Using Fractions and Series to Solve Problems in a Recursive Setting

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Teachers are always seeking ways to connect algebra to real-world situations. The fraction $\frac{1}{1-x}$ and its associated series provide an opportunity to develop such connections. We will develop these connections in three situations:

Situation 1

The manager of a power plant wishes to sell 1 unit of power. The plant itself requires power to operate. In particular, let x be the fraction of the power produced internally, which is needed to keep the plant functioning. For instance, if $x = 1/10$, then 1/10 of the power is used internally while 9/10 may be sold. The question is: How much power must be produced in order to have 1 unit available for sale?

Solution 1

The manager first decides to produce exactly one unit of power; however, he realizes that x-units were needed to produce this power, leaving only $(1 - x)$ units available for sale. To make up the deficiency, he orders x additional units produced. He then learns that $x \cdot x$ or x^2 units were used to produce the x additional units. Of this supplemental production, only $x - x^2$ units are available for sale. The total amount thus available for sale is $(1 - x) + (x - x^2) = (1 - x^2)$.

He now orders that x^2 additional units be produced but finds that x^3 were used in its production. The total amount now available for sale is $(1 - x^2) + (x^2 - x^3) = (1 - x^3)$. Table I shows what would happen if the manager continues to employ this sequence of recursive production steps.

The manager thus determines that it is necessary to produce $1 + x + x^2 + x^3 + x^4 + \ldots$ units of power in order to have 1 unit available for sale.

Situation 2

A woman goes to the bank and wishes to borrow money for one year. Assume that the interest rate is $100x$, where x is the decimal notation and

Table I

No. Units Produced	Cumulative Produced	No. Units Used Internally	Cum. Used Internally	No. Units for Sale	Cum. Units for Sale
1	1	x	x	$1 - x$	$1 - x$
x	$1 + x$	x^2	$x + x^2$	$x - x^2$	$1 - x^2$
x^2	$1 + x + x^2$	x^3	$x + x^2 + x^3$	$x^2 - x^3$	$1 - x^3$
x^3	$1 + x + x^3 + x^4$	x^4	$x + x^2 + x^3 + x^4$	$x^3 - x^4$	$1 - x^4$
•	•	•	•	•	•
•	•	•	•	•	•
•	•	•	•	•	•

$100x$ is percent notation. Suppose that the interest is deducted in advance; that is, if the interest rate is 10 percent ($x = 0.1$) and the woman wishes to borrow \$1, the bank will deduct \$.10 in advance and give her \$.90. She would be required to repay \$1 at the end of the year. The question is: How much money should the woman borrow in order to actually receive \$1?

Solution 2

The borrower first secures a loan of \$1; however, x dollars of interest is deducted in advance. She therefore receives $(1 - x)$ dollars. To make up this deficiency, she then borrows x additional dollars; but $x \cdot x$ or x^2 dollars is deducted in advance from her new loan so she receives only $(x - x^2)$ dollars. Her total proceeds are thus $(1 - x) + (x - x^2) = (1 - x^2)$ dollars.

If she then borrows x^2 dollars to make up this new deficiency, she will receive $(x^2 - x^3)$ dollars, giving her cumulative proceeds of $(1 - x^3)$ dollars.

Table II reveals the results if the borrower continues to employ this sequence of loan operations.

The woman thus determines that it is necessary to borrow $(1 + x + x^2 + x^3 + x^4 + \ldots)$ dollars in order to receive \$1 from the bank.

Situation 3

A cat is chasing a mouse. The ratio of the speed of the mouse to the speed of the cat is x, where $x < 1$. If the cat is 1 metre away from the mouse when the chase begins, how far will the cat have to run to catch the mouse? (Assume they run in the same direction.)

Solution 3

The cat first runs 1 metre, but during this time, the mouse has run x metres away from the cat. The cat then runs x metres more while the mouse is running $x \cdot x$ or x^2 metres away from the cat. To make up this x^2 distance, the cat then runs x^2 metres

Table II

No. Dollars Borrowed	Cumulative Borrowed	No.Dollars Deducted for Interest	Cum. Interest Deducted	No. Dollars Received	Cum. Dollars Received
1	1	x	x	$1- x$	$1 - x$
x	$1 + x$	x^2	$x + x^2$	$x - x^2$	$1 - x^2$
x^2	$1 + x + x^2$	x^3	$x + x^2 + x^3$	$x^2 - x^3$	$1 - x^3$
x^3	$1 + x + x^3 + x^4$	x^4	$x + x^2 + x^3 + x^4$	$x^3 - x^4$	$1 - x^4$
•	•	•	•	•	•
•	•	•	•	•	•
•	•	•	•	•	•

Table III

No. Metres Run—Cat	Total Distance Run—Cat	No.of Metres Run—Mouse	Total Distance Run—Mouse	No. Metres Gained—Cat	Total Distance Gained—Cat
1	1		x	$1 - x$	$1 - x$
x	$1 + x$	x^2	$x + x^2$	$x - x^2$	$1 - x^2$
x^2	$1 + x + x^2$	x^3	$x + x^2 + x^3$	$x^2 - x^3$	$1 - x^3$
x^3	$1 + x + x^3 + x^4$	x^4	$x + x^2 + x^3 + x^4$	$x^3 - x^4$	$1 - x^4$
•	•	•	•	•	•
•	•	•	•	•	•
•	•	•	•	•	•

but finds that the mouse has run x^3 metres further. Table III reveals what would happen if the cat continues to employ this sequential chasing technique.

The cat must run the distance $1 + x + x^2 + x^3 + x^4 + \ldots$ to catch the mouse. (The problem recalls Zeno's Paradox.)

The reader will note that all of these problems have equivalent algebraic formulations. The amount of power to be produced, the amount of money to be borrowed and the distance to be run by the cat are all $1 + x + x^2 + x^3 + x^4 + \ldots$. This is a geometric progression; polynomial division will reveal the progression to be equal to $\frac{1}{1-x}$. The alert reader may have noticed that each of these three problems could have been solved more easily using the fraction form of this equality. For example, if x is the fraction of power used internally, then $(1 - x)$ is the amount available for sale. Thus, if a is the amount of power to be produced to sell 1 unit, we have the equation $(1 - x)(a) = 1$, or $a = \frac{1}{1-x}$. Problems 2 and 3 have similar fractional analysis solutions.

In the particular case for which $x = 10\% = 0.1$, $a = \frac{1}{1-0.1} = \frac{1}{0.9} = 1.1111\ldots$.

Challenges for the Reader

At 12 o'clock, the hands of the clock coincide. When will this next happen? The solution to this problem is again the algebraic solution $a = \frac{1}{1-x}$ or its associated series.

Is 2 = 3?

Here is the sequence of steps:

$4 - 10 = 9 - 15$

$4 - 10 + \frac{25}{4} = 9 - 15 + \frac{25}{4}$

$(2 - \frac{5}{2})^2 = (3 - \frac{5}{2})^2$

$2 - \frac{5}{2} = 3 - \frac{5}{2}$

$2 = 3$

Which step contains the mistake? Explain.