# Understanding Student Responses to Open-Ended Tasks

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Communicating mathematical knowledge is a challenge for students, and assisting students in clearly expressing their mathematical ideas is a challenge for teachers. Open-ended tasks give students opportunities to select their own approaches for both solving problems and expressing mathematical ideas (Billstein 1998; Conway 1999). Students' responses to these tasks give teachers evidence of their students' problem-solving and communication skills.

This article discusses examples of the detailed explanations that students offered in response to a written, open-ended geometry task, especially the ways in which students communicated their knowledge. As this article illustrates, different students may select different methods of communication, such as using text, diagrams or mathematical symbols, to display their solution processes. Examining the student solution processes helps the teacher better understand the students' mathematical knowledge.

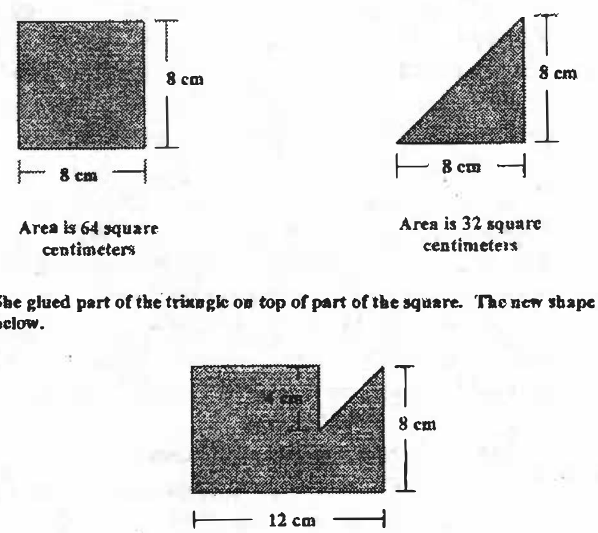
## Student Responses

The irregular area task, shown in Figure 1, was administered to sixth-grade students (Moskal 1997a, 1997b) after they had received instruction in determining the area of squares, rectangles and triangles. The Irregular Area Task was originally developed as part of the Quantitative Understanding: Amplifying Student Achievement and Reasoning (QUASAR) project (see Lane et al. 1995). Although a classroom of students completed this task, only four responses will be discussed here. To maintain confidentiality, both the students' and the teacher's names have been changed.

At the start of the area unit, Ms. Harding showed her students how to subdivide a rectangular figure into unit squares. She explained to them that the total number of unit squares that made up the rectangle was the *area* of the rectangle. Building from this conceptualization of area, she derived with her students the standard formula for finding the area of a rectangle, that is, base × height. She used a similar approach to

**Figure 1**
**Irregular Area Task**

**Teressa was working on an art project. Shown below are a square and a triangle she cut out.**




**What is the area of the new shape? Explain how you found your answer. You may use the drawings in your explanation.**

**Area of new shape: ______________ square centimeters**

develop the formula for finding the area of a triangle. Harding dedicated a great deal of classroom attention to explaining how portions of unit squares could be combined to create complete unit squares when finding the area of a triangular figure. Before administering the Irregular Area Task, however, Harding had not addressed how the area concept could be extended to find the area of figures other than squares, rectangles and triangles. Harding had previously used open-ended tasks in the classroom, and she repeatedly encouraged her students to write complete solutions to open-ended mathematics problems.

## Naser's Response

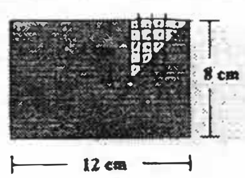
Naser, whose response is shown in Figure 2, used a combination of text and diagrams to explain his answer. His explanation suggests that Naser was attempting to reshape the irregular figure into a large rectangle with a base of 12 cm and a height of 8 cm. He indicated that he has calculated the area of the larger rectangle, but he did not include the outcome of this computation.

Although Naser's overall approach is correct, reforming the figure into a rectangle and removing the area of the missing triangle, he made several mistakes in the solution process. Then he explained that he had subtracted 15 square units from the rectangle to get his answer of 81. Fifteen square units was his calculated area for the missing triangle in the rectangle. Naser had drawn three vertical lines and four horizontal lines, partitioning the square into 20 rectangular regions (see Figure 2). If Naser's goal was to partition the square into unit squares, he should have drawn three lines vertically and three lines horizontally, which would have resulted in 16 square units. Small pencil dots, visible in each rectangle, make up the missing triangular region. Naser apparently counted the rectangles in the missing triangle to arrive at the value of 15. When Naser counted the rectangles in the missing triangle, he may have treated the segments of whole units along the hypotenuse of the triangle as whole units. This scenario would account for an area of 15 $cm^2$, rather than 10 $cm^2$, on the basis of a right triangle with a base of 4 and a height of 5.

Harding expressed disappointment in Naser's response. She was surprised that Naser was able to develop such an "elegant" solution yet not able to find the area of the missing triangle. When asked what made Naser's response "elegant," Harding explained, "He knew to create a larger rectangle. He knew to get rid of the little piece [referring to the triangle] . . . it was pretty good [referring to the solution]." Harding further explained that during instruction, they had not discussed how to find an unknown area by breaking the unknown region into parts with known areas. Naser's response demonstrated that he was able to extend the concepts beyond classroom instruction.

Harding also expressed frustration with a portion of Naser's response. Naser had counted the units along the hypotenuse of the right triangle as whole units. The concept of combining half units to create whole units had been directly addressed in class. Several other students had also made this mistake. Harding believed that the recurrence of this error suggested that her efforts to clarify the difference between half and whole units had been ineffective for many students.

**Figure 2**
**Naser's response**




**What is the area of the new shape? Explain how you found your answer. You may use the drawings in your explanation.**

I made the picture one big rectangel. Than I × 12×8 then I went back to the regular shape and I − 15 square area because thats what was the amont of the little shape in the rectangel and thats how I got my answer 81

**Area of new shape:** 81 **square centimeters**

## Kevin's Response

Kevin reproduced the irregular figure and labeled the length of each side (see Figure 3). The labeling on the text bubbles of Kevin's explanation suggests that he was aware that the sides of a square are congruent. Kevin's response also contains errors. For example, he argued that the hypotenuse of the removed triangle was 4 cm because it was the same as the leg of the missing triangle. He then added the sides of the irregular figure, and because of his incorrect work, obtained a sum of 44 as the result. Kevin found the perimeter of the irregular figure rather than the area. He also explained, "and then I multiply that by 2." This incorrect step results in his final answer of 88. Kevin may have remembered from class that calculating the area involved multiplication. By multiplying by 2, Kevin forced multiplication to appear in his work. Although his final answer to the problem was correct (that is, 88 $cm^2$), he acquired this answer through a series of errors.

At first, Harding was pleased with Kevin's response. She said, "Kevin does not always understand math, but this time he got it." She also said that she was impressed by the detail that he provided in his response. On closer inspection, however, Harding realized that her initial evaluation was inappropriate. She had seen the correct answer and the "detail that filled the page" but had not examined the contents of the response. After reading through Kevin's response, she concluded, "He thinks he is finding a perimeter." Harding also expressed concern about Kevin's efforts to multiply by 2. She did not understand why Kevin would include this step and intended to ask him to explain his work orally the next day. If Kevin had given only his final answer, his teacher may have assumed that he had a firm understanding of the concepts being assessed.

**Figure 3**

**Kevin's response**

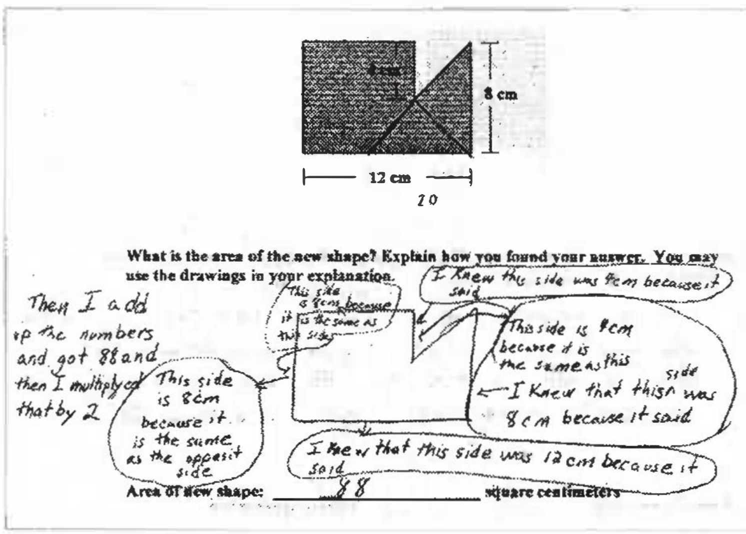
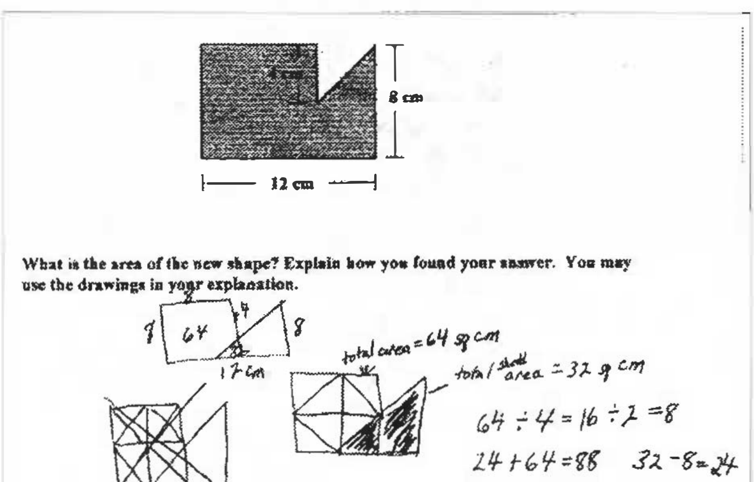



**Figure 4**

**Ning's response**




## Ning's Response

As shown in Figure 4, Ning created a series of diagrams to explain her response. She first divided the irregular figure into the overlapping square and triangle. She labeled each subdivision with the given areas. She then divided the square into four smaller squares with equal areas. She divided each resulting square in half to form two triangles with equal areas. The process that Ning used is clarified through the accompanying calculations. Using symbols, Ning divided the area of the large square (64) by 4, resulting in a value of 16. She then divided the area of the small squares (16) by 2, giving her the area of the small triangles (8). The supporting calculations clarify her pictorial representation.

Ning's next step was to remove the area of overlap (8) between the given square and triangle from the area of the given triangle (32). Apparently, Ning recognized that this amount is accounted for in the areas of both the large square and the large triangle. Finally, she summed the resulting value with the area of the large square, obtaining the correct answer of 88.

Ning's teacher reacted to her response with great surprise. She said, "Ning's English is so flawed; I really don't have a grasp of what Ning knows and doesn't know." Ning's family has recently moved to the United States, and her English skills were not well developed. From

Ning's response, Harding decided, "I need to provide her with more opportunities to explain herself through pictures. It is clear to me now . . . this might be a way to get beyond her language problems." Although Ning's response contained few words, it conveyed a great deal of understanding. Through the use of diagrams and mathematical symbols, Ning was able to effectively communicate her reasoning process.

### Becky's Response

Becky also produced a correct solution in response to the Irregular Area Task. Her explanation is shown in Figure 5. Although she wrote that she had drawn 12 horizontal lines and 8 vertical lines on the diagram of the irregular figure, Becky had drawn 11 vertical lines that partitioned the region into 12 vertical sections and 7 horizontal lines that partitioned the region into 8 horizontal sections. Although Becky incorrectly conveyed her approach in her written response, the diagram that she provided clarified the process she had used.

Becky explained that she had counted 8 unshaded squares. Then she subtracted these 8 from 96, the area of the whole figure. On the left side of Becky's paper, she wrote the calculation 12 × 8 and the resulting answer of 96. Becky's explanation states that she counted the units that made up the missing triangle. Examination of her diagram suggests that she also counted the number of units in the larger rectangle. In Becky's drawing, the large rectangle is subdivided in 96 unit squares. Each of these unit squares contains a pencil dot. It seems that Becky either multiplied 12 × 8 to verify the number of unit squares that she counted or counted the number of unit squares to verify her multiplication.

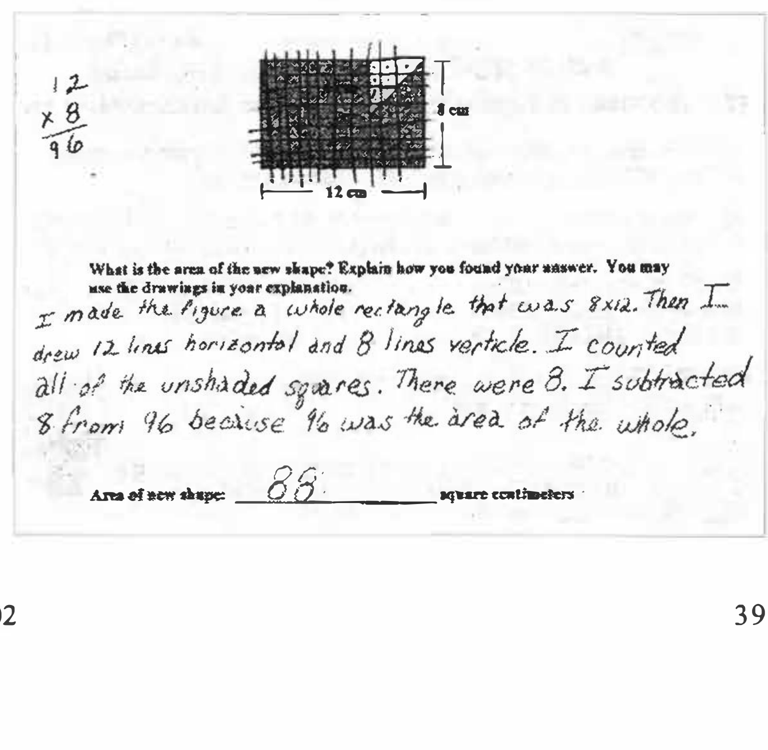
To explain how she acquired the area of the missing triangle (8), she wrote that she had counted the unshaded squares. This explanation does not appear to be supported by her diagram. In the diagram, the missing triangle is half of a 3 × 4 rectangle that has an area of 12. Half of this area results in an area of 6 for the missing triangle. However, if all the segments in the 3 × 4 rectangle are counted, including those along the hypotenuse, then the total is 16 segments. Half of the 16 is 8. Becky's determination that the area of the missing triangle is 8 may have been the result of a misconception concerning how to deal with the units that are divided along the hypotenuse of the triangle. Becky also had a partially erased calculation of 96 − 16 = 80 written on her paper. Examining the answer space suggests that prior to an erasure, Becky had written 80 in the answer space. It is possible that Becky has originally sought to subtract what she believed to be the area of the 3 × 4 rectangle.

Harding thought that Becky had effectively conveyed a "basic" understanding of the area concept. Harding said, "I should be happy with this. But Ning's response was so good . . . Becky is still counting." She explained that many of her other students were also "still counting." She expressed concern that her students would choose to count 96 square units rather than multiply the base by the height. In the future, Harding intended to create a task about a figure with a large enough area that counting would not be possible. She hoped that this type of activity would convince her students that benefits were to be found using multiplication in determining an area. Harding did not identify Becky's potential misunderstanding of how to account for the units along the hypotenuse of a right triangle when finding the area.

## Summary

As these examples show, students as young as sixth grade are capable of providing detailed written explanations that reflect their mathematical reasoning. A teacher should not, however, expect this type of detail the first time that students address

**Figure 5**
**Becky's response**



open-ended tasks. These examples were collected at the end of sixth grade after Harding had emphasized clarity in writing throughout the year. When her students submitted incomplete or unclear written responses, Harding provided written or oral feedback that indicated which portion of the response required further elaboration. Examining the explanations gave Harding evidence of the diverse levels of understanding among her students. Kevin's solution contained a series of errors, including an attempt to find the perimeter of the irregular figure. On the basis of a superficial observation of a correct answer and an explanation that "filled the page," Harding originally concluded that Kevin had a complete understanding of the area concept. After closer inspection, she realized that Kevin was attempting to find the perimeter rather than the area of the irregular figure. When Kevin found an incorrect value for the perimeter, he multiplied it by 2. This example raises an important concern when examining student responses. When students are asked to provide explanations, teachers should take time to read and make sense of their work.

Ning, Becky and Naser all used reasonable approaches to acquire their answers. Ning used multiplication and division to find the area of the irregular figure. Becky used two approaches. In one approach, she attempted to decompose the irregular figure into unit squares in order to count the unit squares. In the other approach, she used multiplication to find the area of the larger rectangle and subtracted what she thought was the area of the missing triangle from this value. Harding did not acknowledge the approach that used multiplication; instead, she expressed disappointment that Becky was still counting. Naser used multiplication to find the area of the large rectangle and a counting strategy to find the area of the missing triangle. Harding was satisfied with Naser's overall approach but was disappointed that he was unable to find the correct area of the missing triangle. All three of these students exhibited flaws in their communications, but their overall explanations offered clear indications of their reasoning processes.

Using text, diagrams and symbols in their responses to this task supported the students' communications. For example, Ning had recently moved to the United States, and her English skills were not well developed. Through the use of symbols, diagrams and some text, Ning was able to provide a convincing argument that supported her correct answer. Text and diagrams are also likely to be appropriate methods for younger students to use to communicate their knowledge. In elementary and middle school, students are still developing their writing skills. The use of diagrams and symbols offers these students additional tools for expressing their knowledge.

Cohen and Fowler (1998) have argued that assessments should elicit evidence not only of what students can do but also of what they understand. The detailed explanations offered by her students allowed Harding to evaluate their understanding of the area concept. Time, practice and feedback had given Harding's students the opportunity to develop their written communication skills. By examining their explanations, Harding saw their varying levels of understanding. Naser, who gave an incorrect answer, displayed greater knowledge of the area concept than did Kevin, who gave the correct answer. The freedom to use text, diagrams and symbols in response to an open-ended task supported these students as they displayed their mathematical knowledge.

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