Babylonian Mathematics in Cuneiform II

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The similarities between our number system and that of the Babylonians are several. Like them, we use a finite number of symbols or digits (we use 10) to express all integers; and we, too, assign importance to the position of a digit, so that for every place it is moved to the left, its value is multiplied by a constant factor (10 for us, 60 for the Babylonians). Like them, we make use of an extension of this rule to express certain fractions (decimal fractions in our case)—that moving a digit one place to the right means to divide its value by the constant factor 10 or 60. The numbers 10 and 60, which play such an important role are the bases for the two number systems, which are called the decimal and the sexagestimal system.

The differences between the two systems are the Babylonian base 60 and the absence of the equivalent of the decimal point in the sexagesimal system.

There is nothing especially outstanding about the numbers 10 and 60. Our predecessors' choice of 10 is just a matter of coincidence, and though the Babylonians were not above counting on their fingers, as we can conclude from their special sign for 10, their choice of 60 as a base also had its motivation outside mathematics. It is not hard to prove that any integer n greater than 1 can serve as the base of a positional or place-value number system (as we call a number system with the common characteristics of the decimal and the sexagesimal number systems).

In such a system, we will need n different symbols or digits whose principal values are 0, 1, 2, . . . , $n - 1$. To move a digit one place to the left will mean to multiply its value by n and to move it one place to the right, even beyond the units' place, will mean to divide its value by n.

We show this by using binary system, as an example. We then have two digits, 0 and 1. The first 10 numbers are written in this system. Thus:

1, 10, 11, 100, 101, 11, 1000, 1001, 1010

In order to translate the binary number 1001011 into decimal notation, we observe that

$1001011 = 1 \cdot 2^6 + 0 \cdot 2^5 + 0 \cdot 2^4 + 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2 + 1 = 75$

If we are to write, say, the number 308 (base 10) in binary form, we see that 308 lies between the following two consecutive powers of 2:

$2^8 = 256$ and $2^9 = 512$

and so

$308 = 2^8 + 52.$

Now, 52 is between

$2^5 = 32$ and $2^6 = 64$

and so

$52 = 2^5 + 20.$

Similarly,

$20 = 2^4 + 4 = 2^4 + 2^2,$

and so

$308 = 2^8 + 2^5 + 2^4 + 2^2$

$= 1 \cdot 2^8 + 0 \cdot 2^7 + 0 \cdot 2^6 + 1 \cdot 2^5 + 1 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2 + 0$

This, written in binary form becomes 100110100.

In our binary example, the multiplication and addition tables are as simple as can be:

·	0	1
0	0	0
1	0	1

+	0	1
0	0	1
1	1	10

Accordingly, a binary multiplication is carried out thus:

$$\begin{array}{r}
1101 \\
\underline{110} \\
0000 \\
1101 \\
\underline{1101} \\
1001110
\end{array}$$

We can now return to the easily seen differences between the sexagesimal and the decimal systems. It should be clear that the base 60, though unfamiliar, does as well as 10. Each base has its advantages and disadvantages. An obvious disadvantage of the larger base 60 is that a multiplication table has a size (59 by 59) that practically prohibits memorization. On the other hand, it is possible to write large numbers with few sexagesimal digits.

One further advantage of the Babylonian base is that more fractions can be written as finite sexagesimal fractions than can be written as finite decimal fractions. I have already described such fractions in my previous article, "Babylonian Mathematics in Cuneiform I" (the section on the reciprocal table) but it is natural to ask the more general question: When does a reduced fraction (namely, in lowest terms) p/q have a finite expansion in a number system with the base n?

A finite decimal fraction can be thought of as a fraction whose denominator is a power of 10, and a finite sexagesimal fraction as one whose denominator is a power of 60. Similarly, a finite fraction in any other number system with the base n is a fraction whose denominator is a power of n. Our question is then: When can a reduced fraction p/q be turned into a fraction with denominator n^x? Since we can only change the denominator of a reduced fraction by multiplying both the numerator and denominator of the fraction by some integer, the answer is that p/q can be turned into a fraction p/n^x precisely if the denominator q contains only prime factors that are also in n^x, and therefore in n.

So, since 2 is itself a prime, the only reduced fractions that can be written as finite binary fractions are those whose denominators are already powers of 2. Those that can be turned into finite decimal fractions are the ones whose denominators have no other prime factors than 2 and 5, since $10 = 2 \cdot 5$. But since $60 = 2^2 \cdot 3 \cdot 5$, the allowable prime factors for finite sexagesimal expansions are 2, 3, and 5. Thus, if we consider the denominators 2, 3, 4,...20, only four of them will produce finite binary fractions and seven will give finite decimal equivalents, while 13 have finite sexagesimal expansions.

The other major difference—namely, the absence of the equivalent of the decimal point—is, to be sure, a flaw in the sexagesimal system. Yet it is not as serious as one might think at first. We only need to remember that when we are concerned with multiplication and division of decimal fractions, we can forget about the decimal points. After all, they have no influence on the sequence of digits in the result, but control only its size. In fact, when we use a slide rule or look up the logarithm of a number (the power to which a fixed number, usually10, must be raised in order to produce a given number), we are in a situation not too different from that of the Babylonians. We only get the digits of the answer and then have to decide the position of the decimal point. At any rate, this deficiency is a small price to pay for the enormous advantage that operations with fractions are usually no more complicated than those with integers.

The origin of the sexagesimal system is not certain. We know, however, that in early times there was a system of weights and measures whereby the larger unit was 60 times smaller. It was customary to write a measure of, say, 72 smaller units as a large 1 followed by a small 12; this represented 1 large and 12 small units. The number 60 may have become important because the principal unit of weight for silver—the mana—was subdivided into 60 shekels. This may have given way to the consideration of sixtieths as a natural subdivision of units, and the preference for the base 60 in general.

Babylonian Arithmetic

One disadvantage of a base as large as 60 is the large size of the multiplication table showing the products of any two one-digit numbers. One may tremble to imagine Babylonian schoolboys trying to memorize a 59 by 59 multiplication table. But we have found quantities of tables of various kinds, including multiplication tables, so it is clear that such memorization was unnecessary.

This is not to say that we have tablets containing the 59 times 59 products, for we do not. What we find are many 9-table type tables arranged according to multiples of p:

1	p
2	$2p$
3	$3p$
.	.
.	.
.	.
19	$19p$
20	$20p$
30	$30p$
40	$40p$
50	$50p$

and sometimes ending with p. We call p the principal number of the multiplication table.

From this table, any multiple of p can readily be found—$47p$ is simply the sum of $40p$ and $7p$. One might think that there were 59 such tables with p = 1, 2, 3,...59. But what we actually find is a selection of principal numbers, which at first is quite puzzling. We have, for example, a multiplication table with p = 44, 26, 40, an enormous number, but none for p = 17. It is the presence of this curious principal number 44, 26, 40 that makes the puzzle pieces fall into place, for 44, 26, 40 is the last number in the standard reciprocal table in Figure 2 in Babylonian Mathematics I, which we have discussed before. It seems that the principal numbers are essentially the numbers we find in a standard reciprocal table. One exception is 7 which quite naturally appears as a principal number even though it is absent from the reciprocal tables, and so the product of any two numbers can quite easily be found. The coincidence of principal numbers and the numbers in the reciprocal table gives a clue as to how the Babylonians actually computed. It is quite clear that the reciprocal table, combined with these multiplication tables, also served for divisions, since r divided by n is r multiplied by the reciprocal of n, or $(\frac{r}{n}) = r \cdot (\frac{1}{n})$.

In our decimal system, we have a variety of rules and shortcuts that make computation easier: to multiply by 5 we divide by 2 and multiply by 10; a number is divisible by 3 (or 9) if the sum of the digits is divisible by 3 (or 9). If one works consistently with the sexagesimal system, one soon finds many such simple devices. Many more rules are possible in the sexagesimal than in the decimal system since the base 60 has so many divisors.

Sexagesimal calculations were further assisted by quite a large variety of tables. We find extended reciprocal tables—even giving the reciprocals to several places—of numbers such as 7 and 11 whose reciprocals do not have finite sexagesimal expansions. There are tables for the computation of compound interest, of squares and square roots, of cubes and several complicated tables that indicate an interest in numerical procedures far beyond the requirements of simple arithmetic.

Thus, it is perfectly clear that the Babylonians found no more difficulties in arithmetical computation than we do today. In this respect they were unique in the classical world, and it is therefore not surprising that when Greek astronomy had reached the stage where extensive calculations were called for, the Greek astronomers turned to the sexagesimal number system for a sensible way of expressing fractions.

This is the reason Babylonian fractions are used even now, for example, in the subdivision of degrees and hours—the units for measuring angle and time, the two basic quantities observed in classical astronomy. The Greeks wrote the measure of angles using the Babylonian system, and so do we when we write 120°12' 20". When we say the time is 2 hours, 30 minutes and 10 seconds, we are actually using the terminology of the Babylonians of 4,000 years ago who would have said, somewhat more simply, that 2, 30, 10 hours have passed since noon.

Bibliography

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Leonhard Euler (1707–1783)

A farmer wanted to buy horses and oxen for a total of 1,770 coins. For a horse he paid 31 coins and for an ox 21 coins. How many horses and oxen did he buy? (Is there more than one solution?)