now keeping perfect time, one was gaining 1 minute every 24 hours and one was losing 1 minute every 24 hours. At 9:00 on March 19, 1998, he set all three clocks to show the correct time. When would the three clocks again show the correct time if they were kept running at the rates they had shown in the test?

Find a positive integer t such that $s^2 - t^2$ is a prime number and $s = 14$.

The Place of Games in the Teaching of Mathematics and the Place of Numero Among Available Mathematics Games

Frank Drysdale and Brian McGuiness

Is playing games a waste of time in a mathematics lesson?

Obviously, it depends on the game. Philosophically, however, is there a place for games in the mathematics curriculum?

Doug Clarke (quoted in "Early Maths Games Pay Off" 2000) and John Gough (2000) have recently argued persuasively for a change in our approach to mathematics teaching. Internationally, they are not alone, especially after mathematics educators have looked at the results of the Second International Maths Study (SIMS) and the Third International Mathematics and Science Study (TIMSS).

The Japanese system produced the best results in SIMS and was also very prominent in TIMSS. However, it also resulted in significantly negative attitudes toward mathematics, among students and teachers alike. The Japanese system allocates much more time to mathematics than does Australia, for instance, in several ways:

- Longer mathematics periods, in elementary schools especially
- Six-day school weeks
- Large amounts of homework
- The majority of students receiving private tuition outside school

Although there is no real evidence from these studies that one teaching method yields better results than another, there seems to be no doubt that, when more time is allocated to the subject, results improve. However, both SIMS and TIMSS reveal a clear negative correlation between performance and attitude: increased time allocation results in higher achievement but also causes students to be less confident and more antagonistic toward the subject.

In England, the National Curriculum that came into being after SIMS and before TIMSS placed a greater emphasis on science, especially at the primary level, with an average of 15 minutes more per week spent on science and 15 minutes less on mathematics. The subsequent TIMSS results showed that students' mathematics performance had deteriorated whereas science performance had improved. But even more interesting was a post-TIMSS survey that showed a rise in attitudes toward mathematics and a corresponding fall in attitudes toward science.

This negative correlation between attainment and attitude suggests that a curriculum placing greater emphasis on mathematics will almost inevitably result in widespread feelings of failure, lower self-confidence and more negative attitudes—with students much more ready to give up mathematics in favor of something less demanding.

It came, then, as no surprise when England's minister for education announced early in 2000 that mathematics was to be a national priority, with all students having at least an hour per day but, significantly, not "more of the same." In an attempt to break the negative correlation between attainment and attitude, he stated that they needed more and better mathematics games.

In Australia, Judy Anderson (1995a) said,

> Positive attitudes to learning mathematics must be fostered. Students will often develop a poor attitude if they see mathematics lessons as boring and if they continually fail assessments. Certainly, students need to become competent in carrying out basic numerical operations, but if these procedures are rehearsed daily, using the same teaching strategies, then it is not surprising that many students will describe mathematics lessons as boring. A variety of teaching strategies should be used, including games.

In a newspaper article titled "Early Maths Games Pay Off" (2000), Doug Clarke was quoted as saying, "Children whose mathematics skills were nurtured by teachers as part of a structured but creative programme (including simple activities such as playing cards) did between 27 per cent and 79 per cent better than other students, in key [mathematics] areas."

In "Mathematics: It's All a Game," Gough (2000) writes,

> Mathematics games are good, because children learn better if they enjoy what they are doing. Unlike worksheets, games present challenges at every turn, and feel worth doing. Apart from practice in arithmetic when scoring, tactical problem solving, spatial thinking and probability, there are social and attitudinal benefits. Attention spans extend. Students learn "game manners": co-operatively taking turns, abiding by rules, playing fairly, and being gracious when they win or lose. It's all learning, and it's all fun and games.

However, all games are not equal! Some so-called mathematics games are merely a different way of testing children on such things as tables and combinations. Instead of asking, "What is 6 times 4?", a teacher might throw two dice and see who is first to say the product or sum of the numbers that come up. The better students will still win, and the weaker ones will still lose, with fear of failure and ridicule being reinforced.

Other games go to the other extreme: the outcome is decided by chance alone. Although such games may provide educational or social benefits, they are ineffective mathematically.

The element of chance is, however, important: it adds excitement, introduces elementary probability and allows the less able player to be successful, at least occasionally. (Psychologists tell us that occasional rewards are, in fact, a more effective stimulus than rewarding every performance.) It is obvious, too, that no one enjoys playing games in which they seem to have no chance of success.

If a mathematics game is to serve its purpose, it needs to be

- enjoyable to the point that children are enthusiastic about playing it,
- constructed in such a way that the more skillful player (the better mathematics student) wins on the majority of occasions but not always, and
- useful in providing opportunities to teach and reinforce specific elements of mathematics.

This is how Numero has established itself—throughout Australia and, increasingly, around the world. Children and adults of all ages and all numeracy levels love playing Numero. School principals have made comments such as "Our children love rainy days because they can stay inside at lunchtime and play Numero!"

Although the game's element of chance means that weaker players sometimes win, the better mathematics students will do so more often. In Western Australia—where there is a weekly newspaper column, "Numero Challenge," in the *West Australian's ED Magazine* and an annual Interschool Numero Challenge—some schools have developed a reputation for outstanding performances and success.

Teachers continue to be amazed by how Numero enhances their work in the classroom, speeding up students' acquisition of basic numeracy skills and giving them the perfect tool for coping with and extending a wide range of abilities. Indeed, as one principal said, "Numero transforms the attitudes of students, not just towards mathematics, but towards school as a whole." One classroom teacher enthused, "My children do more arithmetic calculations in a 15-minute game of Numero than they do in a week of traditional [mathematics]."

It is no surprise then to read from W. Edwards, Numeracy Consultant for the Royal Boroughs of Windsor and Maidenhead in the United Kingdom, "Children really enjoy playing Numero without realising they are practising mental calculation strategies. To them it is just a great game, while teachers are finding that Numero can play a key role in achieving the mental calculation abilities required by the latest National Numeracy Project."

In Western Australia, A. Newhouse (a Mathematical Association of Western Australia committee member) says,

> I have used Numero with students for six years. When played regularly, it becomes a strategy for teachers to develop the number outcomes from the Curriculum Framework. Concepts such as fractions can be easily introduced without the anxiety which often accompanies this aspect of mathematics. Numero is a strategy game, and so it is ideal for developing lateral thinking skills, achieving confidence and self-motivation for their learning. Students really enjoy playing Numero.

Gough (2000) refers to "the latest math game rage from the US" that had been highlighted in recent TV programs. While acknowledging that children enjoy these games, he goes on to say that "the novelty wears off quickly, and waiting for your turn is very time consuming. The materials are very expensive and not easy for children or parents to find." He continues, "All the advantages of [these games] are contained in an Australian card game, Numero, that practises arithmetical thinking, using a special pack of number cards and an easy rummy-like playing method."

In light of her earlier comments about the need for fostering positive attitudes while improving children's numeracy, Judy Anderson (1995b), reviewing Numero for the journal of the Mathematical Association of New South Wales, writes,

> Numero is an innovative card game, designed to provide students with practice in basic number facts, as well as allowing them the opportunity to develop strategies and problem solving skills. Numero provides students with a motivating way to practise their basic number facts, using mental computation, rather than relying on a calculator. I believe that all junior secondary students, as well as primary students, will benefit from playing this game as it would certainly enhance their ability to calculate mentally, develop problem-solving strategies, and make quick decisions.

Let's return to my opening question: Is playing games a waste of time in a mathematics lesson? No— far from it! In fact, it seems to be the best way to break the relationship between improving numeracy and increasing negative attitudes toward mathematics. As C. Serravite, principal of Amaroo Primary School in Collie, Western Australia, says, "Numero is the highlight of every mathematics lesson. It challenges all students, as it allows them to progress at their own pace, as well as having mathematical fun. Numero has become an integral part of the curriculum."

Numero is useful at all levels of primary school and lower secondary school. It can be easily learned even by five-year-olds. Although the few simple rules (which are set out clearly in the instructional guide and video) never vary, the advanced levels can be extended to be more suitable for secondary and even tertiary students, as well as teachers. Numero also fits into the nonacademic stream for upper secondary students, where something is needed to help improve the numeracy levels of students who are bored of and antagonistic toward the methods that have failed them throughout their earlier years.

Numero allows all students in a class to play at levels commensurate with their ability. Within seconds of the packs being distributed, with a minimum of preparation, students are engaged in the game. The teacher is free to move around the class, helping when necessary and motivating students to extend themselves.

Numero can be used to teach and practise all basic arithmetic operations, fractions, decimals, percentages, powers and indices. Children often find themselves mastering such things as multiplying negative numbers long before they are expected to do so.

As one school principal said to his staff before a Numero professional development workshop, "Be warned! Numero is addictive!"

For more information about Numero, contact Ernest Klassen, a mathematics consultant and a teacher at Woodlawn School in Steinbach, Manitoba, at ernestk@mb.sympatico.ca. A student from Australia brought Numero to Ernest Klassen's classroom, and Klassen has demonstrated the game to teachers in Manitoba and Alberta.

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The two sides of a rectangle are *a* metres and *b* metres. If each side is increased by 10 percent of its original length, by what percentage does the area increase?

Ages of John's Three Daughters

Klaus Puhlmann

My friend John and I were traveling on the train from Edson to Vancouver to attend a mathematics conference. Because we had the compartment to ourselves, we could talk about anything and everything. We discussed the math sessions we intended to attend, but soon our conversation entered the personal realm.

John spoke about his family and how proud he was to be the father of three daughters. "How old are your daughters?" I asked John. Rather than answering my question readily, John instead created a mathematical problem for me—with the offer to pay for my meal if I could tell him how old his three daughters are.

John stated, "I have three daughters. The product of their ages is 36; the sum of their ages is my house number. How old are my daughters?" I thought for a while and said, "This isn't enough information to answer your question." John said, "You are right. I forgot to tell you that my oldest daughter and I went to the hospital last night to visit my mother-in-law." This information did not help me at first, but after some brooding and racking my brain, I recognized its importance and was able to determine the ages of John's three daughters.

Do you know how old the three daughters of my friend John are?

Unsolvable Problems: Trisection of an Angle

Klaus Puhlmann

We have all learned in school how easy it is to bisect any angle with a straightedge and compass. However, trisection using a straightedge and compass is impossible for some angles, as P. L. Wantzel proved in 1847.

Certain specific angles can be easily trisected, but there is no general procedure that permits the construction, with Euclidean tools, of an angle that is exactly one-third of a given arbitrary angle. For example, because a 30° angle is easily constructed using only a straightedge and a pair of compasses, a 90° angle can be trisected.

We know that an obtuse angle can always be divided into one or more right angles and an acute angle. Hence, the problem of trisecting an arbitrary angle can be reduced, without loss of validity of the generality of the claim, to the task of trisecting an acute angle.

Let us assume we have an angle $3\theta = 60°$, which is to be trisected. From our knowledge of trigonometry, we know that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ is a trigonometric identity. As we insert $\cos 3\theta = 60° = 1/2$, and substitute $\cos\theta = y/2$, we get $3\cos\theta = 3y/2$ and $4\cos^3\theta = y^3/2$, from which the irreducible cubic equation follows: $y^3 - 3y - 1 = 0$. The roots of this cubic equation cannot be constructed using Euclidean methods. We must therefore conclude that cos 20° is not constructible. Because the angle 60° cannot be trisected, we have thus produced one example that allows us to conclude that it is impossible to trisect an arbitrary angle using a straightedge and compass only.

The first attempts to trisect a general angle arose so long ago that historians are unable to find a record. Comfort with bisecting angles led naturally to attempts at trisecting angles. Only after many attempts at trisecting a general angle, restricted of course to the classical rules and tools, was it apparent that some other mathematical principles blocked the success of this endeavor.

Hippias of Elis, who lived in the fifth century B.C., was one of the first to attempt to solve the trisection problem. Frustrated, he devised a curve called the quadratrix, which allowed him to give an exact solution to the problem. However, this was not achieved with the use of a straightedge and a compass alone; it involved what is often referred to as a nonclassical solution. The history of the trisection problem reveals other nonclassical solutions.

Many people have been drawn to this powerful and fascinating problem, all attempting to solve it only to discover that trisecting a general angle using only a straightedge and compass is impossible. The hundreds of attempts in the past and present have shown that it is impossible to find, by a straightedge and compass construction, a root x of the trisection equation. This and the other unsolvable problems, when stripped of all implications, are hardly worth more than passing attention, but they have yielded fruitful discoveries in other mathematical fields.

Archimedes, Niomedes, Pappus, Leonardo da Vinci, Dürer, Descartes, Ceva, Pascal, Huygens, Leibniz, Newton, Maclaurin, Mascheroni, Gauss, Steiner, Charles, Sylvester, Kempe, Klein, Dickson—all of these, and hundreds more, have attacked the trisection problem directly or created mathematics by which substantial advances could be made toward a full understanding of the situation. Those who are still determined to show that the trisection of any angle is possible using only a compass and straightedge are well advised to first examine the existing proofs to see if mathematical mistakes have been made.

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