In this section, we will share your points of view on teaching and learning mathematics and your responses to anything contained in delta-K. *We appreciate your interest and value the views of those who write.*

Mathematics with Joy

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Each term when I come to a classroom to teach basic level mathematics (for example, calculus, precalculus and so on), I can see fear and horror in my students' eyes. Their bodies are motionless. They are expecting mental torture, as one of my students said. They are in this room only because they have to be here. They have to take this class to get their degree. Sometimes it is their very last class because they were so afraid of taking it before. Sometimes it is their n-th attempt to pass it, $n > 1$. They are coming to this class thinking they will never understand math. It takes some time before students become involved in class discussions, and then they want to continue learning more about their favorite mathematics. I can finally see smiles on their faces. Our class starts out as one that feels dreadfully long but becomes one where time passes like a blur. My guess is that the change happens when students quit being behind and start moving ahead. They start to believe in themselves. I am waiting for this faith to develop in every course because I need it. Faith is necessary for victory.

Learning any mathematical subject can be enjoyable and relaxing. Solving mathematics problems can be amusing and delightful. Mathematics and fun go together naturally. I find it useful to bring interesting and nontrivial problems to class that are related to a particular topic and ask them to solve them when they have time. Each of the problems has a simple but gorgeous solution. Students always find the time to solve interesting problems that demand creativity. They do not give up and will remain engaged in this once they start. Of course, I always give bonus points to their final score if they succeed. This is worthwhile to me because getting students to think about these problems and having them arrive at the correct answers will lead students to discover how awesome and exciting mathematics really is. They may even find that they have learned so many new ways of solving mathematics problems that their skills have improved and their enjoyment of mathematics has increased.

Warm-Up Questions

At the beginning of each class, we play brain warm-up games with three types of questions:

1. Simple true and false
2. Statements that can be true or false, with the goal of finding out when these statements are true
3. Small problems with short answers

These questions help students review previous concepts and prepare them for understanding new ones. All tasks require a quick response and good understanding of the material studied. Knowing these facts also helps with problem solving.

Here are some questions that cover the three problem types identified above. A solution key is provided to show different properties touched on in each question. The following sample questions are restricted to relations and functions.

True/False Questions (Type 1)

Statement: If $y = f(x)$ is even, then $y = f(-x)$ is odd.
Answer: False, $f(-x)$ is even too: $f(-x) = f(x)$.
Statement: The inverse of an even function is even.
Answer: False, an even function does not have an inverse at all. It's not one-to-one.
Statement: $f(x)$ times $f^{-1}(x) = x$.
Answer: False (it is true for the composition $f(x) \circ f^{-1}(x) = x$ of these two functions).

When True/When False Questions (Type 2)

Statement: If $y = f(x)$ is defined on a set of real numbers, then $y = | f(x) |$ is always even.
Answer: False. It's true when $f(x)$ is odd or even.
Statement: If $y = f(x)$ is even (odd) then $y = f(x) + 4$ is even (odd).
Answer: True for even functions, false for odd functions.
Statement: A graph of a polynomial function can have no x-intercepts.
Answer: True for the polynomial functions of even degree; false for the polynomial functions of odd degree.
Statement: A symmetric function decreases or increases, but not both.
Answer: True for the functions symmetric with respect to the origin; false for the functions symmetric with respect to the y-axis.

Short Problems (Type 3)

Question: Give an example of a function symmetric with respect to x-axis.
Answer: Impossible, a function cannot be symmetric with respect to x-axis. Only a relation can have such symmetry.
Question: Is it possible for a function to be odd or even at the same time?
Answer: Yes, the function is $y = 0$.
Question: Are the graphs of (1) $y = x$, (2) $y = \sqrt{x^2}$, (3) $y = |x|$, (4) $y = \sqrt{x}\,\sqrt{x}$, and (5) $y = \frac{x^2}{x}$ identical?
Answer: No {(2) and (3) are identical, (4) is the non-negative half of (1), the difference between (1) and (5) is an open dot at $x = 0$}.
Question: How many roots does $y = \cos 2$ have?
Answer: None. It's a straight line.
Question: Is it possible for one function to have two horizontal asymptotes?
Answer: Yes, it's easy to construct a split function with such a property.

Bonus Problems

Let us take a look at some of the bonus problems mentioned earlier.

Domain! Domain! How much trouble it brings. The first problem presented is devoted to the domain:

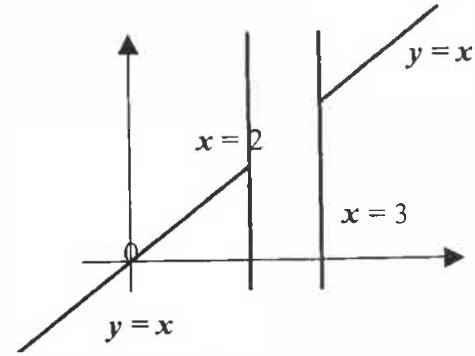
Problem 1

Sketch the graph of the curve $y \sqrt{x^2 - 5x + 6} = x \sqrt{(x - 2)(x - 3)}$.

Looking for the answer, one has to be careful not only with the domain. Lots of students were very close to the correct answer but only a few of them found it.

Solution

Domain: $x \leq 2$, $x \geq 3$.



If $x < 2$, $x > 3$, then dividing both sides by $\sqrt{x^2 - 5x + 6}$ will result in $y = x$;
If $x = 2$, $x = 3$, then $y \cdot 0 = 2 \cdot 0$ or y can be any number and the equation becomes $x = 2$. The same with $x = 3$.

Based on the domain idea, it is easy to construct different modifications of the problem.

The next problem helps students understand and review relations between a function and its inverse.

Problem 2

Find a function such as the function and its inverse have the same equations.

Solution

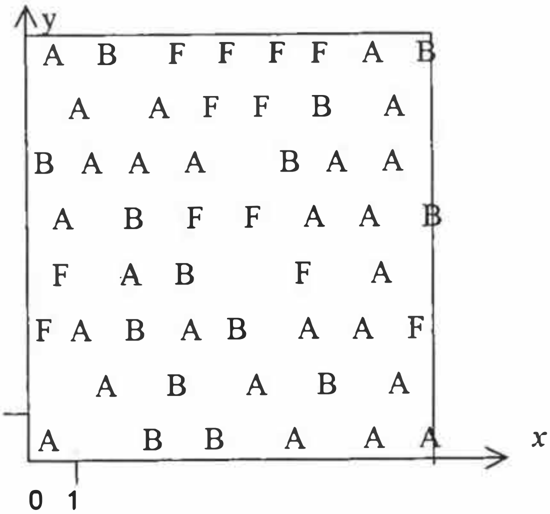
There are infinitely many such functions. The simplest is $y = x$. Others can be combined into different families, like any function of the form $axy + b(x+y) + c = 0$ where a, b, c are any real numbers. For example, the function $y = (x-3)/(2x-1)$ has the same equation as its inverse and can be written in the above form as $2xy - (x+y) + 3 = 0$.

It is absolutely essential that students understand straight lines and know how to find their equations. The following problem deals with lines.

Problem 3

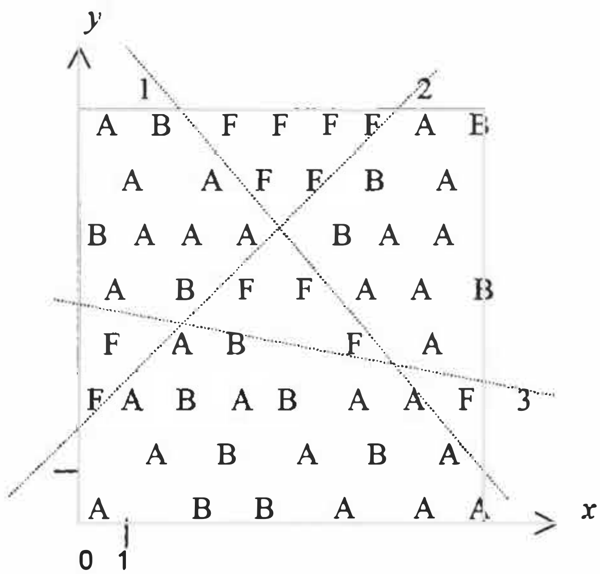
To get A's and B's (two good passing grades) on this course, let us use three lines that separate A's and B's from F's (Fail) and find the equations of these lines.

Last time I assigned this problem to a group of 160 students, I received more than 70 different sets of equations and spent the whole weekend checking their solutions. I was a little bit upset but only because not everybody had responded.



Solution

This is just one of thousands of possible solutions.



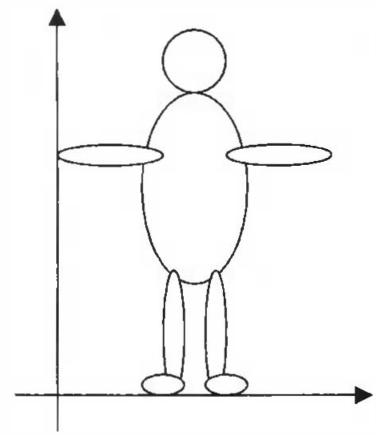
First line: $9y - 10x - 999 = 0$
Second line: $8y - 9x - 16 = 0$
Third line: $11y + 2x - 66 = 0$

The same idea can be used to divide a set of *nm* items in *m* groups of *n* different items in each group using th]e minimum number of lines.

If you want to practise conics and their equations in general form, ask your students to draw a funny little man using conics and describe the equations they used (*Problem 4*).

Solution

This is just one of thousands of possible solutions. I know it is not the best one.



Head: $(x-4)^2 + (y-10)^2 = 1$

Body: $\frac{(x-4)^2}{1.5^2} + \frac{(y-6)^2}{3^2} = 1;$

Hands: $\frac{(x-1.5)^2}{1.5^2} + \frac{(y-7)^2}{0.5^2} = 1; \frac{(x-6.5)^2}{1.5^2} + \frac{(y-7)^2}{0.5^2} = 1$

Legs: $\frac{(x-3)^2}{0.5^2} + \frac{(y-2)^2}{2^2} = 1; \frac{(x-5)^2}{0.5^2} + \frac{(y-2)^2}{2^2} = 1$

There are many more interesting problems related to the topic of functions and relations.

Some Favorite Bonus Problems

In conclusion, I would like present just three of my favorite problems related to different and important notations and definitions in mathematics. Some of them are tricky.

Problem 1

Let $0 < m < n$. What is closer to 1: n/m or m/n?

Solution

The distance between 1 and n/m is $|1 - n/m| = |(m-n)/m| = (n-m)/m$.

The distance between 1 and m/n is $|1 - m/n| = |(n-m)/n| = (n-m)/n$.

Comparing $|1 - n/m|$ and $|1 - m/n|$, we obtain:

$|1 - n/m| - |1 - m/n| = (n-m)/m - (n-m)/n = (n-m)^2 / (mn) > 0$.

Hence, $|1 - n/m| > |1 - m/n|$ and therefore m/n is closer to 1.

Look at this problem and its solution. What has been involved? Answer: the absolute value, the distance between two points, operations with fractions and more. Do students who solve this problem deserve 0.5 added to their total of a 100? I think they do.

One can make up a tale from this problem: "Once upon a time there was a beautiful princess named 1. Two knights named *n/m* and *m/n* were deeply in love with her and wanted to marry her. The princess said she would marry the knight closest to her. Help her in her choice."

The next problem involves a system of equations in several variables with a nontraditional method of solving.

Problem 2

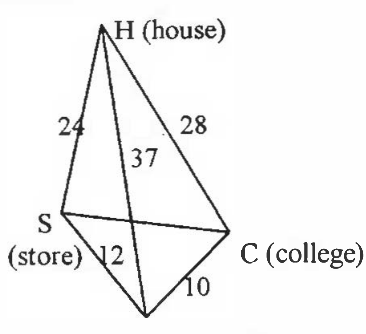
A farmer has three children. They helped their father pick up apples. Both John and Ann picked up 11 kg of apples per hour, Ann and Tim together picked up 33 kg per hour, and Tim and John together picked up 20 kg per hour. How many kg of apples did the children pick up during one hour?

Solution

The problem does not make sense at all. If both John and Ann picked up 11 kg per hour, then Ann picked up at most 11 kg and Tim picked up at least 22 kg (33 kg picked up by Ann and Tim together—33 – 11 = 22 kg). But Tim and John picked up 20 kg per hour, which is less than the 22 kg picked up by Tim himself.

Problem 3

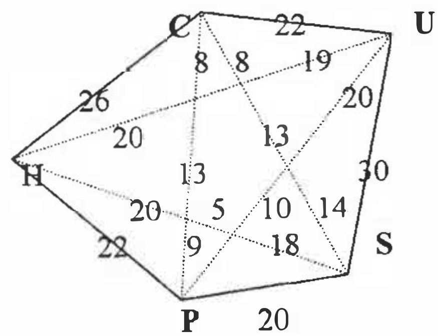
A carrier has to deliver 3 packages from a post office (P) to a store (S), a house (H) and a college C. None of any three buildings lies on a straight line. The distance between the post office (P) and the store (S) is 12 km, or PS = 12 km. The remaining distances are PH = 37 km, PC = 10 km, SC = 19 km, SH = 24 km and CH = 28 km. Find the shortest route.



Solution

There is no solution. The problem itself contains incorrect data. The points H, S and P cannot form a triangle, since the sum of the lengths of its two smaller sides, HS and SP, would be less than the length of its largest side, HP: $12 + 24 < 37$.

This problem can be made more complicated if additional delivery points are considered. For example, we can use the following pentagon,



where the distance between the post office (P) and the hospital (H) is 22 km, between the college (C) and P is 30 km, between the university (U) and P is 40 km, between the school (S) and P is 20 km, between H and C is 26 km, between H and U is 44 km, between H and S is 43 km, between C and U is 22 km, between C and S is 35 km, and between U and S is 30 km.

At least it is really easy to check the students' answers to the last two problems.

One can see that plenty of similar tricky problems may be constructed for any mathematical topic under study.

Giving such problems encourages students to be enthusiastic about mathematics, learn more about it and, finally, begin to love this beautiful subject—the queen of science. If they love it, deeper understanding will develop.

The editor would appreciate receiving comments on this article and invites readers to submit suggestions and new ideas or share their experiences on how to make mathematics enjoyable for students.