delta-k

JOURNAL OF THE
MATHEMATICS COUNCIL
OF THE ALBERTA
TEACHERS' ASSOCIATION

Volume 39, Number 2 June 2002

GUIDELINES FOR MANUSCRIPTS

delta-K is a professional journal for mathematics teachers in Alberta. It is published to

- promote the professional development of mathematics educators, and
- stimulate thinking, explore new ideas and offer various viewpoints.

Submissions are requested that have a classroom as well as a scholarly focus. They may include

- personal explorations of significant classroom experiences;
- descriptions of innovative classroom and school practices;
- reviews or evaluations of instructional and curricular methods, programs or materials;
- discussions of trends, issues or policies;
- a specific focus on technology in the classroom; and
- a focus on the curriculum, professional and assessment standards of the NCTM.

Manuscript Guidelines

1. All manuscripts should be typewritten, double-spaced and properly referenced.
2. Preference will be given to manuscripts submitted on 3.5-inch disks using WordPerfect 5.1 or 6.0 or a generic ASCII file. Microsoft Word and AmiPro are also acceptable formats.
3. Pictures or illustrations should be clearly labeled and placed where you want them to appear in the article. A caption and photo credit should accompany each photograph.
4. If any student sample work is included, please provide a release letter from the student's parent allowing publication in the journal.
5. Limit your manuscripts to no more than eight pages double-spaced.
6. A 250–350-word abstract should accompany your manuscript for inclusion on the Mathematics Council's website.
7. Letters to the editor or reviews of curriculum materials are welcome.
8. *delta-K* is not refereed. Contributions are reviewed by the editor(s), who reserve the right to edit for clarity and space. **The editor shall have the final decision to publish any article.** Send manuscripts to Klaus Puhlmann, Editor, PO Box 6482, Edson, Alberta T7E 1T9; fax 723-2414, e-mail klaupuhl@gyrd.ab.ca.

Submission Deadlines

delta-K is published twice a year. Submissions must be received by August 31 for the fall issue and December 15 for the spring issue.

MCATA Mission Statement

Providing leadership to encourage the continuing enhancement of teaching, learning and understanding mathematics.

delta-K

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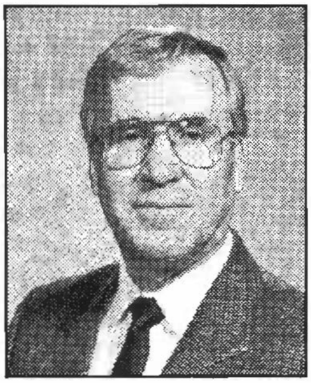
EDITORIAL



Both the president's message and the articles in the National Council of Teachers of Mathematics section of this issue deal with issues that are very important in the teaching and learning of mathematics—mathematical literacy and communication.

Both of these issues are closely linked to the ability to read. As I reflect on that connection between mathematical literacy/communication and reading mathematics, it is obvious that it is not only a challenge for mathematics teachers but that it is also poorly understood. What does reading mathematics actually mean? Does it describe our ability to comprehend numerical data? Or is it being able to comprehend the words in a mathematics textbook?

Being competent in reading describes one's ability to understand written texts of various kinds with respect to their content, purpose and formal structure and to be able to discern the larger picture. This would imply that being competent in reading is not only an important ability in achieving one's personal goals but also an essential foundation for the development of one's own knowledge and abilities—in short, an important prerequisite for our meaningful participation in life.

In this context, reading mathematics is a constructive process in which readers interact with the text, using their prior knowledge and experience to make connections, generate new ideas and hypothesis, and make sense of what they read. Reading mathematics is the ability to make sense of everything that is on a page, whether the page is a worksheet, a spreadsheet, an overhead transparency, a computer screen, a test, or a page in a mathematical journal—in other words, any resource that students might use to learn and apply mathematics.

From this, it follows that reading mathematics requires the same skills as reading in chemistry, biology or other content areas. These skills include

- decoding and comprehending what is read;
- analyzing and evaluating content in the context of one's existing knowledge, experience and interpretive framework; and
- making inferences and generating conclusions based on the reader's unique interpretation of the content.

If reading mathematics requires the same skills one uses to read other content-area material, why is reading a mathematics textbook so difficult for students? These are some possible reasons:

- Mathematics texts demand that readers use additional, content-specific reading skills. Students must be able to not only read from left to right as in other subject areas but also from right to left, top to bottom, bottom to top and even diagonally.
- Mathematics texts contain more concepts per word, per sentence and per paragraph than any other kind of text. In addition, these concepts are often abstract and their meanings are often very difficult to visualize.
- Mathematics texts are generally written in a very terse or compact style. Each sentence contains a lot of information, and there is little redundancy. Sentences and words often have precise meaning and connect logically to other sentences.
- Mathematics reading requires students to be proficient at decoding not only words but also numeric and non-numeric symbols. For example, a mathematical sentence like $\{x \mid x < 7, x \in R\}$ would read: The set whose members are all x, that satisfy the condition that x is less than seven, with all x belonging to the set of real numbers.
- Mathematics texts are often laid out in a way that appears incomprehensible to students.
- Mathematics texts are written above the grade level for which they are intended. This applies to the choice of vocabulary and the sentence structure.

Although reading mathematics requires the same skills as reading generally, it does present teachers with incredibly more challenges than reading in other subject areas. Ensuring that students understand, "speak" and are fluent in the unique and precise language of mathematics is a critical step in overcoming this challenge.

Klaus Puhlmann

From the President's Pen

Mathematical literacy is a term you have likely heard or read about lately. K–12 mathematics teachers need to talk about what mathematical literacy means to us and to the teaching and learning of mathematics. Here are some questions to discuss:

- What does it mean to be mathematically literate in today's society?
- What knowledge, skills and attitudes are needed to be mathematically literate?
- How does the Alberta Program of Studies address mathematical literacy?
- How is this different from being mathematically literate 20 years ago?
- Will this change for future graduates?
- How would you determine if a child is mathematically literate in Grade 1, 5 or 8?

If we consider the definition for mathematical literacy from the Program for International Student Assessment (PISA), then students require more than skills, fluency and procedures to be mathematically literate. PISA states that mathematics literacy is

> an individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded mathematical judgements and to engage in mathematics, in ways that meet the needs of that individual's current and future life as a constructive, concerned and reflective citizen.

For me, *mathematical literacy* is a broader, more generous term than the popular, more frequently used term *numeracy*. Other areas of education are using the term *literacy* to describe the understanding and fluency in their area—technological literacy, scientific literacy and media literacy—and mathematics is another form of literacy. Why confuse the issue with a different name?

Richard L. Venezky (as quoted in Harris and Hodges 1995) describes literacy in the following way:

> In current usage, the term implies an interaction between social demands and individual competence. Thus, the levels of literacy required for social functioning can vary and have varied across cultures and across time within the same culture.

Elliot Eisner (1997), a proponent for the arts in education, has more for us to think about when we think of reading something.

> In order to be read, a poem, an equation, a painting, a dance, a novel, or a contract each requires a distinctive form of literacy, when literacy means, as I intend it to mean, a way of conveying meaning through and recovering meaning from the form of representation in which it appears.

His definition of literacy is also generous and easily includes mathematics as a form of literacy. Although his example seems discrete, we in mathematics education know the depth of literacy that is also required to powerfully read, understand and solve an equation.

The idea of mathematical literacy is not new or an addition to the work we are already doing in our mathematics classrooms. The K–12 Program of Studies already includes it, requiring us to examine the nature and processes of mathematics, and grade level outcomes. The *Alberta Mathematics Program of Studies* (Alberta Learning 1996) states that

> students need to become mathematically literate in order to explore problem-solving situations, accommodate changing conditions, and actively create new knowledge in striving for self-fulfillment.

This discussion is just beginning. We must keep it going to clearly articulate our understanding of a mathematically literate student, graduate and successful member of society. This understanding includes all the rich dimensions of a powerful mathematical thinker and recognizes that mathematical literacy does not selectively focus on what is easily measured or tested.

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Harris, T. L., and R. E. Hodges, eds. *The Literacy Dictionary: The Vocabulary of Reading and Writing*. Newark, Del.: International Reading Association, 1995.

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Sandra Unrau

Thomas Alva Edison (1847–1931)

His guests were often wondering why the gate onto his property required so much strength to open. Being the genius that he was, he told one of his guests the gate was connected to a pump mechanism that pumped 20 litres of water into his cistern when the gate opened. One day he changed the pump capacity from 20 to 25 litres and thus he required 12 fewer visitors to fill the empty cistern. What was the capacity of the cistern?

The Right Angle

Deanna Shostak

Learning and Teaching Resources Branch

Online Elementary Mathematics Resource (Grade 6) Now Available

As part of the new LearnAlberta.ca portal, Alberta Learning has developed an Online Elementary Mathematics Resource. This resource offers interactive lessons related to key mathematics concepts, a glossary of mathematical terms, printable parent notes and activity sheets, problem-solving strategies and number operations.

The Online Elementary Mathematics Resource directly relates to classroom learning and is designed to help students, parents and teachers in a variety of ways.

Teachers can use this mathematics resource to

- enhance lesson plans,
- illustrate concepts in class and
- assist students with extra review.

Students can use this mathematics resource to

- review what is learned in class,
- reinforce skills through extra practice and
- refer to examples that may help with assigned homework.

Parents can use this mathematics resource to

- gain a better understanding of what their child is learning in school and
- help their child at home.

The Online Elementary Mathematics Resource (Grade 6) is now available for use by parents, students and teachers free of charge through the website at www.learnalberta.ca/math. Students who do not have a computer at home may wish to visit a public library or another institution that offers public access to the Internet.

Grades 4 and 5 mathematics concept lessons are under development and are expected to be available in the fall 2002. Grades 1 to 3 lessons are targeted for availability in 2003 and 2004.

For more information about the Online Elementary Mathematics Resource, please contact Debbie Duvall, Project Manager, Learning and Teaching Resources Branch, Alberta Learning at Debbie.Duvall@gov.ab.ca or phone (780) 422-3257.

Learner Assessment Branch

Diploma exams in Pure Mathematics 30, Applied Mathematics 30, Mathematics 30 (old) and Mathematics 33 will be available in both June and August. The diploma exams for both Pure and Applied Mathematics 30 will be released in June, but secured in August. All administrations of Mathematics 30 (old) and Mathematics 33 are secure. This means that the examinations will arrive in your school in sealed envelopes bearing the student's name, and no one other than the student writing the exam will see the contents of the envelope. Mathematics 30 (old) diploma exams are only available until August 2002, and Mathematics 33 diploma exams are available until August 2003. Mathematics 33 exams written during the 2002/03 school year will only be available to students who are repeating the course or who have been granted special permission. A request for special status must be made in writing to Raja Panwar, Director, Curriculum Branch, Alberta Learning, 11160 Jasper Avenue, Edmonton T5K 0L2.

The Applied Mathematics 30 diploma exam will be worth 20 percent this year, while the weighting of all the other math diploma exams will be worth 50 percent. For students who wrote a Pure Mathematics 30 diploma examination worth 20 percent and wish to upgrade their final grade by writing a diploma examination worth 50 percent, special blending rules will apply. Alberta Learning will calculate the student's final grade by taking the higher school awarded mark (SAM) and blending it with last year's diploma examination mark (DEM) (using an 80 percent, 20 percent blend) and also blending it with this year's diploma examination (using a 50 percent, 50 percent blend). The higher mark will appear on the student's transcript. The example below illustrates this new blending rule.

	June 2001 (20% diploma)	January 2002 (50% diploma)	January 2002 SAM with June 2001 DEM
School Awarded Mark	75%	78%	
Diploma Exam Mark	60%	64%	
Final Grade	72%	71%	74%

Although both marks in January 2002 were higher, the 50 percent blend results in a lower final grade for the student. Therefore, Alberta Learning will calculate this student's final grade by using the higher SAM (January 2002) and the June 2001 diploma exam (worth 20 percent). The student's results statement will contain the most current year's marks, but the best overall final grade (74 percent) will appear on the student's transcript.

If further clarification is required, please contact the math/science unit, Learner Assessment Branch, (780) 427-0010.

Calculator Policy

The list of approved calculators and instructions for clearing these calculators can be found at www.learning.gov.ab.ca/k_12/testing/diploma/bulletins/default.asp. All science and mathematics diploma examinations, including Mathematics 30 (old) and Mathematics 33, require the use of a scientific calculator or graphing calculator approved by Alberta Learning. This list is updated annually. Please note that the TI-82 will be deleted from this list for the 2002/03 school year.

Information Bulletins

These bulletins provide students and teachers with information about diploma examinations for the 2002/03 school year. The bulletins for both Applied Mathematics 30 and Pure Mathematics 30 were reviewed by teachers across the province in the spring 2002 and recommendations were made to change some of the curriculum standards and example questions. These new bulletins can be downloaded from www.learning.gov.ab.ca/k_12/testing/diploma/bulletins/default.asp before September 2002.

The Learning Technologies Branch

The Learning Technologies Branch (LTB) is responsible for providing leadership and consultation in identifying, developing, implementing and evaluating effective distance learning strategies and techniques in Alberta schools. Recently, LTB has developed a number of additional secondary and elementary mathematics resources.

Secondary Mathematics Resources

The following print resource for secondary mathematics has been developed by LTB and is available from the Learning Resources Centre (LRC):

- Applied Mathematics 20 Student Module Pack (2001) (Product #440868)

The following electronic resource has been developed by LTB and is available from LRC:

- Applied Mathematics 20 Multimedia Segments (CD-ROM v.1.0, Windows/Mac) (2001) (Product #460361)

LTB has also developed a resource called Tools for Teachers. Tools for Teachers is an LTB site that contains LTB distance learning materials linked to the Program of Studies objectives. Mathematics materials available are Mathematics 31, Pure Mathematics 10–30 and Applied Mathematics 10. Multimedia segments for Applied Mathematics 10 and 20 are available on Tools for Teachers. Also, Electronic Teachers' Guides for some recently completed courses are available for downloading from this site. The website address is www.tools4teachers.ab.ca and access information is given on the site.

LTB is currently developing print resources and multimedia segments for Applied Mathematics 30.

Elementary Mathematics Resources

The following print resources for elementary mathematics have been developed by LTB and are available from LRC:

- Mathematics 2 Student Pack (2001) (Product #456815)
- Mathematics 5 Student Pack (2001) (Product #442822)

LTB is currently developing print resources for Mathematics 3 and 6.

For more information, visit our website at www.learning.gov.ab.ca/ltb.

In this section, we will share your points of view on teaching and learning mathematics and your responses to anything contained in delta-K. *We appreciate your interest and value the views of those who write.*

Mathematics with Joy

Natali Hritonenko

Each term when I come to a classroom to teach basic level mathematics (for example, calculus, precalculus and so on), I can see fear and horror in my students' eyes. Their bodies are motionless. They are expecting mental torture, as one of my students said. They are in this room only because they have to be here. They have to take this class to get their degree. Sometimes it is their very last class because they were so afraid of taking it before. Sometimes it is their n-th attempt to pass it, $n > 1$. They are coming to this class thinking they will never understand math. It takes some time before students become involved in class discussions, and then they want to continue learning more about their favorite mathematics. I can finally see smiles on their faces. Our class starts out as one that feels dreadfully long but becomes one where time passes like a blur. My guess is that the change happens when students quit being behind and start moving ahead. They start to believe in themselves. I am waiting for this faith to develop in every course because I need it. Faith is necessary for victory.

Learning any mathematical subject can be enjoyable and relaxing. Solving mathematics problems can be amusing and delightful. Mathematics and fun go together naturally. I find it useful to bring interesting and nontrivial problems to class that are related to a particular topic and ask them to solve them when they have time. Each of the problems has a simple but gorgeous solution. Students always find the time to solve interesting problems that demand creativity. They do not give up and will remain engaged in this once they start. Of course, I always give bonus points to their final score if they succeed. This is worthwhile to me because getting students to think about these problems and having them arrive at the correct answers will lead students to discover how awesome and exciting mathematics really is. They may even find that they have learned so many new ways of solving mathematics problems that their skills have improved and their enjoyment of mathematics has increased.

Warm-Up Questions

At the beginning of each class, we play brain warm-up games with three types of questions:

1. Simple true and false
2. Statements that can be true or false, with the goal of finding out when these statements are true
3. Small problems with short answers

These questions help students review previous concepts and prepare them for understanding new ones. All tasks require a quick response and good understanding of the material studied. Knowing these facts also helps with problem solving.

Here are some questions that cover the three problem types identified above. A solution key is provided to show different properties touched on in each question. The following sample questions are restricted to relations and functions.

True/False Questions (Type 1)

Statement: If $y = f(x)$ is even, then $y = f(-x)$ is odd.
Answer: False, $f(-x)$ is even too: $f(-x) = f(x)$.
Statement: The inverse of an even function is even.
Answer: False, an even function does not have an inverse at all. It's not one-to-one.
Statement: $f(x)$ times $f^{-1}(x) = x$.
Answer: False (it is true for the composition $f(x) \circ f^{-1}(x) = x$ of these two functions).

When True/When False Questions (Type 2)

Statement: If $y = f(x)$ is defined on a set of real numbers, then $y = | f(x) |$ is always even.
Answer: False. It's true when $f(x)$ is odd or even.
Statement: If $y = f(x)$ is even (odd) then $y = f(x) + 4$ is even (odd).
Answer: True for even functions, false for odd functions.
Statement: A graph of a polynomial function can have no x-intercepts.
Answer: True for the polynomial functions of even degree; false for the polynomial functions of odd degree.
Statement: A symmetric function decreases or increases, but not both.
Answer: True for the functions symmetric with respect to the origin; false for the functions symmetric with respect to the y-axis.

Short Problems (Type 3)

Question: Give an example of a function symmetric with respect to x-axis.
Answer: Impossible, a function cannot be symmetric with respect to x-axis. Only a relation can have such symmetry.
Question: Is it possible for a function to be odd or even at the same time?
Answer: Yes, the function is $y = 0$.
Question: Are the graphs of (1) $y = x$, (2) $y = \sqrt{x^2}$, (3) $y = |x|$, (4) $y = \sqrt{x}\,\sqrt{x}$, and (5) $y = \frac{x^2}{x}$ identical?
Answer: No {(2) and (3) are identical, (4) is the non-negative half of (1), the difference between (1) and (5) is an open dot at $x = 0$}.
Question: How many roots does $y = \cos 2$ have?
Answer: None. It's a straight line.
Question: Is it possible for one function to have two horizontal asymptotes?
Answer: Yes, it's easy to construct a split function with such a property.

Bonus Problems

Let us take a look at some of the bonus problems mentioned earlier.

Domain! Domain! How much trouble it brings. The first problem presented is devoted to the domain:

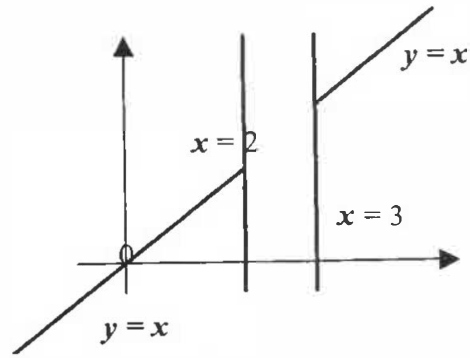
Problem 1

Sketch the graph of the curve $y\sqrt{x^2 - 5x + 6} = x\sqrt{(x-2)(x-3)}$.

Looking for the answer, one has to be careful not only with the domain. Lots of students were very close to the correct answer but only a few of them found it.

Solution

Domain: $x \leq 2$, $x \geq 3$.



If $x < 2$, $x > 3$, then dividing both sides by $\sqrt{x^2 - 5x + 6}$ will result in $y = x$;
If $x = 2$, $x = 3$, then $y \cdot 0 = 2 \cdot 0$ or y can be any number and the equation becomes $x = 2$. The same with $x = 3$.

Based on the domain idea, it is easy to construct different modifications of the problem.

The next problem helps students understand and review relations between a function and its inverse.

Problem 2

Find a function such as the function and its inverse have the same equations.

Solution

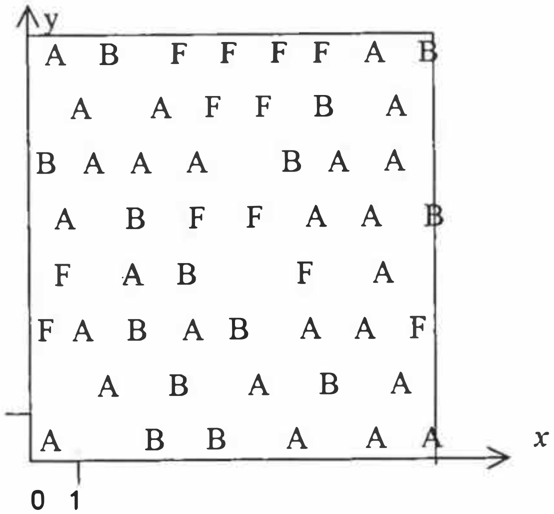
There are infinitely many such functions. The simplest is $y = x$. Others can be combined into different families, like any function of the form $axy + b(x+y) + c = 0$ where a, b, c are any real numbers. For example, the function $y = (x-3)/(2x-1)$ has the same equation as its inverse and can be written in the above form as $2xy - (x+y) + 3 = 0$.

It is absolutely essential that students understand straight lines and know how to find their equations. The following problem deals with lines.

Problem 3

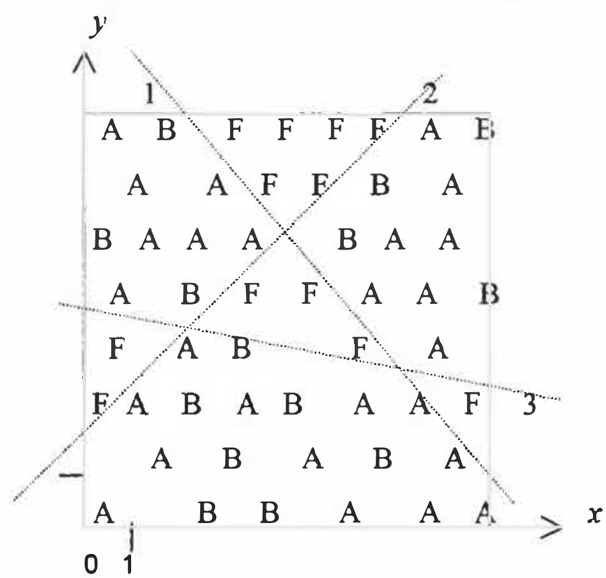
To get A's and B's (two good passing grades) on this course, let us use three lines that separate A's and B's from F's (Fail) and find the equations of these lines.

Last time I assigned this problem to a group of 160 students, I received more than 70 different sets of equations and spent the whole weekend checking their solutions. I was a little bit upset but only because not everybody had responded.



Solution

This is just one of thousands of possible solutions.



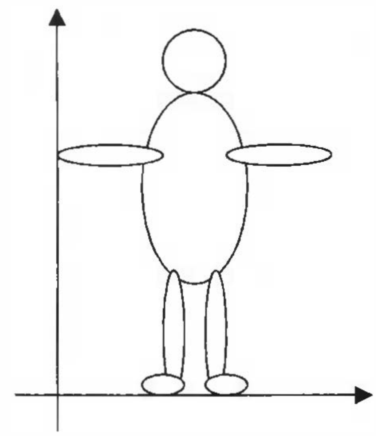
First line: $9y - 10x - 999 = 0$
Second line: $8y - 9x - 16 = 0$
Third line: $11y + 2x - 66 = 0$

The same idea can be used to divide a set of nm items in m groups of n different items in each group using th]e minimum number of lines.

If you want to practise conics and their equations in general form, ask your students to draw a funny little man using conics and describe the equations they used (*Problem 4*).

Solution

This is just one of thousands of possible solutions. I know it is not the best one.



Head: $(x-4)^2 + (y-10)^2 = 1$

Body: $\frac{(x-4)^2}{1.5^2} + \frac{(y-6)^2}{3^2} = 1;$

Hands: $\frac{(x-1.5)^2}{1.5^2} + \frac{(y-7)^2}{0.5^2} = 1; \frac{(x-6.5)^2}{1.5^2} + \frac{(y-7)^2}{0.5^2} = 1$

Legs: $\frac{(x-3)^2}{0.5^2} + \frac{(y-2)^2}{2^2} = 1; \frac{(x-5)^2}{0.5^2} + \frac{(y-2)^2}{2^2} = 1$

There are many more interesting problems related to the topic of functions and relations.

Some Favorite Bonus Problems

In conclusion, I would like present just three of my favorite problems related to different and important notations and definitions in mathematics. Some of them are tricky.

Problem 1

Let $0 < m < n$. What is closer to 1: n/m or m/n?

Solution

The distance between 1 and n/m is $|1 - n/m| = |(m-n)/m| = (n-m)/m$.

The distance between 1 and m/n is $|1 - m/n| = |(n-m)/n| = (n-m)/n$.

Comparing $|1 - n/m|$ and $|1 - m/n|$, we obtain:
$|1 - n/m| - |1 - m/n| = (n-m)/m - (n-m)/n = (n-m)^2 / (mn) > 0$.

Hence, $|1 - n/m| > |1 - m/n|$ and therefore m/n is closer to 1.

Look at this problem and its solution. What has been involved? Answer: the absolute value, the distance between two points, operations with fractions and more. Do students who solve this problem deserve 0.5 added to their total of a 100? I think they do.

One can make up a tale from this problem: "Once upon a time there was a beautiful princess named 1. Two knights named *n/m* and *m/n* were deeply in love with her and wanted to marry her. The princess said she would marry the knight closest to her. Help her in her choice."

The next problem involves a system of equations in several variables with a nontraditional method of solving.

Problem 2

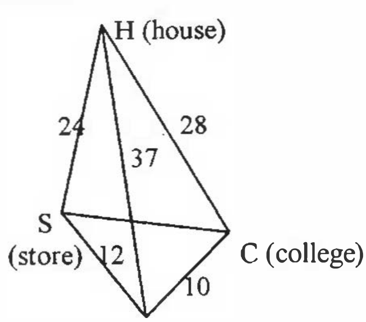
A farmer has three children. They helped their father pick up apples. Both John and Ann picked up 11 kg of apples per hour, Ann and Tim together picked up 33 kg per hour, and Tim and John together picked up 20 kg per hour. How many kg of apples did the children pick up during one hour?

Solution

The problem does not make sense at all. If both John and Ann picked up 11 kg per hour, then Ann picked up at most 11 kg and Tim picked up at least 22 kg (33 kg picked up by Ann and Tim together—33 – 11 = 22 kg). But Tim and John picked up 20 kg per hour, which is less than the 22 kg picked up by Tim himself.

Problem 3

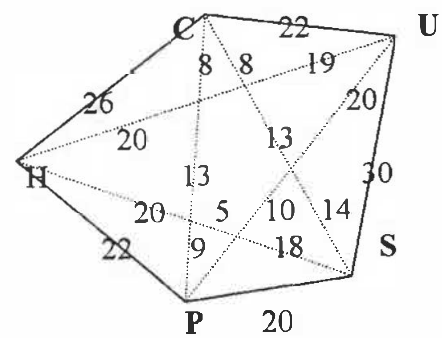
A carrier has to deliver 3 packages from a post office (P) to a store (S), a house (H) and a college C. None of any three buildings lies on a straight line. The distance between the post office (P) and the store (S) is 12 km, or PS = 12 km. The remaining distances are PH = 37 km, PC = 10 km, SC = 19 km, SH = 24 km and CH = 28 km. Find the shortest route.



Solution

There is no solution. The problem itself contains incorrect data. The points H, S and P cannot form a triangle, since the sum of the lengths of its two smaller sides, HS and SP, would be less than the length of its largest side, HP: 12 + 24 < 37.

This problem can be made more complicated if additional delivery points are considered. For example, we can use the following pentagon,



where the distance between the post office (P) and the hospital (H) is 22 km, between the college (C) and P is 30 km, between the university (U) and P is 40 km, between the school (S) and P is 20 km, between H and C is 26 km, between H and U is 44 km, between H and S is 43 km, between C and U is 22 km, between C and S is 35 km, and between U and S is 30 km.

At least it is really easy to check the students' answers to the last two problems.

One can see that plenty of similar tricky problems may be constructed for any mathematical topic under study.

Giving such problems encourages students to be enthusiastic about mathematics, learn more about it and, finally, begin to love this beautiful subject—the queen of science. If they love it, deeper understanding will develop.

The editor would appreciate receiving comments on this article and invites readers to submit suggestions and new ideas or share their experiences on how to make mathematics enjoyable for students.

Communication is an important process standard in school mathematics; hence, the mathematics curriculum emphasizes the continued development of language and symbolism to communicate mathematical ideas. Communication includes regular opportunities to discuss mathematical ideas and to explain strategies and solutions using words, mathematical symbols, diagrams and graphs. While all students need extensive experience in expressing mathematical ideas orally and in writing, some students may have the desire or should be encouraged by teachers to publish their work in journals.

delta-K *invites students to share their work with others beyond their classroom. Submissions could include papers on a particular mathematical topic, a mathematics project, an elegant solution to a mathematical problem, an interesting problem, an interesting discovery, a mathematical proof, a mathematical challenge, an alternative solution to a familiar problem, poetry about mathematics, a poster or anything deemed to be of mathematical interest.*

Teachers are encouraged to review students' work prior to submission. Please attach a dated statement that permission is granted to the Mathematics Council of the Alberta Teachers' Association to publish the work in delta-K. *The student author (or the parents if the student is under 18 years of age), must sign this statement, indicate the student's grade level and provide an address and telephone number.*

This issue's submissions are high school mathematics projects, completed as course requirements.

One Pure Mathematics 30 project was selected for publication. Shari Monner, a Grade 12 student at Holy Family CyberHigh School in High Prairie, submitted "Pure Mathematics 30 Student Project: Sunrise and Sunset." Her teacher is Sheryl Heikel.

One Applied Mathematics 30 project was selected for publication. Mark Fredrick, a Grade 12 student at Barrhead Composite High School in Barrhead, submitted "Applied Mathematics 30 Student Project: Medical Research: Huntington's Disease." His teacher is Leahan Schaffrick.

Pure Mathematics 30 Student Project: Sunrise and Sunset

Shari Monner

The topic to be investigated relates to the theme of sunrise and sunset.

An almanac on the Internet may be used to generate source data for times of sunrise and sunset for various dates equally dispersed throughout the year. To study the effect of latitude on sunrise and sunset, an additional set of data should be obtained for a second location, at a different latitude. For example, Edmonton is located at N53°36' W113°30' and Mexico City is located at N19°24' W99°12'.

Part A

1. **Use the data for sunrise and sunset times in Edmonton to create lists in a graphing calculator or spreadsheet program. Note that times are written in decimal form, so 18:24 is shown as 18.40 h.**

   L1, L2 and L3 for Edmonton are shown in Appendix A.

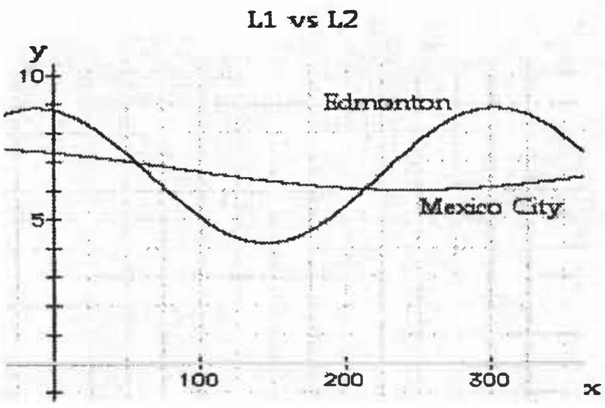
2. **Use the data in L2 and L3 to create data for a new list, L4, that shows the number of hours of daylight as a function of day number *n*. On your graphing calculator, graph: i. L1 versus L2, ii. L1 versus L3, iii. L1 versus L4. Sketch the graphs on the axes given.**

   L4 for Edmonton is shown in Appendix A and the graphs are shown below.

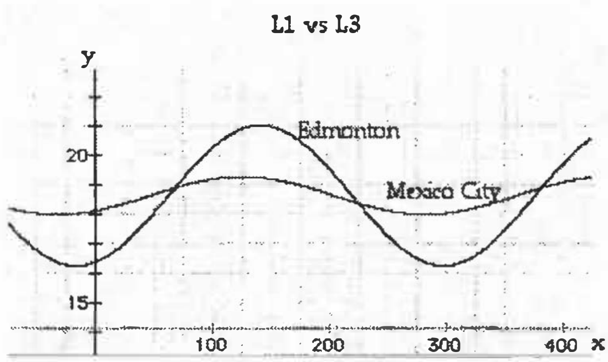
3. **Using the data given for Mexico City, repeat steps 1 and 2 above. Sketch the graphs on the same axes as above.**

L1, L2, L3, and L4 for Mexico City are shown in Appendix A and the graphs are shown below.

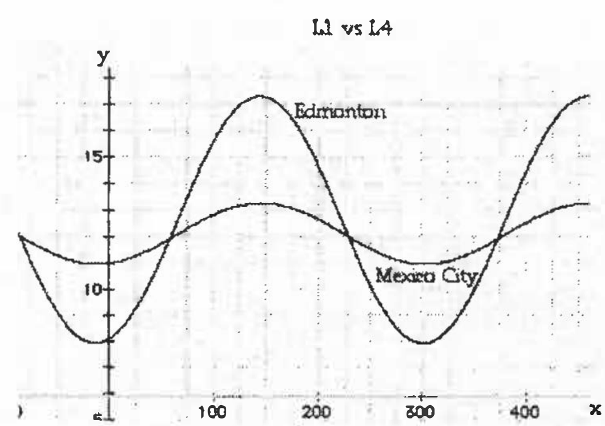
Day number versus time of sunrise



Day number versus time of sunset



Day number versus hours of sunlight



4. Find the equation and explain how a student could obtain a function of the form $f(n) = a \sin b\,(n - c) + d$, correct to four decimals to represent

i. the number of hours of sunlight as a function of the day number, *n*, for both locations

Edmonton equation:
$y = 4.6767 \sin(0.0168x - 1.3160) + 12.1638$
$f(n) = 4.6767 \sin 0.0168(n - 78.3333) + 12.1638$

Mexico equation:
$y = 1.1544 \sin(0.0169x - 1.3408) + 12.1184$
$f(n) = 1.1544 \sin 0.0169(n - 79.3373) + 12.1184$

ii. time of sunrise as a function of *n* for Edmonton

Edmonton sunrise equation:
$y = 2.3303 \sin(0.0165x + 1.8040) + 6.5145$
$f(n) = 2.3303 \sin 0.0165(n + 109.3333) + 6.5145$

iii. the time of sunset as a function of *n* for Mexico City

Mexico sunset equation:
$y = 0.6446 \sin(0.0157x - 0.9300) + 18.6255$
$f(n) = 0.6446 \sin 0.0157(n - 59.2357) + 18.6255$

$y = a \sin(bx - c) + d$ functions are calculated by regression* on a graphing calculator, $f(n) = a \sin b(n - c) + d$ is then calculated from $y = a \sin(bx - c) + d$ using $c = \frac{c}{b}$ to put the equation in the correct form.

*Regression steps used to determine equations are listed in Appendix B

5. For Mexico City, consider June 20 as the first day of summer and consider December 22 as the first day of winter. Using the data from these days, determine algebraically a sine equation in the form of $y = a \sin b(n + c) + d$, for sunlight hours. State all the features of this sinusoidal equation that can be inferred from the parameters of $y = a \sin b(n + c) + d$.

June 20		**December 22**	
day	171	day	356
rise	5.98	rise	7.10
set	19.28	set	18.07
hours	13.30	hours	10.97

maximum	13.3 at $x = 171$
minimum	10.97 at $x = 356$
period	365 (number of days in a year)
amplitude	$13.3 - 10.97 = 2.33$ $\frac{2.33}{2} = 1.165$
vertical displacement	$10.97 + 1.165 = 12.135$
horizontal displacement	$\frac{365 - 171}{2} = \frac{185}{2} = 92.5$ $171 - 92.5 = 78.5$

$y = a \sin b(n - c) + d$
$a = 1.165$
$b = \frac{2\pi}{365}$
$c = -78.5$
$d = 12.135$
$y = 1.165 \sin \frac{2\pi}{365}(n - 78.5) + 12.135$

The variable a in the above equation shows how varied the hours of sunlight are in Mexico City. The higher the value of a, the more variation there is during the year in hours of sunlight.

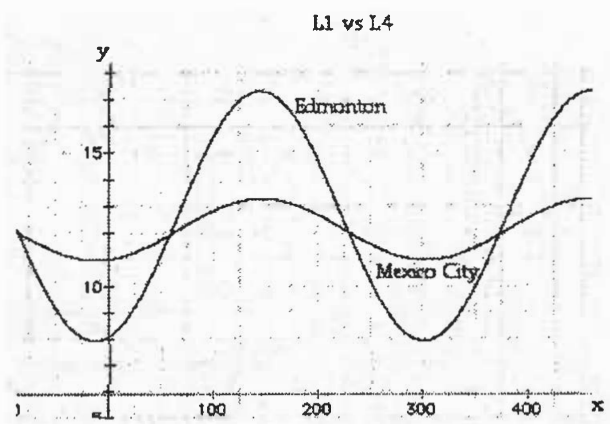
The variable b in the above equation is taken from the period, which is the number of days in a year. The cycle of hours of sunlight per day is repeated annually.

The variable c in the above equation shows the number of days from the beginning of the year to the first day of spring.

The variable d in the above equation is the average hours of sunlight per day in a year in Mexico City.

6. **Explain how the latitude of a location affects hours of sunlight.**

Hours of Sunlight Per Day



As you can see from the above graph, the latitude of a location drastically affects the variation in hours of sunlight per day. Mexico City has less variation than Edmonton in its hours of sunlight per day because Edmonton is much farther from the equator than Mexico City.

7. **One factor that affects a region's growing season is hours of sunlight. If the growing season is made up of days with at least 15 hours of daylight, what are the approximate dates of the start and end of the growing season in Edmonton? Explain your method for determining these dates.**

   The start date of the growing season in Edmonton is April 28, and the end date is August 14.

Using the regression method shown in Appendix B, I found the equation for the sinusoidal function that corresponds with the data in a graph of day number versus hours of daylight in Edmonton. I graphed this function on the same grid as $y = 15$ and used the calculate–intersect function on my graphing calculator. When the sinusoidal function is above the line $y = 15$, the day numbers corresponding to the y coordinates are days in the growing season.

Part B

1. **Determine the number of hours of sunlight to the nearest minute that exist in Edmonton on May 24 by doing the solution algebraically using the equation found in Part A, 4i.**

   $f(n) = 4.6767 \sin 0.0168(n - 78.3333) + 12.1638$
   May 24 = Day 124 = n

   $f(n) = 4.676 \sin 0.0168(124 - 78.3333) + 12.1638$
   $f(124) = 15.41$
   $0.41 \times 60 = 24.6$

   On May 24, the number of hours of sunlight was 15:25.

2. **Describe how you could check your solution to 1 above using a graphing calculator.**

   Graph the function, $f(n) = 4.6767 \sin 0.0168(n - 78.3333) + 12.1638$, trace to $x = 124$ and record the value of y; this value should be equal to 15:25 in decimal form, which is 15.41.

3. **What other day would have the same amount of sunlight hours as May 24?**

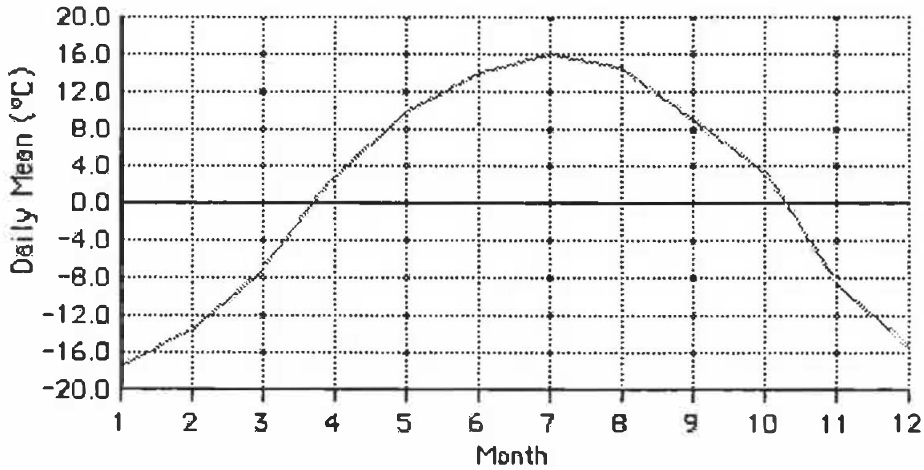
   Either August 7 or 8 would have the same number of sunlight hours as May 24. August 7 had 15:27 hours and August 8 had 15:23 hours.

Part C

Many phenomena exhibit a periodic pattern. Locate a source of data other than sunrise and sunset times that appear to be periodic. (Some suggestions are tides, predator–prey populations and the distance of the moon from the Earth.)

1. **Using this data, sketch a graph that represents the periodic relationship. See Appendix C for data information.**

Daily Mean Temperature of Peace River, Alberta (1944–90)



2. Using an appropriate regression model, determine an equation that will represent the data.

Regression steps used are listed in Appendix B.

$y = 17.9901 \sin(0.4685x - 1.6847) - 1.3507$

3. Analyze the equation for its "fit" with the graph of the data.

It doesn't show the highs and lows appropriately because the amplitude is a bit too high, although the above equation is as close to every point on the graph as possible, the minimum temperature is a bit too low in comparison to the data. It is, however, close enough to give an idea of the likely average temperature in each month in years to come.

Summary

Sinusoidal functions can be used to predict many occurrences based on previously collected data. Sunrise and sunset times, fluctuations in these times and average temperature changes in the year all demonstrate the seasons, which occur year after year. This is yet another example of the predictability of natural occurrences: evidence of a creator.

Acknowledgments

Thanks to my teacher Sheryl Heikel for answering all my questions and providing me with appropriately scaled graphs. Thanks to my father George Monner for helping me find the raw data on temperature changes. Thanks to my classmate Mark Glass for his mathematical consultation.

Appendix A

Edmonton Data

L1	L2	L3	L4
1	8.85	16.42	7.57
6	8.82	16.52	7.70
11	8.77	16.63	7.86
16	8.70	16.77	8.07
21	8.60	16.92	8.32
26	8.48	17.07	8.59
31	8.35	17.23	8.88
36	8.22	17.40	9.18
41	8.05	17.57	9.52
46	7.88	17.73	9.85
51	7.72	17.90	10.18
56	7.53	18.07	10.54
61	7.33	18.22	10.89
66	7.13	18.38	11.25
71	6.93	18.53	11.60
76	6.73	18.68	11.95
81	6.53	18.85	12.32
86	6.33	19.00	12.67
91	6.13	19.15	13.02
96	5.93	19.30	13.37
101	5.73	19.45	13.72
106	5.53	19.62	14.09
111	5.35	19.77	14.42
116	5.17	19.92	14.75
121	4.98	20.07	15.09
126	4.83	20.22	15.39
131	4.67	20.37	15.70
136	4.53	20.50	15.97
141	4.40	20.63	16.23
146	4.30	20.75	16.45
151	4.20	20.87	16.67
156	4.13	20.95	16.82
161	4.08	21.03	16.95
166	4.07	21.08	17.01
171	4.07	21.12	17.05
176	4.08	21.13	17.05
181	4.13	21.12	16.99
186	4.20	21.08	16.88
191	4.28	21.02	16.74
196	4.38	20.93	16.55
201	4.50	20.83	16.33
206	4.63	20.70	16.07
211	4.77	20.57	15.80
216	4.90	20.42	15.52
221	5.03	20.25	15.22
226	5.18	20.08	14.90
231	5.33	19.90	14.57
236	5.48	19.72	14.24
241	5.62	19.52	13.90
246	5.77	19.33	13.56
251	5.92	19.12	13.20
256	6.07	18.92	12.85
261	6.20	18.72	12.52
266	6.35	18.52	12.17
271	6.50	18.30	11.80
276	6.65	18.10	11.45
281	6.80	17.90	11.10
286	6.95	17.72	10.77
291	7.10	17.52	10.42
296	7.27	17.33	10.06
301	7.42	17.15	9.73
306	7.58	16.98	9.40
311	7.75	16.83	9.08
316	7.90	16.70	8.80
321	8.05	16.57	8.52
326	8.20	16.45	8.25
331	8.35	16.37	8.02
336	8.48	16.30	7.82
341	8.60	16.25	7.65
346	8.68	16.23	7.55
351	8.77	16.23	7.46
356	8.82	16.27	7.45
361	8.85	16.32	7.47

L1 = Day number
L2 = Time of sunrise
L3 = Time of sunset
L4 = Hours of sunlight

L1 versus L2

Equation:
$y = 2.33 \sin(0.02x + 1.81) + 6.52$

L1 versus L3

Equation:
$y = 2.38 \sin(0.02x - 1.24) + 18.63$

L1 versus L4
Equation:
$y = 4.68 \sin(0.02x - 1.31) + 12.16$

Mexico City Data

L1	L2	L3	L4
1	7.18	18.17	10.99
6	7.20	18.21	11.01
11	7.22	18.27	11.05
16	7.23	18.33	11.10
21	7.23	18.38	11.15
26	7.22	18.43	11.21
31	7.20	18.48	11.28
36	7.17	18.53	11.36
41	7.13	18.57	11.44
46	7.10	18.62	11.52
51	7.05	18.65	11.60
56	6.98	18.68	11.70
61	6.93	18.70	11.77
66	6.87	18.73	11.86
71	6.80	18.77	11.97
76	6.73	18.78	12.05
81	6.65	18.80	12.15
86	6.58	18.83	12.25
91	6.52	18.85	12.33
96	6.45	18.87	12.42
101	6.38	18.88	12.50
106	6.32	18.92	12.60
111	6.25	18.93	12.68
116	6.20	18.97	12.77
121	6.15	18.98	12.83
126	6.10	19.02	12.92
131	6.05	19.05	13.00
136	6.02	19.08	13.06
141	6.00	19.12	13.12
146	5.98	19.15	13.17
151	5.97	19.18	13.21
156	5.97	19.22	13.25
161	5.97	19.25	13.28
166	5.97	19.27	13.30
171	5.98	19.28	13.30
176	6.00	19.30	13.30
181	6.03	19.32	13.29

L1	L2	L3	L4
186	6.05	19.32	13.27
191	6.08	19.32	13.24
196	6.12	19.30	13.18
201	6.15	19.28	13.13
206	6.18	19.27	13.09
211	6.20	19.23	13.03
216	6.23	19.18	12.95
221	6.27	19.15	12.88
226	6.28	19.10	12.82
231	6.32	19.03	12.71
236	6.33	18.97	12.64
241	6.35	18.90	12.55
246	6.37	18.83	12.46
251	6.38	18.77	12.39
256	6.40	18.68	12.28
261	6.42	18.62	12.20
266	6.43	18.53	12.10
271	6.45	18.47	12.02
276	6.47	18.40	11.93
281	6.48	18.32	11.84
286	6.52	18.25	11.73
291	6.53	18.20	11.67
296	6.57	18.13	11.56
301	6.60	18.08	11.48
306	6.63	18.05	11.42
311	6.67	18.00	11.33
316	6.72	17.98	11.26
321	6.77	17.97	11.20
326	6.82	17.95	11.13
331	6.87	17.95	11.08
336	6.92	17.95	11.03
341	6.97	17.97	11.00
346	7.02	18.00	10.98
351	7.07	18.03	10.96
356	7.10	18.07	10.97
361	7.15	18.12	10.97

L1 = Day number
L2 = Time of sunrise
L3 = Time of sunset
L4 = Hours of sunlight

L1 versus L2

Equation:
$y = 0.74 \sin(0.01x + 2.27) + 6.75$

L1 versus L3

Equation:
$y = 0.64 \sin(0.02x - 0.93) + 18.63$

L1 versus L4

Equation:
$y = 1.15 \sin(0.02x - 1.34) + 12.12$

Appendix B

Regression steps: (Key Strokes in Bold)

1. STAT → 1: Edit (Under EDIT menu) → Enter Lists 1–4
2. STAT → [Right Arrow] to CALC menu → [Up Arrow] to C: SinReg → ENTER → 2nd → 1 (L1) →, → 2nd → 2 (L2) (or 3 for L3 and so on . . .) → ENTER
3. Y= → [Up Arrow] (to Plot1*) → ENTER → [Down Arrow] (to Y1=) → VARS → 5: Statistics. . . → [Right Arrow] (to EQ menu) → 1: RegEQ → GRAPH

*Plot 1 is preset to show L1 versus L2, Plot 2 to show L1 versus L3 and Plot 3 to show L1 versus L4

Appendix C

Temperature Data Acquired from Environment Canada's website www.cmc.ec.gc.ca/climate/normals/ALTAP005.HTM.

Month	Daily Mean (C)
1	−17.5
2	−13.3
3	−7.2
4	3.0
5	9.9
6	14.1
7	15.9
8	14.6
9	9.2
10	3.4
11	-8.5
12	−15.2

Heron of Alexandria (c. 50?)

There were four water fountains. The first fountain filled the cistern in one day, the second filled the cistern in two days, the third in three days and the fourth in four days. How long does it take for all four water fountains to fill the cistern together?

Applied Mathematics 30 Student Project: Medical Research: Huntington's Disease

Mark Fredrick

Huntington's disease (HD) is a neurological disease characterized by movement disorder, dementia and psychiatric disturbances. A single abnormal or mutated gene produces HD. The impact of a gene depends partly on whether it is dominant or recessive. If a gene is dominant, only one of the paired chromosomes is required to produce HD's effect. HD is autosomal dominant, which means that only one copy of the defective gene, inherited from either parent, is needed to produce the disease.

Part A

1. In addition to the terms introduced, the following terms will also help in completing this project:

Punnet square: a chart showing the possible combinations of alleles in offspring.
Heterozygous: a person who has one or more pairs of unlike genes.
Homozygous: a person possessing like pairs of genes for any hereditary characteristics.
Huntington's disease: a neurological disease that is produced by an abnormal gene.

2. A couple, of which one is heterozygous for the HD gene, is planning to have children.
   a. Construct the sample space (Punnet square) for possible offspring of this couple.

	H	h
h	hH	hh
h	hH	hh

   b. What is the probability that this couple will have a child who has HD?
   $\frac{2}{4} = \frac{1}{2}$ or 50%
   Therefore, there is a 50 percent chance that the child will have HD.
   c. If this couple were to have three children, what would be the probability that at least one of the children would have HD?
   Binompdf (3, 0.50) = (1.25, .375, .375, .125)
   P = (1 or 2 or 3) = .375/1 + .375/2 + .375/3 = .875 or 85%
   Therefore, the probability that at least one child will have HD is 85 percent.

3. A different couple, both of whom are heterozygous for the HD gene, is planning to have children.
   a. Construct the sample space (Punnet square) for possible offspring of this couple.

	H	h
H	HH	Hh
h	hH	hh

   b. What is the probability that these parents will have a child who has HD?
   $\frac{3}{4}$ or 75%
   There is a 75 percent chance that the child will have HD.
   c. If this couple were to have three children, what would be the probability that none of the children would have HD?
   Binompdf (3, 0.75) = 9.015625, .140625, .421875, .421875)
   P(0) = .015625 or 1.6%
   Therefore, the probability that none of the children will have HD is 1.6 percent.

4. A sample of 1,007 people was taken. Each person has only one parent who is heterozygous for the HD gene. Construct the symmetric 95 percent confidence interval for the number of people who would have HD in this sample.

Mean $= \mu = n \times p$
$\mu = 1{,}007 \times 0.50$
$\mu = 503.5$
Standard Deviation $= \sigma = \sqrt{n \times p(1 - P)}$
$\sigma = \sqrt{(1{,}007 \times 0.50 \times (1 - 0.50))}$
$\sigma = 15.9$

95% confidence interval is $\pm 1.96\sigma$

Lower bound $= \mu - 1.96\sigma$
$= 503.5 - 1.96(15.9)$
$= 472.3$

Upper bound $= \mu + 1.96\sigma$
$= 503.5 + 1.96(15.9)$
$= 534.7$

Therefore, with the 95 percent confidence interval, we know that between 472 and 535 of the 1,007 people will have HD.

Most patients with HD take medication on a regular basis. The patient's body excretes the medication at a certain rate. If the patient takes the medication as prescribed, over time the amount of medication in the body will be maintained at a constant level.

Part B

1. A doctor prescribes 16 ml of medication to a certain patient to be taken every four hours over an extended period of time. The patient's body excretes 25 percent of the amount of the medication every four hours. Show the amount of medication that remains in the body after each dose by constructing a spreadsheet or by producing a table with your graphing calculator. Use four decimal places for any columns that contain decimal values. Extend your spreadsheet until it shows that the total amount of medication in the body is maintained at a relatively constant level.

Part B #1

Time (hr)	Amount Excreted (ml)	Residual Amount (ml)	Dosage (ml)	Total Amount in the Body (ml)
0	0.0000	0.0000	16.0000	16.0000
4	4.0000	12.0000	16.0000	28.0000
8	7.0000	21.0000	16.0000	37.0000
12	9.2500	27.7500	16.0000	43.7500
16	10.9375	32.8125	16.0000	48.8125
20	12.2031	36.6094	16.0000	52.6094
24	13.1523	39.4570	16.0000	55.4570
28	13.8643	41.5928	16.0000	57.5928
32	14.3982	43.1946	16.0000	59.1946
36	14.7986	44.3959	16.0000	60.3959
40	15.0990	45.2970	16.0000	61.2970
44	15.3242	45.9727	16.0000	61.9727
48	15.4932	46.4795	16.0000	62.4795
52	15.6199	46.8597	16.0000	62.8597
56	15.7149	47.1447	16.0000	63.1447
60	15.7862	47.3586	16.0000	63.3586
64	15.8396	47.5189	16.0000	63.5189
68	15.8797	47.6392	16.0000	63.6392
72	15.9098	47.7294	16.0000	63.7294
76	15.9323	47.7970	16.0000	63.7970
80	15.9493	47.8478	16.0000	63.8478
84	15.9619	47.8858	16.0000	63.8858
88	15.9715	47.9144	16.0000	63.9144
92	15.9786	47.9358	16.0000	63.9358
96	15.9839	47.9518	16.0000	63.9518
100	15.9880	47.9639	16.0000	63.9639
104	15.9910	47.9729	16.0000	63.9729
108	15.9932	47.9797	16.0000	63.9797
112	15.9949	47.9848	16.0000	63.9848
116	15.9962	47.9886	16.0000	63.9886
120	15.9971	47.9914	16.0000	63.9914
124	15.9979	47.9936	16.0000	63.9936
128	15.9984	47.9952	16.0000	63.9952
132	15.9988	47.9964	16.0000	63.9964
136	15.9991	47.9973	16.0000	63.9973
140	15.9993	47.9980	16.0000	63.9980
144	15.9995	47.9985	16.0000	63.9985

148	15.9996	47.9989	16.0000	63.9989
152	15.9997	47.9991	16.0000	63.9991
156	15.9998	47.9994	16.0000	63.9994
160	15.9998	47.9995	16.0000	63.9995
164	15.9999	47.9996	16.0000	63.9996
168	15.9999	47.9997	16.0000	63.9997
172	15.9999	47.9998	16.0000	63.9998
176	15.9999	47.9998	16.0000	63.9998
180	16.0000	47.9999	16.0000	63.9999
184	16.0000	47.9999	16.0000	63.9999
188	16.0000	47.9999	16.0000	63.9999
192	16.0000	48.0000	16.0000	64.0000
196	16.0000	48.0000	16.0000	64.0000
200	16.0000	48.0000	16.0000	64.0000
204	16.0000	48.0000	16.0000	64.0000
208	16.0000	48.0000	16.0000	64.0000
212	16.0000	48.0000	16.0000	64.0000
216	16.0000	48.0000	16.0000	64.0000
220	16.0000	48.0000	16.0000	64.0000
224	16.0000	48.0000	16.0000	64.0000
228	16.0000	48.0000	16.0000	64.0000
232	16.0000	48.0000	16.0000	64.0000
236	16.0000	48.0000	16.0000	64.0000
240	16.0000	48.0000	16.0000	64.0000

2. When does the total amount of medication in the patient's body appear to level off? What amount of medication is in the patient's body at this point? Explain your answer.

   The total amount of medication in the patient's body levels off after 192 hours. The amount of medication in the patient's body is 64.0000 ml. This is because of a balanced dose and excretion.

3. If the patient forgets to take one dose of the medication in the first 36 hours, what would the effect be on the level of medication in the patient's body?

Part B #3a

Time (hr)	Amount Excreted (ml)	Residual Amount (ml)	Dosage (ml)	Total Amount in the Body (ml)
0	0.0000	0.0000	16.0000	16.0000
4	4.0000	12.0000	16.0000	28.0000
8	7.0000	21.0000	16.0000	37.0000
12	9.2500	27.7500	16.0000	43.7500
16	10.9375	32.8125	16.0000	48.8125
20	12.2031	36.6094	16.0000	52.6094
24	13.1523	39.4570	16.0000	55.4570
28	13.8643	41.5928	16.0000	57.5928
32	14.3982	43.1946	16.0000	59.1946
36	14.7986	44.3959	0.0000	44.3959
40	11.0990	33.2970	16.0000	49.2970
44	12.3242	36.9727	16.0000	52.9727
48	13.2432	39.7295	16.0000	55.7295
52	13.9324	41.7972	16.0000	57.7972

56	14.4493	43.3479	16.0000	59.3479
60	14.8370	44.5109	16.0000	60.5109
64	15.1277	45.3832	16.0000	61.3832
68	15.3458	46.0374	16.0000	62.0374
72	15.5093	46.5280	16.0000	62.5280
76	15.6320	46.8960	16.0000	62.8960
80	15.7240	47.1720	16.0000	63.1720
84	15.7930	47.3790	16.0000	63.3790
88	15.8448	47.5343	16.0000	63.5343
92	15.8836	47.6507	16.0000	63.6507
96	15.9127	47.7380	16.0000	63.7380
100	15.9345	47.8035	16.0000	63.8035
104	15.9509	47.8526	16.0000	63.8526
108	15.9632	47.8895	16.0000	63.8895
112	15.9724	47.9171	16.0000	63.9171
116	15.9793	47.9378	16.0000	63.9378
120	15.9845	47.9534	16.0000	63.9534
124	15.9883	47.9650	16.0000	63.9650
128	15.9913	47.9738	16.0000	63.9738
132	15.9934	47.9803	16.0000	63.9803
136	15.9951	47.9852	16.0000	63.9852
140	15.9963	47.9889	16.0000	63.9889
144	15.9972	47.9917	16.0000	63.9917
148	15.9979	47.9938	16.0000	63.9938
152	15.9984	47.9953	16.0000	63.9953
156	15.9988	47.9965	16.0000	63.9965
160	15.9991	47.9974	16.0000	63.9974
164	15.9993	47.9980	16.0000	63.9980
168	15.9995	47.9985	16.0000	63.9985
172	15.9996	47.9989	16.0000	63.9989
176	15.9997	47.9992	16.0000	63.9992
180	15.9998	47.9994	16.0000	63.9994
184	15.9998	47.9995	16.0000	63.9995
188	15.9999	47.9996	16.0000	63.9996
192	15.9999	47.9997	16.0000	63.9997
196	15.9999	47.9998	16.0000	63.9998
200	16.0000	47.9999	16.0000	63.9999
204	16.0000	47.9999	16.0000	63.9999
208	16.0000	47.9999	16.0000	63.9999
212	16.0000	47.9999	16.0000	63.9999
216	16.0000	48.0000	16.0000	64.0000
220	16.0000	48.0000	16.0000	64.0000
224	16.0000	48.0000	16.0000	64.0000
228	16.0000	48.0000	16.0000	64.0000
232	16.0000	48.0000	16.0000	64.0000
236	16.0000	48.0000	16.0000	64.0000
240	16.0000	48.0000	16.0000	64.0000

If the patient forgot to take the medication within the first 36 hours, the level of medication in his body would only be 63.9996 ml instead of 64.0000 ml after 192 hours, and the medication would not level off at 64.0000 ml until 216 hours.

If the patient forgets to take one dose of the medication sometime on the sixth day, what would the effect be on the level of medication in the patient's body?

Part B 3b

Time (hr)	Amount Excreted (ml)	Residual Amount (ml)	Dosage (ml)	Total Amount in the Body (ml)
0	0.0000	0.0000	16.0000	16.0000
4	4.0000	12.0000	16.0000	28.0000
8	7.0000	21.0000	16.0000	37.0000
12	9.2500	27.7500	16.0000	43.7500
16	10.9375	32.8125	16.0000	48.8125
20	12.2031	36.6094	16.0000	52.6094
24	13.1523	39.4570	16.0000	55.4570
28	13.8643	41.5928	16.0000	57.5928
32	14.3982	43.1946	16.0000	59.1946
36	14.7986	44.3959	16.0000	60.3959
40	15.0990	45.2970	16.0000	61.2970
44	15.3242	45.9727	16.0000	61.9727
48	15.4932	46.4795	16.0000	62.4795
52	15.6199	46.8597	16.0000	62.8597
56	15.7149	47.1447	16.0000	63.1447
60	15.7862	47.3586	16.0000	63.3586
64	15.8396	47.5189	16.0000	63.5189
68	15.8797	47.6392	16.0000	63.6392
72	15.9098	47.7294	16.0000	63.7294
76	15.9323	47.7970	16.0000	63.7970
80	15.9493	47.8478	16.0000	63.8478
84	15.9619	47.8858	16.0000	63.8858
88	15.9715	47.9144	16.0000	63.9144
92	15.9786	47.9358	16.0000	63.9358
96	15.9839	47.9518	16.0000	63.9518
100	15.9880	47.9639	16.0000	63.9639
104	15.9910	47.9729	16.0000	63.9729
108	15.9932	47.9797	16.0000	63.9797
112	15.9949	47.9848	16.0000	63.9848
116	15.9962	47.9886	16.0000	63.9886
120	15.9971	47.9914	16.0000	63.9914
124	15.9979	47.9936	16.0000	63.9936
128	15.9984	47.9952	0.0000	47.9952
132	11.9988	35.9964	16.0000	51.9964
136	12.9991	38.9973	16.0000	54.9973
140	13.7493	41.2480	16.0000	57.2480
144	14.3120	42.9360	16.0000	58.9360
148	14.7340	44.2020	16.0000	60.2020
152	15.0505	45.1515	16.0000	61.1515
156	15.2879	45.8636	16.0000	61.8636
160	15.4659	46.3977	16.0000	62.3977
164	15.5994	46.7983	16.0000	62.7983
168	15.6996	47.0987	16.0000	63.0987
172	15.7747	47.3240	16.0000	63.3240
176	15.8310	47.4930	16.0000	63.4930
180	15.8733	47.6198	16.0000	63.6198
184	15.9049	47.7148	16.0000	63.7148
188	15.9287	47.7861	16.0000	63.7861
192	15.9465	47.8396	16.0000	63.8396

196	15.9599	47.8797	16.0000	63.8797
200	15.9699	47.9098	16.0000	63.9098
204	15.9774	47.9323	16.0000	63.9323
208	15.9831	47.9492	16.0000	63.9492
212	15.9873	47.9619	16.0000	63.9619
216	15.9905	47.9715	16.0000	63.9715
220	15.9929	47.9786	16.0000	63.9786
224	15.9946	47.9839	16.0000	63.9839
228	15.9960	47.9880	16.0000	63.9880
232	15.9970	47.9910	16.0000	63.9910
236	15.9977	47.9932	16.0000	63.9932
240	15.9983	47.9949	16.0000	63.9949

If the patient forgets to take the medication on the sixth day, the level of medication in his body would only be 63.8396 ml instead of 64.0000 ml after 192 hours.

4. After 10 days, the method of treatment is changed, but before a new treatment can be started, all of the current medication must be excreted from the patient's body. Use a spreadsheet or your graphing calculator to determine the amount of time it will take, to the nearest day, for the patient's body to excrete all the medication. Show how you determine this answer and justify it.

Part B #4

Time (hr)	Amount Excreted (ml)	Residual Amount (ml)	Dosage (ml)	Total Amount in the Body (ml)
0	0.0000	0.0000	16.0000	16.0000
4	4.0000	12.0000	16.0000	28.0000
8	7.0000	21.0000	16.0000	37.0000
12	9.2500	27.7500	16.0000	43.7500
16	10.9375	32.8125	16.0000	48.8125
20	12.2031	36.6094	16.0000	52.6094
24	13.1523	39.4570	16.0000	55.4570
28	13.8643	41.5928	16.0000	57.5928
32	14.3982	43.1946	16.0000	59.1946
36	14.7986	44.3959	16.0000	60.3959
40	15.0990	45.2970	16.0000	61.2970
44	15.3242	45.9727	16.0000	61.9727
48	15.4932	46.4795	16.0000	62.4795
52	15.6199	46.8597	16.0000	62.8597
56	15.7149	47.1447	16.0000	63.1447
60	15.7862	47.3586	16.0000	63.3586
64	15.8396	47.5189	16.0000	63.5189
68	15.8797	47.6392	16.0000	63.6392
72	15.9098	47.7294	16.0000	63.7294
76	15.9323	47.7970	16.0000	63.7970
80	15.9493	47.8478	16.0000	63.8478
84	15.9619	47.8858	16.0000	63.8858
88	15.9715	47.9144	16.0000	63.9144
92	15.9786	47.9358	16.0000	63.9358
96	15.9839	47.9518	16.0000	63.9518
100	15.9880	47.9639	16.0000	63.9639

104	15.9910	47.9729	16.0000	63.9729
108	15.9932	47.9797	16.0000	63.9797
112	15.9949	47.9848	16.0000	63.9848
116	15.9962	47.9886	16.0000	63.9886
120	15.9971	47.9914	16.0000	63.9914
124	15.9979	47.9936	16.0000	63.9936
128	15.9984	47.9952	16.0000	63.9952
132	15.9988	47.9964	16.0000	63.9964
136	15.9991	47.9973	16.0000	63.9973
140	15.9993	47.9980	16.0000	63.9980
144	15.9995	47.9985	16.0000	63.9985
148	15.9996	47.9989	16.0000	63.9989
152	15.9997	47.9991	16.0000	63.9991
156	15.9998	47.9994	16.0000	63.9994
160	15.9998	47.9995	16.0000	63.9995
164	15.9999	47.9996	16.0000	63.9996
168	15.9999	47.9997	16.0000	63.9997
172	15.9999	47.9998	16.0000	63.9998
176	15.9999	47.9998	16.0000	63.9998
180	16.0000	47.9999	16.0000	63.9999
184	16.0000	47.9999	16.0000	63.9999
188	16.0000	47.9999	16.0000	63.9999
192	16.0000	48.0000	16.0000	64.0000
196	16.0000	48.0000	16.0000	64.0000
200	16.0000	48.0000	16.0000	64.0000
204	16.0000	48.0000	16.0000	64.0000
208	16.0000	48.0000	16.0000	64.0000
212	16.0000	48.0000	16.0000	64.0000
216	16.0000	48.0000	16.0000	64.0000
220	16.0000	48.0000	16.0000	64.0000
224	16.0000	48.0000	16.0000	64.0000
228	16.0000	48.0000	16.0000	64.0000
232	16.0000	48.0000	16.0000	64.0000
236	16.0000	48.0000	16.0000	64.0000
240	16.0000	48.0000	16.0000	64.0000
244	16.0000	48.0000	0.0000	48.0000
248	12.0000	36.0000	0.0000	36.0000
252	9.0000	27.0000	0.0000	27.0000
256	6.7500	20.2500	0.0000	20.2500
260	5.0625	15.1875	0.0000	15.1875
264	3.7969	11.3906	0.0000	11.3906
268	2.8477	8.5430	0.0000	8.5430
272	2.1357	6.4072	0.0000	6.4072
276	1.6018	4.8054	0.0000	4.8054
280	1.2014	3.6041	0.0000	3.6041
284	0.9010	2.7030	0.0000	2.7030
288	0.6758	2.0273	0.0000	2.0273
292	0.5068	1.5205	0.0000	1.5205
296	0.3801	1.1403	0.0000	1.1403
300	0.2851	0.8553	0.0000	0.8553
304	0.2138	0.6414	0.0000	0.6414
308	0.1604	0.4811	0.0000	0.4811
312	0.1203	0.3608	0.0000	0.3608
316	0.0902	0.2706	0.0000	0.2706

320	0.0677	0.2030	0.0000	0.2030
324	0.0507	0.1522	0.0000	0.1522
328	0.0381	0.1142	0.0000	0.1142
332	0.0285	0.0856	0.0000	0.0856
336	0.0214	0.0642	0.0000	0.0642
340	0.0161	0.0482	0.0000	0.0482
344	0.0120	0.0361	0.0000	0.0361
348	0.0090	0.0271	0.0000	0.0271
352	0.0068	0.0203	0.0000	0.0203
356	0.0051	0.0152	0.0000	0.0152
360	0.0038	0.0114	0.0000	0.0114
364	0.0029	0.0086	0.0000	0.0086
368	0.0021	0.0064	0.0000	0.0064
372	0.0016	0.0048	0.0000	0.0048
376	0.0012	0.0036	0.0000	0.0036
380	0.0009	0.0027	0.0000	0.0027
384	0.0007	0.0020	0.0000	0.0020
388	0.0005	0.0015	0.0000	0.0015
392	0.0004	0.0011	0.0000	0.0011
396	0.0003	0.0009	0.0000	0.0009
400	0.0002	0.0006	0.0000	0.0006
404	0.0002	0.0005	0.0000	0.0005
408	0.0001	0.0004	0.0000	0.0004
412	0.0001	0.0003	0.0000	0.0003
416	0.0001	0.0002	0.0000	0.0002
420	0.0001	0.0002	0.0000	0.0002
424	0.0000	0.0001	0.0000	0.0001
428	0.0000	0.0001	0.0000	0.0001
432	0.0000	0.0001	0.0000	0.0001
436	0.0000	0.0000	0.0000	0.0000

Therefore, it will take 18 days (436 hours) for the patient to excrete all the medication.

If the patient must stop the medication after 10 days and wait until it is all excreted from his body, then the patient must wait 18 days (436 hours) until all the medication is excreted.

Size of the Box

There are 1,000,000 steel balls, each of which has a diameter of 1 mm. They are to be placed in a box. What is the size of the box, and can one person carry the box?

NCTM Standards in Action The Process Standard: Communication

Klaus Puhlmann

NCTM's *Principles and Standards for School Mathematics* (2000) identifies 10 standards that form an essential and comprehensive foundation for students from Kindergarten to Grade 12 as they learn mathematics. The standards are divided into five content standards (number and operation, algebra, geometry, measurement, and data analysis and probability) and five process standards (problem solving, reasoning and proof, communication, connections and representation). This article focuses on one process standard: communication.

The *Alberta Program of Studies for K–12 Mathematics* has several main goals for students learning mathematics. These include preparing students to

- use mathematics confidently to solve problems,
- communicate and reason mathematically,
- appreciate and value mathematics,
- commit themselves to lifelong learning and
- become mathematically literate adults, using mathematics to contribute to society.

As indicated above, the focus in this article will be on an important process standard: communication.

Our teaching and learning of mathematics must shift away from having a single-minded focus on arriving at the answer to a problem to students being able to communicate effectively how an answer was obtained. This shift requires that students have the opportunity to read, explore, investigate, write, listen to, discuss and explain ideas in their own language of mathematics. Communication plays an important role in the construction and connection of ideas. But what is mathematical communication? What are the various forms of communicating mathematical thinking? What is the value of communication in mathematical learning? Teaching students to communicate mathematical ideas clearly and effectively, orally and in writing must be a conscious decision on the part of the teachers; it must be woven into the lesson planning. Teachers need to ensure that students communicate with other students and themselves. It is critical that teachers see communication as an essential part of mathematics and mathematics education. The way in which it is woven into the lessons can and should take various forms, recognizing that as students progress through the grades, the mathematics that they communicate should become more complex and abstract. Teachers need to recognize and respect the diverse ways that students communicate, and students must come to understand the different ways that ideas can be presented to different audiences and for different purposes.

Teachers must make an important decision in their planning about using the teacher–student relationship to facilitate communication. Stories or student samples that can be used to facilitate student learning are all important considerations. Decisions about mathematical discourse are not only meant to make it meaningful for students but to encourage them to actively participate in constructing knowledge.

According to the NCTM standards, as laid out in *Principles and Standards for School Mathematics* (2000, 60), instructional programs from pre-Kindergarten through Grade 12 should enable all students to

- organize and consolidate their mathematical thinking through communication;
- communicate their mathematical thinking coherently and clearly to peers, teachers and others;
- analyze and evaluate the mathematical thinking and strategies of others; and
- use the language of mathematics to express mathematical ideas precisely.

The most prevalent form of communicating in mathematics classrooms in the past has been through listening. It is encouraging that students are now experiencing a more balanced approach, involving writing, reading and listening. Writing and reading are communication skills that have been used too infrequently in the mathematics classroom. We need to more frequently pose questions like: Why is your answer correct? What have you learned from this exercise? What is another method to find the solution? What have you learned about a particular topic today? How is this idea or concept related to what we have learned before? Posing these and other questions as activities, assignments or mathematics journal entries is just another way to facilitate communication and give students opportunities to reflect on their learning.

As teachers we need to, first of all, value the power of communicating mathematics. We need to clarify what strategies we want to use to assign, manage and assess students' journal writing; select appropriate reading/writing assignments that contribute to the learning of mathematics; encourage and facilitate reflective thinking; incorporate technology or other resources to enhance mathematical learning; and assess students' communication and understanding of mathematics.

Students must have opportunities to test their ideas with others in the mathematics classroom or elsewhere to see if their ideas can be understood and if they are confident about what they know. To foster these communication skills, students must feel secure and supported when expressing their ideas. When students graduate, their ultimate goal is to be confident in engaging in a dialogue about mathematics and to present their ideas clearly, completely and precisely. This goal will only be achieved if communication receives the emphasis that it deserves throughout the grades. Much of what students are exposed to in mathematics depends on their skill to write properly, yet little time is being devoted to teaching and practising that skill in mathematics. "The process of learning to write mathematically is similar to that of learning to write in any genre" (NCTM 2000, 62). Guided practice is important and so is our attention to teaching students the specifics of mathematical arguments, including the use and special meaning of mathematical language and the representations and standards of explanation and proof.

Students also need to be exposed to the thinking of others who might view problems from a different perspective. This allows students to make sense of the thinking of others and to incorporate it into their own interpretive framework. Having students exposed to the thinking of others enhances their own critical thinking skills, which is important in mathematics.

As it is the goal to ultimately teach students the proper use of the mathematical language and the clear and precise expression of mathematical ideas, opportunities must be provided for students to practise this throughout the grades. This also implies that we need to teach students to become good collaborative workers so that they can exchange mathematical ideas effectively with others.

Teachers must help students develop skills in mathematical communication that will not only serve them well both inside and outside the classroom but also enhance their overall level of mathematical literacy.

The three articles that follow deal with the issue of communication in mathematics. Constructive and practical suggestions are being offered in these three articles for use by teachers in their classroom. The first article deals with the question, How can we communicate in the language of mathematics? The second article demonstrates how open-ended tasks are being used to allow students to select their own approaches and to communicate mathematical ideas. The third article talks about using communication to develop students' mathematical literacy. It suggests that allowing students to develop skills in communicating mathematically is pivotal to becoming mathematically literate.

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Communicating in the Language of Mathematics

Larry Buschman

As the classroom mathematics curriculum expands to encompass the entire range of skills included in the NCTM's *Curriculum and Evaluation Standards for School Mathematics* (1989), the process by which a student arrives at the answer to a problem becomes as important as the answer itself. Answers alone often fail to reveal the nature of a student's thinking, the strategies used in the problem-solving process or the level of understanding. Additionally, the standards document includes the expectation that students will be able to "relate their everyday language to mathematical language and symbols" (NCTM 1989, 26).

Using oral or written communication as a tool with which students can reflect their understanding of mathematics helps them make connections and personalize mathematical concepts. When students communicate mathematical information, they remember it, understand it, and use it to uncover and find even more information (Perkins 1992).

Teachers need to know how to help students grow into accomplished communicators of mathematics who can describe their thinking processes clearly. Teachers must help students make their thinking visible to others by encouraging them to talk and write about the process they use to solve problems.

The author's past efforts to encourage students to discuss and explain their problem-solving process have focused on (1) journal writing, (2) student-authored story problems, (3) the mathematician's chair, (4) cooperative-learning activities, and (5) parent newsletters. However, two articles in the May 1992 issue of *Educational Leadership*—"Creating Tests Worth Taking" by Grant Wiggins and "Evaluating Problem Solving in Mathematics" by Walter Szetela and Cynthia Nicol—present more ideas to add to the existing student-communication activities occurring in a second-grade classroom. The goal of embedding speaking and writing into the daily mathematical activities of students is being met through the following activities.

Mathematics Journal

Ask students to keep a mathematics journal, which can constitute a major part of the daily curriculum but, when added to a student's portfolio, can also furnish an ongoing record of the student's mathematical growth.

In our classroom, students begin each day by recording statistical data related to the date, weather and various problem-solving activities (Figure 1): (1) day, date, number of school days attended in the current school year and number of school days remaining; (2) at least five numbered sentences that equal the date; (3) the weather report—temperature, precipitation, wind speed, wind direction and cloud type; (4) predictions for the next day's weather and the color, size and shape of the next day's calendar piece; (5) answers to various measurement activities, such as the time shown on a Judy clock, the amount of money in a container, the weight of an object or group of objects, or an estimation of the quantity, weight or length of an object using a standard for comparison; and (6) the solution to a open-ended mathematics problem.

At the end of each day, students use their journal to reflect on the day's mathematical activities. Students are asked to think about how they would answer such questions as "How did you help another person?" and "What did you learn that you did not know before?"

Student-Authored Story Problems

Ask students to create original story problems for someone else—a classmate, a teacher, a student in another classroom or a family member—to solve.

The directions to the student include the following:

- Write a story problem using you imagination or the information in a picture, newspaper advertisement, poster or short story.
- Have other people solve your problem.
- After seeing the solutions to your problem, lead a class discussion about your problem and the solutions.

Mathematician's Chair

Ask students to sit in a chair that has been designated the "mathematician's chair" and to share original problems that they have authored or solutions to a problem written by someone else.

A mathematician's chair is very similar to an author's chair, except that students share mathematics problems and solutions rather than stories or books. Expect students, while in the mathematician's chair, to use effective speaking skills and to communicate their thoughts clearly and completely. Also, expect classmates to use effective listening skills and to give the author useful and usable feedback:

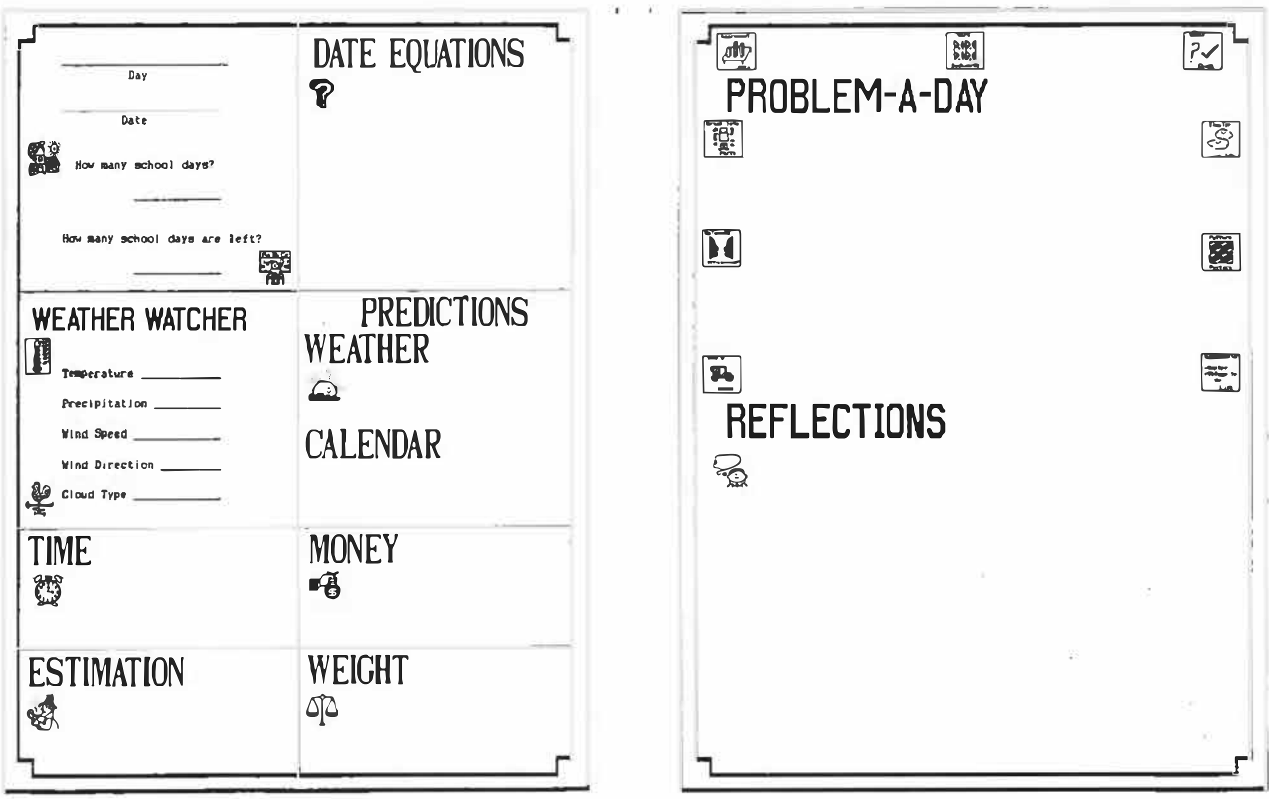
- What did you think about the problem?
- Do you agree or disagree with the solution?
- How could the author improve the problem or solution?
- How could the author change the problem to create a new problem or change the solution to arrive at a new way to solve the problem?

Cooperative Learning

Have students engage in cooperative problem solving by asking them to describe the process that they will use to solve a problem, to work collaboratively on the problem, and to reflect on the effectiveness of the group and the contributions of individual members.

Talking with peers in cooperative-learning groups is especially important for young children. Students become comfortable with new words when they are free to experiment with language in a nonthreatening environment. To communicate their thinking to others more effectively, students must have frequent opportunities to hear and speak mathematics with peers, teachers and parents.

Figure 1
Mathematics journal



Family Newsletter

Ask students to write weekly or monthly "Family Newsletters."

Give the following directions to the students:

- Pretend you are a reporter for a newspaper. Write a story about something that occurred in mathematics class since the last "Family Newsletter." Use your journal and portfolio to help write the story.
- Ask at least two other students to listen to the story so that they can offer ideas for improvement.

Mathematics Communication Structures

"Mathematics communication structures" were created to add variety to students' communication tasks. Each structure listed subsequently was designed to give students a framework that supports and enhances the process of mathematical communication.

Structure 1

Present a problem and the answer arrived at by an imaginary person. Have the student write a letter to this person, explaining agreement or disagreement with the answer.

- Directions to the student: (1) Write a letter to the person who solved this problem. (2) Explain why you agree or disagree with the answer.
- Sample problem: Which number does not belong? Kristina thinks the answer is 6.

6	12
10	13

Structure 2

Present an already solved problem with a significant error. Have the student comment on the error by reacting to a series of questions about the solution.

- Directions to the student: (1) Read the problem and look at how this person solved the problem. (2) Answer each question that follows the problem.
- Sample problem: Chris and Bob have to be home by 9 p.m. It is now 7 p.m. How many hours may they play before they have to go home? Travis solved the problem this way.

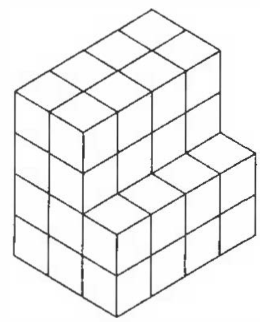
$$\begin{array}{r} 9 \\ +\,7 \\ \hline 16 \end{array}$$ I say 16 hours.

Explain whether Travis's reasoning was correct or incorrect.

Structure 3

Present a problem with all the facts and conditions, but have students write a different question for the problem. Have the students solve the new problem and explain why their new question made the problem more or less difficult to solve.

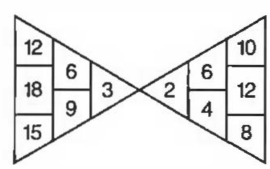
- Directions to the student: (1) Solve the problem as stated. (2) Write a different question but do not change any other part of the problem situation. (3) Solve the new problem and explain why this problem was easier or harder to solve than the original.
- Sample problem: This shape is made by stacking cubes on top of one another. How many cubes are in the bottom layer?



Structure 4

Present a problem and a partial solution. Have the students complete the solution.

- Directions to the student: (1) Finish the solution to this problem. (2) Describe another way to solve the problem.
- Sample problem: List all the ways that you could score 18 points by throwing two darts. Here is Aaron's partial solution:
  One dart hit the 3 and the other dart hit the 15.



Structure 5

Present a problem with facts unrelated to the question. Have the students identify these facts and rewrite the problem, leaving out any irrelevant information.

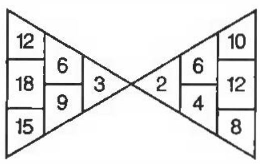
- Directions to the student: (1) Which facts are not needed to solve the problem? (2) Rewrite the problem, leaving out any unneeded information. (3) Solve the problem.

- Sample problem: Farmer Brown has 10 cows and 23 horses. Farmer Smith has 16 cows and 17 horses. If Farmer Brown and Farmer Smith put all their cows in the same barn, how many cows would be in the barn? (Written by Melissa Santoyo, Jefferson Elementary School.)

Structure 6

Present a problem and have the students explain how to solve the problem using only words. Have the students construct and solve a similar problem.

- Directions to the student: (1) Using only words, tell how you would solve this problem. (2) Write a similar problem and describe all the ways that the two problems are alike and different. (3) Solve the problem you wrote.
- Sample problem: Jill threw three darts at the target shown. What is the largest score she could make?



Structure 7

After the students have solved a problem, have them write a new problem with a different context, preserving the original problem structure.

- Directions to the student: (1) Solve the problem. (2) Tell how you solved the problem. (3) Write a new problem that can be solved in the same way. Give your problem to another person and check the solution.
- Sample problem: Tim and Bill are going camping for three days. The guidebook says that four campers need six litres of water for each day. How much water do you think that Tim and Bill should take on their camping trip? Explain your answer.

Structure 8

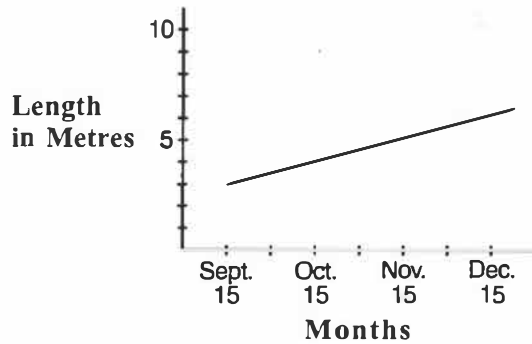
Present a problem without numerals. Have the students estimate the missing numbers, research appropriate numerals and solve the problem. The problem should be based on a real-world situation—the missing information should be available to the student by gathering these data.

- Directions to the student: (1) Estimate the answer. (2) Complete the problem using the data you have gathered. (3) Solve the problem and tell how you found the answer.
- Sample problem: Mr. Buschman drives______ miles to work each day. How many miles does he drive to work during October?

Structure 9

Present the students with a graph or table and have them write a story that represents the data on the graph or table.

- Directions to the student: Write a story about the data in this graph or table.
- Sample problem: Ben's shadow at the same time each time.



Structure 10

Present the students with trends or sample data. Ask them to make predictions to complete the table and write a story that includes the data in the table.

- Directions to the student: (1) Look carefully at the data. (2) Make predictions using the data and fill in the missing information in the table. (3) Explain why you think your predictions are accurate. (4) Write a story that includes the data in the table.
- Sample problem: Rainfall amounts measured in inches.

	Jan.	Feb.	March	April	May	June
Portland	6.4	6.8		3.4		1.0
Jefferson	5.9	6.2	4.4	3.1		
Baker	4.3		1.7	0.5		0.0
Astoria	10.6	11.3			7.5	4.3

Structure 11

Present the student with a real classroom problem that requires a group of students to share actual objects in the classroom or on the playground. Have the student develop and test a plan for sharing the objects.

- Directions to the student: (1) Make a plan for sharing the objects in the problem. (2) Test your plan to see how it works. (3) Have another student tell why he or she thinks your plan is fair or unfair. (4) Write your teacher a letter in which you describe your plan and tell how well your plan worked.

- Sample problem: You have 10 students in your group, but your group only has two bags of pattern blocks. You will get to use the pattern blocks for 20 minutes each day this week. Only one student can use a bag of pattern blocks at a time. Develop a plan for how your group can share the two bags of pattern blocks.

Structure 12

Ask the students to write and publish an original story problem in the form of a "letter problem" (Figure 2). Once completed, letter problems are placed in classrooms throughout the school for other students to solve. Solutions are mailed to the problem's author through the in-school mail-delivery system.

- Directions to the student authoring a letter problem: (1) Write five original story problems. Meet with two other students and use consensus building to choose the best problem. (2) Edit and publish your problem. (3) Place several copies of your problem in chosen classrooms. (4) Read all solutions and reply to each person, telling why you agree or disagree with the solution.
- Directions to the student solving a letter problem: (1) Take one letter problem from the display. Solve the problem, write in detail how you found the answer and tell why you think your answer is both correct and complete. (2) Mail your solution to the author of the problem through the in-school mail-delivery system. You will receive a reply in a few days.

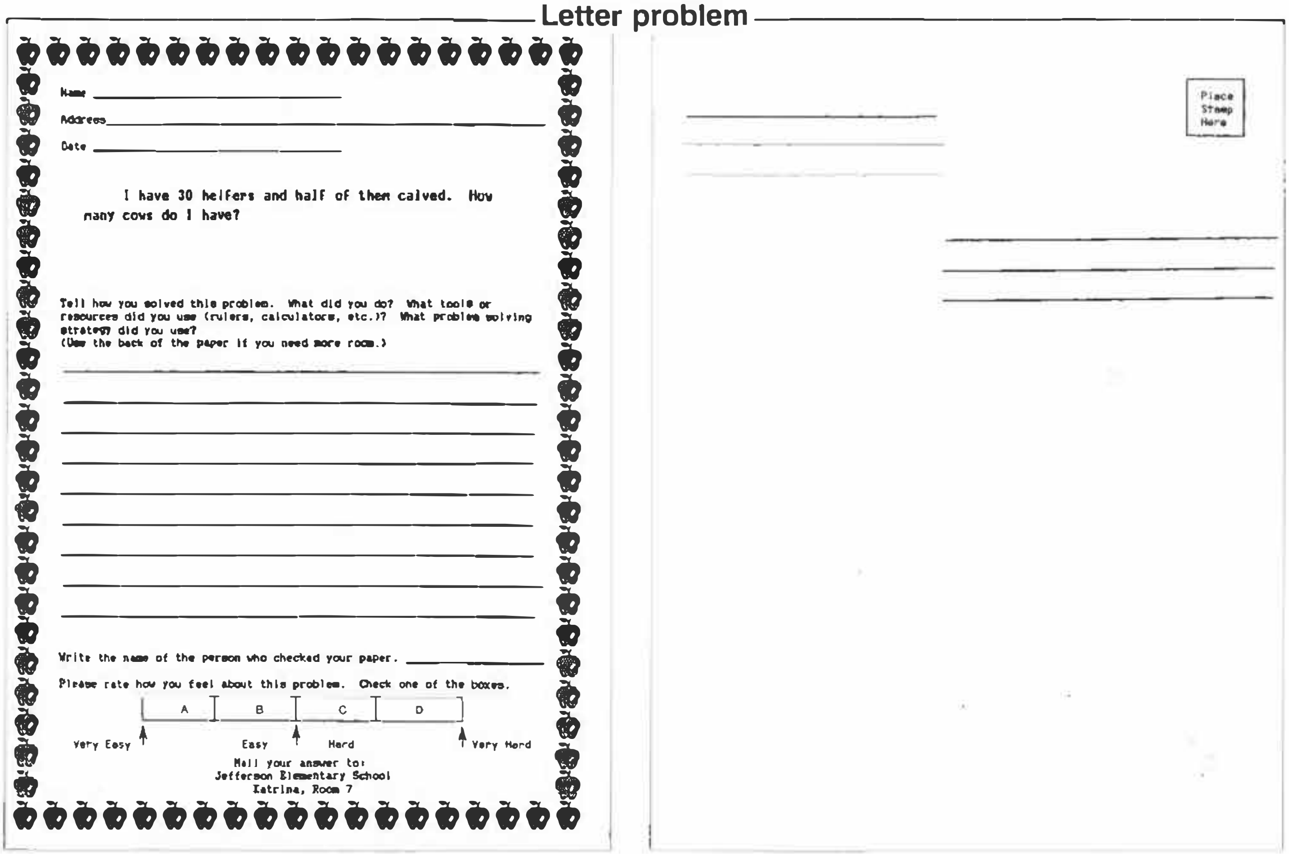
Structure 13

Present the students with a very open-ended problem and have them request the information needed to solve the problem.

- Directions to the student: (1) Request any information you need to answer the question. (2) When you think you have enough information, solve the problem. (3) If you find that you need more information, request help from your teacher.
- Sample problem: How much will it cost for the second-grade field trip? When introducing this type of problem to students, begin with a whole class activity using oral requests and responses

Figure 2

Letter problem


Name

Address

Date

I have 30 heifers and half of them calved. How many cows do I have?

Tell how you solved this problem. What did you do? What tools or resources did you use (rulers, calculators, etc.)? What problem solving strategy did you use?
(Use the back of the paper if you need more room.)

Write the name of the person who checked your paper.

Please rate how you feel about this problem. Check one of the boxes.

A B C D

Very Easy Easy Hard Very Hard

Mail your answer to:
Jefferson Elementary School
Katrina, Room 7

Place Stamp Here

which are recorded and displayed on the overhead projector. As students become comfortable with making requests, ask them to work problems individually or in small groups and to make their requests in writing. By varying the type of information given to students, teachers can control the level of difficulty of the problem. For example, if a student asked, "How much does the bus for the field trip cost?" one of the following replies could be given: $150.00; each bus costs $75.00; each bus costs $55.00 for the driver and $20.00 for gasoline; or each bus costs $8.35 an hour for the driver and $0.05 a mile for the gasoline.

Structure 14

Ask the students to revise a fairy tale or folk tale to include numerical information. This new version of the tale can then be used as a source for generating story problems.

- Directions to the student: (1) Pick a favorite fairy tale or folk tale. Read the tale you have chosen to an adult. (2) Rewrite the tale by adding numerical information. (3) Write five story questions that someone else could answer using the added information.
- Sample tale: "The Five Bears" by Andrea Kachel, Grade 2, Jefferson Elementary School.

  Once upon a time there were five bears. The Papa bear was the oldest. He was 39 years old. Next came the Mama bear. She was 35 years old. Next came their son Andrew. He was 18 years old. Next came Andrea. She was 9 years old. Last came Jessica. She was 3 years old and a real brat. Every day Mama bear would leave for work at 7:00 in the morning and she would get home at 5:00 in the afternoon. The children left for school at 8:00 in the morning and came home on the hot, noisy, bumpy bus at 3:30. Papa bear took baby Jessica to the day care at 9:00 in the morning on his way to work in the honey factory.

Conclusion

Students need time to observe, to work together and to construct an understanding of the language of mathematics and to make it their own. Personal knowledge becomes useful and usable in social situations when combined with the knowledge of others. Thoughts, ideas and the meanings of words are focused and clarified when individuals engage in conversation.

As soon as students use words, they make their understanding of mathematics more precise and more general at the same time. Only by using words in many situations and contexts do students come to understand the full meaning of each word. When students write or talk about mathematics problems, they test, expand and extend their understanding of mathematics. When students write or speak, they do not use language just to express their thoughts; they use the process of communicating with others to engage in a conversation with their own mind.

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Understanding Student Responses to Open-Ended Tasks

Barbara M. Moskal

Communicating mathematical knowledge is a challenge for students, and assisting students in clearly expressing their mathematical ideas is a challenge for teachers. Open-ended tasks give students opportunities to select their own approaches for both solving problems and expressing mathematical ideas (Billstein 1998; Conway 1999). Students' responses to these tasks give teachers evidence of their students' problem-solving and communication skills.

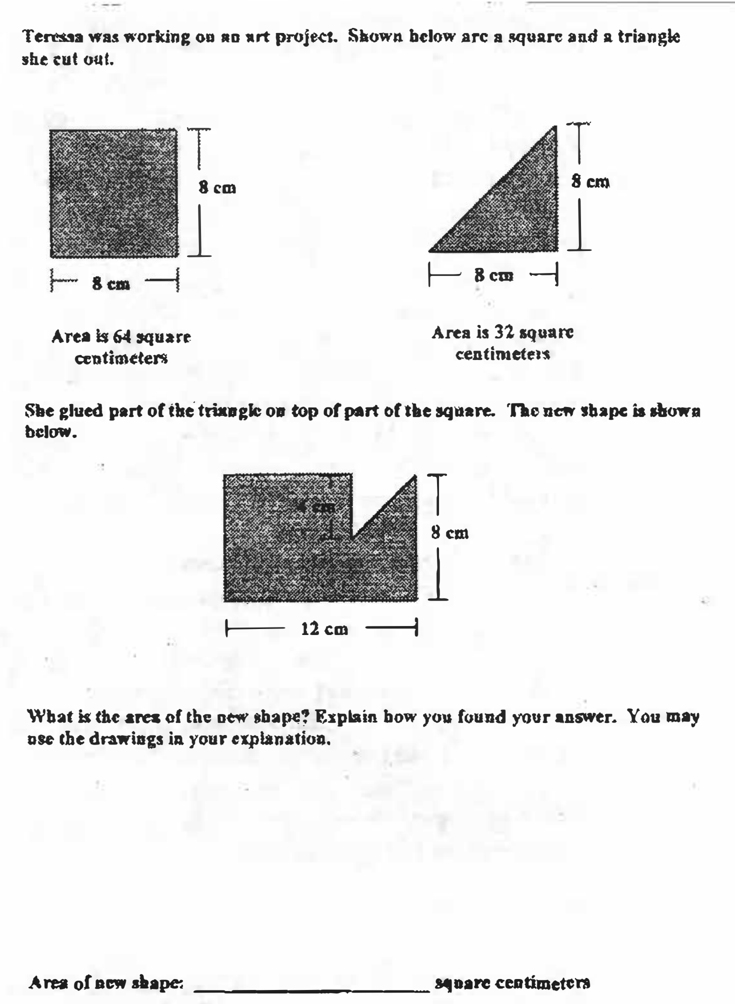
This article discusses examples of the detailed explanations that students offered in response to a written, open-ended geometry task, especially the ways in which students communicated their knowledge. As this article illustrates, different students may select different methods of communication, such as using text, diagrams or mathematical symbols, to display their solution processes. Examining the student solution processes helps the teacher better understand the students' mathematical knowledge.

Student Responses

The irregular area task, shown in Figure 1, was administered to sixth-grade students (Moskal 1997a, 1997b) after they had received instruction in determining the area of squares, rectangles and triangles. The Irregular Area Task was originally developed as part of the Quantitative Understanding: Amplifying Student Achievement and Reasoning (QUASAR) project (see Lane et al. 1995). Although a classroom of students completed this task, only four responses will be discussed here. To maintain confidentiality, both the students' and the teacher's names have been changed.

At the start of the area unit, Ms. Harding showed her students how to subdivide a rectangular figure into unit squares. She explained to them that the total number of unit squares that made up the rectangle was the *area* of the rectangle. Building from this conceptualization of area, she derived with her students the standard formula for finding the area of a rectangle, that is, base × height. She used a similar approach to

Figure 1
Irregular Area Task



develop the formula for finding the area of a triangle. Harding dedicated a great deal of classroom attention to explaining how portions of unit squares could be combined to create complete unit squares when finding the area of a triangular figure. Before administering the Irregular Area Task, however, Harding had not addressed how the area concept could be extended to find the area of figures other than squares, rectangles and triangles. Harding had previously used open-ended tasks in the classroom, and she repeatedly encouraged her students to write complete solutions to open-ended mathematics problems.

Naser's Response

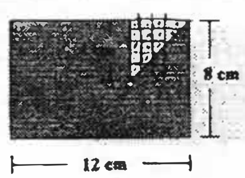
Naser, whose response is shown in Figure 2, used a combination of text and diagrams to explain his answer. His explanation suggests that Naser was attempting to reshape the irregular figure into a large rectangle with a base of 12 cm and a height of 8 cm. He indicated that he has calculated the area of the larger rectangle, but he did not include the outcome of this computation.

Although Naser's overall approach is correct, reforming the figure into a rectangle and removing the area of the missing triangle, he made several mistakes in the solution process. Then he explained that he had subtracted 15 square units from the rectangle to get his answer of 81. Fifteen square units was his calculated area for the missing triangle in the rectangle. Naser had drawn three vertical lines and four horizontal lines, partitioning the square into 20 rectangular regions (see Figure 2). If Naser's goal was to partition the square into unit squares, he should have drawn three lines vertically and three lines horizontally, which would have resulted in 16 square units. Small pencil dots, visible in each rectangle, make up the missing triangular region. Naser apparently counted the rectangles in the missing triangle to arrive at the value of 15. When Naser counted the rectangles in the missing triangle, he may have treated the segments of whole units along the hypotenuse of the triangle as whole units. This scenario would account for an area of 15 cm^2, rather than 10 cm^2, on the basis of a right triangle with a base of 4 and a height of 5.

Harding expressed disappointment in Naser's response. She was surprised that Naser was able to develop such an "elegant" solution yet not able to find the area of the missing triangle. When asked what made Naser's response "elegant," Harding explained, "He knew to create a larger rectangle. He knew to get rid of the little piece [referring to the triangle] . . . it was pretty good [referring to the solution]." Harding further explained that during instruction, they had not discussed how to find an unknown area by breaking the unknown region into parts with known areas. Naser's response demonstrated that he was able to extend the concepts beyond classroom instruction.

Harding also expressed frustration with a portion of Naser's response. Naser had counted the units along the hypotenuse of the right triangle as whole units. The concept of combining half units to create whole units had been directly addressed in class. Several other students had also made this mistake. Harding believed that the recurrence of this error suggested that her efforts to clarify the difference between half and whole units had been ineffective for many students.

Figure 2
Naser's response



What is the area of the new shape? Explain how you found your answer. You may use the drawings in your explanation.

I made the picture one big rectangel. Than I x 12 x 8 then I went back to the regular shape and I − 15 square area because thats what was the amont of the little shape in the rectangel and thats how I got my answer 81

Area of new shape: 81 square centimeters

Kevin's Response

Kevin reproduced the irregular figure and labeled the length of each side (see Figure 3). The labeling on the text bubbles of Kevin's explanation suggests that he was aware that the sides of a square are congruent. Kevin's response also contains errors. For example, he argued that the hypotenuse of the removed triangle was 4 cm because it was the same as the leg of the missing triangle. He then added the sides of the irregular figure, and because of his incorrect work, obtained a sum of 44 as the result. Kevin found the perimeter of the irregular figure rather than the area. He also explained, "and then I multiply that by 2." This incorrect step results in his final answer of 88. Kevin may have remembered from class that calculating the area involved multiplication. By multiplying by 2, Kevin forced multiplication to appear in his work. Although his final answer to the problem was correct (that is, 88 cm^2), he acquired this answer through a series of errors.

At first, Harding was pleased with Kevin's response. She said, "Kevin does not always understand math, but this time he got it." She also said that she was impressed by the detail that he provided in his response. On closer inspection, however, Harding realized that her initial evaluation was inappropriate. She had seen the correct answer and the "detail that filled the page" but had not examined the contents of the response. After reading through Kevin's response, she concluded, "He thinks he is finding a perimeter." Harding also expressed concern about Kevin's efforts to multiply by 2. She did not understand why Kevin would include this step and intended to ask him to explain his work orally the next day. If Kevin had given only his final answer, his teacher may have assumed that he had a firm understanding of the concepts being assessed.

Figure 3
Kevin's response

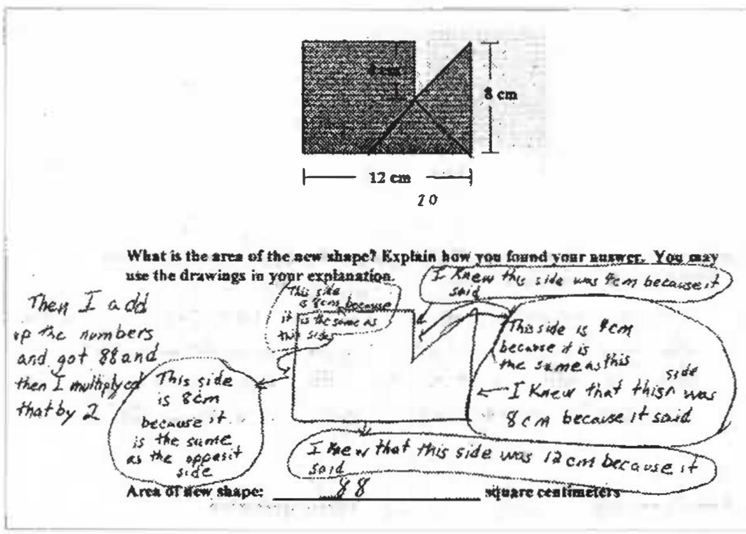
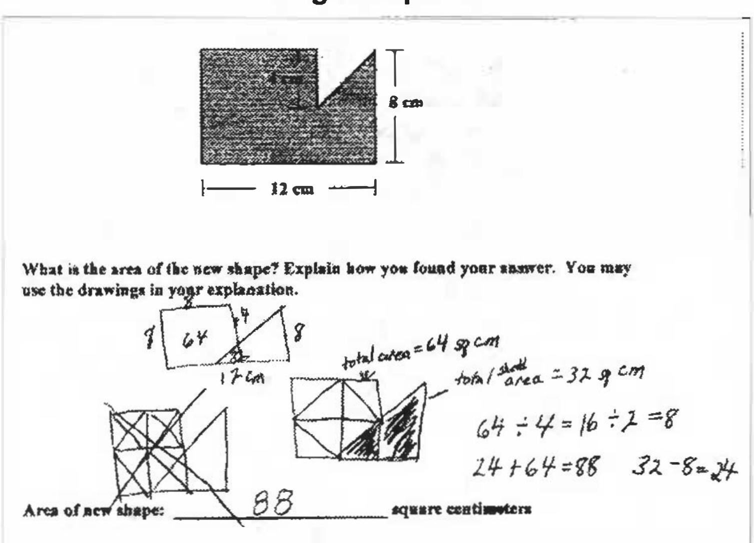


Figure 4
Ning's response



Ning's Response

As shown in Figure 4, Ning created a series of diagrams to explain her response. She first divided the irregular figure into the overlapping square and triangle. She labeled each subdivision with the given areas. She then divided the square into four smaller squares with equal areas. She divided each resulting square in half to form two triangles with equal areas. The process that Ning used is clarified through the accompanying calculations. Using symbols, Ning divided the area of the large square (64) by 4, resulting in a value of 16. She then divided the area of the small squares (16) by 2, giving her the area of the small triangles (8). The supporting calculations clarify her pictorial representation.

Ning's next step was to remove the area of overlap (8) between the given square and triangle from the area of the given triangle (32). Apparently, Ning recognized that this amount is accounted for in the areas of both the large square and the large triangle. Finally, she summed the resulting value with the area of the large square, obtaining the correct answer of 88.

Ning's teacher reacted to her response with great surprise. She said, "Ning's English is so flawed; I really don't have a grasp of what Ning knows and doesn't know." Ning's family has recently moved to the United States, and her English skills were not well developed. From

Ning's response, Harding decided, "I need to provide her with more opportunities to explain herself through pictures. It is clear to me now . . . this might be a way to get beyond her language problems." Although Ning's response contained few words, it conveyed a great deal of understanding. Through the use of diagrams and mathematical symbols, Ning was able to effectively communicate her reasoning process.

Becky's Response

Becky also produced a correct solution in response to the Irregular Area Task. Her explanation is shown in Figure 5. Although she wrote that she had drawn 12 horizontal lines and 8 vertical lines on the diagram of the irregular figure, Becky had drawn 11 vertical lines that partitioned the region into 12 vertical sections and 7 horizontal lines that partitioned the region into 8 horizontal sections. Although Becky incorrectly conveyed her approach in her written response, the diagram that she provided clarified the process she had used.

Becky explained that she had counted 8 unshaded squares. Then she subtracted these 8 from 96, the area of the whole figure. On the left side of Becky's paper, she wrote the calculation 12 × 8 and the resulting answer of 96. Becky's explanation states that she counted the units that made up the missing triangle. Examination of her diagram suggests that she also counted the number of units in the larger rectangle. In Becky's drawing, the large rectangle is subdivided in 96 unit squares. Each of these unit squares contains a pencil dot. It seems that Becky either multiplied 12 × 8 to verify the number of unit squares that she counted or counted the number of unit squares to verify her multiplication.

To explain how she acquired the area of the missing triangle (8), she wrote that she had counted the unshaded squares. This explanation does not appear to be supported by her diagram. In the diagram, the missing triangle is half of a 3 × 4 rectangle that has an area of 12. Half of this area results in an area of 6 for the missing triangle. However, if all the segments in the 3 × 4 rectangle are counted, including those along the hypotenuse, then the total is 16 segments. Half of the 16 is 8. Becky's determination that the area of the missing triangle is 8 may have been the result of a misconception concerning how to deal with the units that are divided along the hypotenuse of the triangle. Becky also had a partially erased calculation of 96 – 16 = 80 written on her paper. Examining the answer space suggests that prior to an erasure, Becky had written 80 in the answer space. It is possible that Becky has originally sought to subtract what she believed to be the area of the 3 × 4 rectangle.

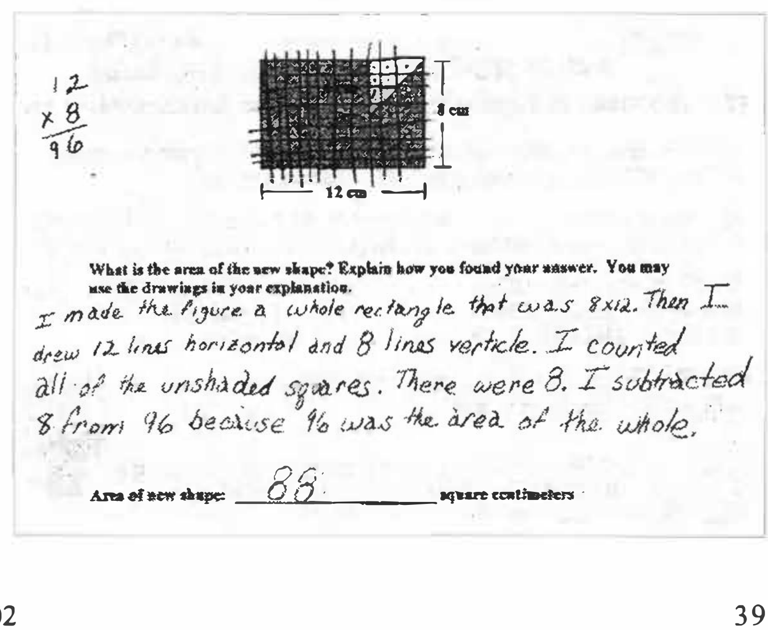
Harding thought that Becky had effectively conveyed a "basic" understanding of the area concept. Harding said, "I should be happy with this. But Ning's response was so good . . . Becky is still counting." She explained that many of her other students were also "still counting." She expressed concern that her students would choose to count 96 square units rather than multiply the base by the height. In the future, Harding intended to create a task about a figure with a large enough area that counting would not be possible. She hoped that this type of activity would convince her students that benefits were to be found using multiplication in determining an area. Harding did not identify Becky's potential misunderstanding of how to account for the units along the hypotenuse of a right triangle when finding the area.

Summary

As these examples show, students as young as sixth grade are capable of providing detailed written explanations that reflect their mathematical reasoning. A teacher should not, however, expect this type of detail the first time that students address

Figure 5

Becky's response



open-ended tasks. These examples were collected at the end of sixth grade after Harding had emphasized clarity in writing throughout the year. When her students submitted incomplete or unclear written responses, Harding provided written or oral feedback that indicated which portion of the response required further elaboration. Examining the explanations gave Harding evidence of the diverse levels of understanding among her students. Kevin's solution contained a series of errors, including an attempt to find the perimeter of the irregular figure. On the basis of a superficial observation of a correct answer and an explanation that "filled the page," Harding originally concluded that Kevin had a complete understanding of the area concept. After closer inspection, she realized that Kevin was attempting to find the perimeter rather than the area of the irregular figure. When Kevin found an incorrect value for the perimeter, he multiplied it by 2. This example raises an important concern when examining student responses. When students are asked to provide explanations, teachers should take time to read and make sense of their work.

Ning, Becky and Naser all used reasonable approaches to acquire their answers. Ning used multiplication and division to find the area of the irregular figure. Becky used two approaches. In one approach, she attempted to decompose the irregular figure into unit squares in order to count the unit squares. In the other approach, she used multiplication to find the area of the larger rectangle and subtracted what she thought was the area of the missing triangle from this value. Harding did not acknowledge the approach that used multiplication; instead, she expressed disappointment that Becky was still counting. Naser used multiplication to find the area of the large rectangle and a counting strategy to find the area of the missing triangle. Harding was satisfied with Naser's overall approach but was disappointed that he was unable to find the correct area of the missing triangle. All three of these students exhibited flaws in their communications, but their overall explanations offered clear indications of their reasoning processes.

Using text, diagrams and symbols in their responses to this task supported the students' communications. For example, Ning had recently moved to the United States, and her English skills were not well developed. Through the use of symbols, diagrams and some text, Ning was able to provide a convincing argument that supported her correct answer. Text and diagrams are also likely to be appropriate methods for younger students to use to communicate their knowledge. In elementary and middle school, students are still developing their writing skills. The use of diagrams and symbols offers these students additional tools for expressing their knowledge.

Cohen and Fowler (1998) have argued that assessments should elicit evidence not only of what students can do but also of what they understand. The detailed explanations offered by her students allowed Harding to evaluate their understanding of the area concept. Time, practice and feedback had given Harding's students the opportunity to develop their written communication skills. By examining their explanations, Harding saw their varying levels of understanding. Naser, who gave an incorrect answer, displayed greater knowledge of the area concept than did Kevin, who gave the correct answer. The freedom to use text, diagrams and symbols in response to an open-ended task supported these students as they displayed their mathematical knowledge.

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Using Communication to Develop Students' Mathematical Literacy

David K. Pugalee

When students are given the opportunity to communicate about mathematics, they engage thinking skills and processes that are crucial in developing mathematical literacy. The importance of communication is evidenced through NCTM's recognition of this skill as one of the five process standards in mathematics, in both the 1989 and 2000 *Standards* documents (NCTM 1989, 2000). Students who are supported in their "speaking, writing, reading and listening in mathematics classes reap dual benefits: they communicate to learn mathematics, and they learn to communicate mathematically" (NCTM 2000, 60). Communication, then, should be a fundamental component in implementing a balanced and effective mathematics program.

Consider a classroom episode in which students solve a version of a common problem, use a rubric (see Figure 1) to rate the written response of a peer and discuss their rationale for the score. Students were asked to describe their process for finding the dimensions of a pool having a perimeter of 18 and an area of 18 square meters. Several students drew diagrams showing how their use of square tiles led to the solution.

Figure 1
General rubric for open-ended responses

0 Answer is unresponsive, unrelated or inappropriate. Nothing is correct.
1 Answer addresses item but is only partially correct; something is correct related to the question.
2 Answer deals correctly with most aspects of the question, but something is missing. Answer may deal with all aspects but have minor errors.
3 All parts of the question are answered accurately and completely. All directions are followed.

Source: Adapted from the *1997–98 North Carolina Open-Ended Assessment, Grade 8*

The students' responses verified that they recognized the need to meet the two constraints of the problem. Hillary wrote, "You can figure this out by first solving for perimeter and saying to yourself—what times 2 plus what times 2 equals 18? After you have an idea of the dimensions, you can plug it into an area formula." Virginia organized her thinking differently and constructed a table listing such dimensions as 6 and 3, 2 and 7, 5 and 4, 8 and 1. For each dimension, she recorded the perimeter and area, concluding that "my study shows that the correct answer is 3 and 6." Lila used a set of 18 color tiles to explore the problem. She moved them around on her desk to form rectangles with a variety of dimensions. Later in her journal, Lila wrote that she "struggled with many different ways" before she decided to "fix the 18 area squares to get 18 around." These multiple methods demonstrate the richness of written communication as a means of helping students organize and consolidate their mathematical thinking.

In this episode, students discussed solutions and detailed decisions for rating their peers' papers. The teacher, Mrs. Weatherman, frequently directed students to give information about the strengths and weaknesses of the responses. One student challenged his partner, "You explained the perimeter very well, but nearly forgot the area. You need something in the problem about how to find area." Another student posed questions to get his partner to consider how her response could be more coherent and clear. "You covered all criteria but could use more detail. Why does it work? Why didn't you choose any other numbers? What factors don't work? Have you tried them all?" The oral dialogue that extended the written work gave students a forum for examining not only their mathematical skills but their ability to express their reasoning with details sufficient to convey the validity of their approaches.

An Overview of the Communication Standard

The previous discussion of a well-known problem shows how communication-rich environments, as described in NCTM's *Standards* documents, encourage students to reflect on their own thinking and gain insights as ideas become explicit through written and oral communication. Each of the five process standards in *Principles and Standards for School Mathematics* (NCTM 2000) presents the major areas of student competence that are important in sustaining and promoting students' mathematical growth. Whereas the earlier *Standards* document presented a differentiated list of skills for students at each grade-level division, the 2000 version, *Principles and Standards*, outlines coherent focus areas that highlight the skills that students from pre-Kindergarten through Grade 12 should have. The grade-band chapters then discuss the attributes of communication at the various levels and suggest how teachers can support communication. The Communication Standard for Grades pre-K–12 (NCTM 2000, 63) stresses that mathematics instructional programs should enable students to

- organize and consolidate their mathematical thinking through communication;
- communicate their mathematical thinking coherently and clearly to peers, teachers and others;
- analyze and evaluate the mathematical thinking and strategies of others; and
- use the language of mathematics to express mathematical ideas precisely.

Expectations at the various grade bands differ primarily in the level of complexity and abstractness of students' communication. In addition, the norms for evaluating students' thinking become more stringent at ascending levels. Beyond presenting and explaining their problem-solving strategies, middle school students should be able to "analyze, compare and contrast the meaningfulness, efficiency, and elegance of a variety of strategies" (NCTM 2000, 268). Another distinguishing feature of communication at the middle school level involves classroom social norms. Adolescents are self-conscious and may be hesitant to expose their thinking to others; therefore, teachers should establish a classroom community that makes students feel free to share their thoughts without fear of ridicule.

Principles and Standards also includes a rich discussion of the teacher's role in promoting the development of the process standards. One fundamental role for the teacher in promoting communication is to create a classroom environment of mutual trust and respect in which students can critique mathematical thinking without personally criticizing their peers. This community atmosphere requires the teacher to be an active facilitator, guiding students as they explore mathematics together. The teacher monitors and facilitates discussions and directs students' conversations so that important learning objectives are met. Second, the teacher selects engaging tasks that require the students to think and reason about important mathematical ideas and concepts. The tasks might have more than one method of solution and use multiple representations. The tasks should further require that students justify, conjecture, interpret and correlate important mathematical ideas. The teacher also has an important role in guaranteeing that all students have opportunities to contribute at some level. Further, the teacher's feedback, about not only the mathematical content and ideas but also the quality of the communication, encourages classroom communities in which communication becomes a tool for thoughtful inquiry.

Illustrating the Four Focus Areas for Communication

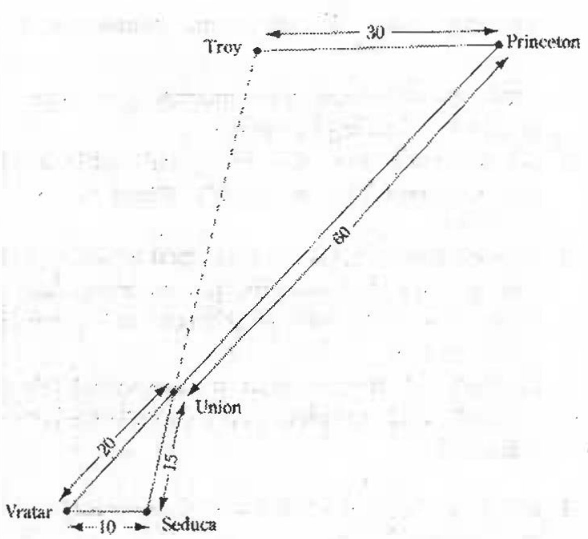
Let's revisit the four focus areas for the communication standard by continuing our illustration of how Mrs. Weatherman's classes worked to extend their mathematical communication. Since state and national tests focus on writing, one particular interest area involved helping students extend their written communication skills through both writing about their solutions and critiquing one another's work in class discussions. The problem in Figure 2 presented students with rich opportunities to develop their abilities to communicate mathematically.

This problem, like the pool problem described previously, required students to *organize and consolidate their mathematical thinking through communication*. Madison began her solution by stating that "this problem is solved by viewing the roads as two similar triangles," Hillary wrote, "I came up with this conclusion because the triangles are similar; therefore, each individual side is proportional." Both of these students then used proportions to solve for the missing side (that is, the distance from Troy to Union). Writing descriptions of mathematical processes encouraged students to reflect on their thinking.

Through sharing their mathematical thinking, students were asked to *communicate their mathematical thinking coherently and clearly to peers, teachers and others.* As part of the focus on communication, the teacher presented papers that had earned the highest rating, a 3, either from the teacher or in peer assessments, as models of good written communication. These papers were the impetus that allowed the class to engage in discussions about the approaches used in the problems. The comments written by the teacher on papers were intended to extend the students' abilities to communicate effectively. Sometimes the teacher simply wrote, "Where could you strengthen your description?" At other times, the papers contained more explicit comments, such as, "I see some computations in the middle of the paper. Could you elaborate on what you were

Figure 2
The Road Problem

The Department of Transportation for Montrose County is interested in building a four-lane highway to connect Troy to Union. The distance from Princeton to Union is 60 miles and from Vratar to Union is 20 miles. One other road between Union and Seduca is 15 miles long. The distance from Troy to Princeton is 30 miles and from Vratar to Seduca is 10 miles. Find the distance of the proposed highway from Troy to Union. Write a description of how you solved the problem.



doing and how that information was used in solving the problem?" and "You have included a diagram on your paper. Could you go back and incorporate that diagram into your discussion so that it is clear how the diagram supports your line of reasoning?" The clarity and cohesion of writing are evident in one student's work on the Road Problem: "I came to the conclusion that 45/15 is equal to 30/10. This problem could also be stated (Union–Troy)/(Union–Seduca) = (Troy–Princeton)/(Vratar–Seduca.)"

Communication provides opportunities for students to *analyze and evaluate their mathematical thinking and strategies of others.* Although many students stated that the Road Problem presented similar triangles, students generally did not present any justification for their thinking. For example, Mallory "rotate[d] the big triangle using the Union axis 180° counter clockwise." She drew a diagram of the smaller triangle, now embedded in the larger one, with Union as the common vertex. Classroom discussion revolved around students' using properties of similar triangles to find their solutions but failing to state why the triangles were similar. At best, the students should have indicated that their assumptions of similarity provided a model for finding a "good" estimate instead of the exact distance from Union to Troy. Yet students were quick to assert that "these triangles are similar in shape but different in size." The comments on the students' papers, including the ones selected for a follow-up discussion, focused on this assumption. Through communication, students' assumptions were analyzed and the importance of justifying their mathematical reasoning was reaffirmed.

Students were able to *analyze and evaluate the mathematical thinking and strategies of others.* Using communication, students extended their use of the language of mathematics. Jonathan wrote, "The side from Troy to Union is unknown, so it is *x* (or any other variable). You must then match corresponding sides." Amanda justified her calculations by writing, "I know to solve proportions you take the cross products." Communication aided students in thinking about how to express mathematical ideas as those concepts became more visible. Amy summarized the importance of communication when she wrote, "Sometimes in math we just do, we don't even stop to ask 'why,' but on these papers you have to know why. So now, we know why we do things." In this example, reading the detailed responses of a peer and engaging in discussions of the ratings gave students occasions to consider other approaches to the problem. This process enabled students to consider, evaluate and build on the thinking of others. This complex interaction involved listening and talking and enabled students to develop facility with mathematical concepts by examining the methods and ideas that others used to determine relative strengths and limitations in those approaches.

Conclusion

Giving students opportunities to develop skills in communicating mathematically should be a natural outgrowth of a well-balanced mathematics program. As a result, students will become comfortable in expressing to others the results of their thinking in both written and oral form. Middle school students must also build skills in evaluating the thinking of others in a mathematical community. This foundation is essential at the secondary level, where students can further develop the ability "to structure logical chains of thought, express themselves coherently and clearly, listen to the ideas of others, and think about their audience when they write or speak" (NCTM 2000, 348–49). Providing such experiences is pivotal in developing communication processes that promote mathematical literacy for all students.

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Edited by Carole E. Malloy, University of North Carolina at Chapel Hill, Chapel Hill, North Carolina.

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Two regular polygons are given. There are twice as many sides in the second polygon as there are in the first. The measure of the interior angles of the first polygon is 10° smaller than the measure of the interior angles of the second polygon. What is the number of sides and the measure of the interior angles of the two regular polygons?

Calendar Math

Art Jorgensen

1. If the second day of a month falls on a Tuesday, what day will the 22nd fall on?
2. Tom is 3 years older than Karen will be next year. Karen is 10 years old now. How old is Tom?
3. Bill lives on one corner of a rectangle, and Sig lives on the opposite corner. The rectangle is 3 km by 4 km. Bill wants to take the shortest distance to Sig's place. How far will he have to go?
4. Susan and Olive live 6 km from town. Susan starts to walk at 11 a.m. She walks at 4 km per hour. Olive leaves at 12 noon on her bike. She goes 10 km per hour. How much later than Susan will she arrive?
5. Add the next 4 digits to the following sequence. 1, 0, 2, 1, 3, 2, 4, __, __, __, __
6. Rose buys a coat for $80. When selling it, she marks it up 25 percent. However, to sell it she has to give a discount of 25 percent. How much does she make or lose on the coat?
7. Willie says that if he had 5 more marbles he would have 4 times as many as Bob. Bob has 20 marbles. How many does Willie have?
8. I have 90 cents consisting of dimes and quarters. In total, I have six coins. How many dimes and how many quarters do I have?
9. Complete this division:

```
     4□
 4 ) 187
     160
      27
     □□
      □R
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10. Marcel picked 1 litre of blueberries on Monday. Then he doubled the number of litres he picked each day until Friday. How many litres did he pick?
11. If Marcel got 73 cents for each litre of blueberries, how much did he earn?
12. This morning the temperature was +4°C. By night it had fallen 15°C. What was the temperature at night?
13. James enjoys collecting hockey cards. He had 4 boxes, each containing 97 cards. How many cards did he have in total?
14. It snowed 5 cm on Monday, 7 cm on Tuesday, 3 cm on Wednesday, 0 cm on Thursday and 8 cm on Friday. What was the average snowfall per day?
15. Draw a line 5 cm long. What is the length in millimetres?
16. There are 117 more people in Dunceville than in Snoozeville. If there are 210 people in Snoozeville, how many are there in Dunceville?
17. On a flight from Edmonton to Calgary, there were 173 passengers on board. Fifty-seven asked for pillows. How many did not ask for pillows?
18. On a flight, there were 235 passengers on board. Eighty-three wanted to read the *Journal* and 52 wanted to read the *Sun*, while the rest slept. How many slept?
19. Express each whole number from 10 to 20 as the sum of not more than 4 square numbers. For example, 14 = 9 + 4 + 1.
20. Find 3 fractions such that when you add 1 to each numerator and denominator, the result is ½. Are there more?
21. Find 2 numbers whose sum is 12 and difference is 8.
22. If the complement of an angle is 41 degrees, what is its supplement?
23. Cynthia wants a candy bar cut into 12 equal pieces for her friends. How many cuts will she have to make?
24. What number squared and doubled is the same?
25. A grasshopper is 4 cm long and an earthworm is 7.5 cm long. How much longer is the earthworm?
26. A rabbit can eat 2.5 centimetres of a carrot in a minute. How long will it take a rabbit to eat a carrot that is a decimetre long?

27. Sandra can run a kilometre in 12 minutes. It takes Juanita 1½ times as long to run a kilometre. How long does it take Juanita to run a kilometre?
28. These numbers are bozos: 5 and 8; 6 and 7; 3 and 10; 2, 8 and 3.
These numbers are not bozos: 4 and 3; 3; 9 and 6; 1 and 11; 7, 2 and 2.
What makes a bozo?
29. A truck delivered 268 pieces of sod. It was placed on 4 plastic platforms. How many pieces of sod were on each platform?
30. Last week gasoline was 78 cents a litre. This week, because of a price war, the price is 59 cents a litre. It takes 43 litres to fill my car. How much did I save this week?
31. Wesley changes oil in his car every 5,000 kilometres. He last changed oil when the odometer read 57,633 kilometres. When will he have to change oil again?

Answers

1. Monday
2. 14 years
3. 5 km
4. 6 minutes
5. 3, 5, 4, 6
6. She loses $5
7. 75 marbles
8. 4 dimes, 2 quarters
9.

```
     4[6]
 4 ) 187
     160
      27
     [2][4]
       [3]R
```

10. 31 litres
11. $22.63
12. –11°C
13. 388 cards
14. 4.6 cm per day
15. 50 mm
16. 327 people
17. 116 passengers did not ask for pillows
18. 100 passengers
19. 10 = 9 + 1
11 = 9 + 1 + 1
12 = 4 + 4 + 4
13 = 4 + 4 + 4 + 1
14 = 9 + 4 + 1
15 = 9 + 4 + 1 + 1
16 = 4 + 4 + 4 + 4
17 = 16 + 1
18 = 9 + 9
19 = 9 + 9 + 1
20 = 16 + 4
Answers may vary.
20. 3/7, 4/9, 5/11. Yes, there are more.
21. 10 and 2
22. 131 degrees
23. 11 cuts
24. 2
25. 3.5 cm longer
26. 4 minutes
27. 18 minutes
28. They add up to 13.
29. 67 pieces
30. $8.17
31. 62,633 km

These problems vary considerably in level of difficulty. However, with slight modifications they can be made suitable for most grades. Have fun.

One hundred and five planks are to be piled in six layers. Each successive layer is to be one less plank than the previous layer below. How many planks are there in the bottom layer?

Using Fractions and Series to Solve Problems in a Recursive Setting

David R. Duncan and Bonnie H. Litwiller

Teachers are always seeking ways to connect algebra to real-world situations. The fraction $\frac{1}{1-x}$ and its associated series provide an opportunity to develop such connections. We will develop these connections in three situations:

Situation 1

The manager of a power plant wishes to sell 1 unit of power. The plant itself requires power to operate. In particular, let x be the fraction of the power produced internally, which is needed to keep the plant functioning. For instance, if $x = 1/10$, then 1/10 of the power is used internally while 9/10 may be sold. The question is: How much power must be produced in order to have 1 unit available for sale?

Solution 1

The manager first decides to produce exactly one unit of power; however, he realizes that x-units were needed to produce this power, leaving only $(1 - x)$ units available for sale. To make up the deficiency, he orders x additional units produced. He then learns that $x \cdot x$ or x^2 units were used to produce the x additional units. Of this supplemental production, only $x - x^2$ units are available for sale. The total amount thus available for sale is $(1 - x) + (x - x^2) = (1 - x^2)$.

He now orders that x^2 additional units be produced but finds that x^3 were used in its production. The total amount now available for sale is $(1 - x^2) + (x^2 - x^3) = (1 - x^3)$. Table I shows what would happen if the manager continues to employ this sequence of recursive production steps.

The manager thus determines that it is necessary to produce $1 + x + x^2 + x^3 + x^4 + \ldots$ units of power in order to have 1 unit available for sale.

Situation 2

A woman goes to the bank and wishes to borrow money for one year. Assume that the interest rate is $100x$, where x is the decimal notation and

Table I

No. Units Produced	Cumulative Produced	No. Units Used Internally	Cum. Used Internally	No. Units for Sale	Cum. Units for Sale
1	1	x	x	$1 - x$	$1 - x$
x	$1 + x$	x^2	$x + x^2$	$x - x^2$	$1 - x^2$
x^2	$1 + x + x^2$	x^3	$x + x^2 + x^3$	$x^2 - x^3$	$1 - x^3$
x^3	$1 + x + x^3 + x^4$	x^4	$x + x^2 + x^3 + x^4$	$x^3 - x^4$	$1 - x^4$
•	•	•	•	•	•
•	•	•	•	•	•
•	•	•	•	•	•

$100x$ is percent notation. Suppose that the interest is deducted in advance; that is, if the interest rate is 10 percent ($x = 0.1$) and the woman wishes to borrow \$1, the bank will deduct \$.10 in advance and give her \$.90. She would be required to repay \$1 at the end of the year. The question is: How much money should the woman borrow in order to actually receive \$1?

Solution 2

The borrower first secures a loan of \$1; however, x dollars of interest is deducted in advance. She therefore receives $(1 - x)$ dollars. To make up this deficiency, she then borrows x additional dollars; but $x \cdot x$ or x^2 dollars is deducted in advance from her new loan so she receives only $(x - x^2)$ dollars. Her total proceeds are thus $(1 - x) + (x - x^2) = (1 - x^2)$ dollars.

If she then borrows x^2 dollars to make up this new deficiency, she will receive $(x^2 - x^3)$ dollars, giving her cumulative proceeds of $(1 - x^3)$ dollars.

Table II reveals the results if the borrower continues to employ this sequence of loan operations.

The woman thus determines that it is necessary to borrow $(1 + x + x^2 + x^3 + x^4 + ...)$ dollars in order to receive \$1 from the bank.

Situation 3

A cat is chasing a mouse. The ratio of the speed of the mouse to the speed of the cat is x, where $x < 1$. If the cat is 1 metre away from the mouse when the chase begins, how far will the cat have to run to catch the mouse? (Assume they run in the same direction.)

Solution 3

The cat first runs 1 metre, but during this time, the mouse has run x metres away from the cat. The cat then runs x metres more while the mouse is running $x \cdot x$ or x^2 metres away from the cat. To make up this x^2 distance, the cat then runs x^2 metres

Table II

No. Dollars Borrowed	Cumulative Borrowed	No. Dollars Deducted for Interest	Cum. Interest Deducted	No. Dollars Received	Cum. Dollars Received
1	1	x	x	$1 - x$	$1 - x$
x	$1 + x$	x^2	$x + x^2$	$x - x^2$	$1 - x^2$
x^2	$1 + x + x^2$	x^3	$x + x^2 + x^3$	$x^2 - x^3$	$1 - x^3$
x^3	$1 + x + x^3 + x^4$	x^4	$x + x^2 + x^3 + x^4$	$x^3 - x^4$	$1 - x^4$
•	•	•	•	•	•
•	•	•	•	•	•
•	•	•	•	•	•

Table III

No. Metres Run—Cat	Total Distance Run—Cat	No.of Metres Run—Mouse	Total Distance Run—Mouse	No. Metres Gained—Cat	Total Distance Gained—Cat
1	1		x	$1 - x$	$1 - x$
x	$1 + x$	x^2	$x + x^2$	$x - x^2$	$1 - x^2$
x^2	$1 + x + x^2$	x^3	$x + x^2 + x^3$	$x^2 - x^3$	$1 - x^3$
x^3	$1 + x + x^3 + x^4$	x^4	$x + x^2 + x^3 + x^4$	$x^3 - x^4$	$1 - x^4$
•	•	•	•	•	•
•	•	•	•	•	•
•	•	•	•	•	•

but finds that the mouse has run x^3 metres further. Table III reveals what would happen if the cat continues to employ this sequential chasing technique.

The cat must run the distance $1 + x + x^2 + x^3 + x^4 + \ldots$ to catch the mouse. (The problem recalls Zeno's Paradox.)

The reader will note that all of these problems have equivalent algebraic formulations. The amount of power to be produced, the amount of money to be borrowed and the distance to be run by the cat are all $1 + x + x^2 + x^3 + x^4 + \ldots$. This is a geometric progression; polynomial division will reveal the progression to be equal to $\frac{1}{1-x}$. The alert reader may have noticed that each of these three problems could have been solved more easily using the fraction form of this equality. For example, if x is the fraction of power used internally, then $(1 - x)$ is the amount available for sale. Thus, if a is the amount of power to be produced to sell 1 unit, we have the equation $(1 - x)(a) = 1$, or $a = \frac{1}{1-x}$. Problems 2 and 3 have similar fractional analysis solutions.

In the particular case for which $x = 10\% = 0.1$, $a = \frac{1}{1-0.1} = \frac{1}{0.9} = 1.1111\ldots$.

Challenges for the Reader

At 12 o'clock, the hands of the clock coincide. When will this next happen? The solution to this problem is again the algebraic solution $a = \frac{1}{1-x}$ or its associated series.

Is 2 = 3?

Here is the sequence of steps:

$4 - 10 = 9 - 15$

$4 - 10 + \frac{25}{4} = 9 - 15 + \frac{25}{4}$

$(2 - \frac{5}{2})^2 = (3 - \frac{5}{2})^2$

$2 - \frac{5}{2} = 3 - \frac{5}{2}$

$2 = 3$

Which step contains the mistake? Explain.

Babylonian Mathematics in Cuneiform II

Sandra M. Pulver

The similarities between our number system and that of the Babylonians are several. Like them, we use a finite number of symbols or digits (we use 10) to express all integers; and we, too, assign importance to the position of a digit, so that for every place it is moved to the left, its value is multiplied by a constant factor (10 for us, 60 for the Babylonians). Like them, we make use of an extension of this rule to express certain fractions (decimal fractions in our case)—that moving a digit one place to the right means to divide its value by the constant factor 10 or 60. The numbers 10 and 60, which play such an important role are the bases for the two number systems, which are called the decimal and the sexagestimal system.

The differences between the two systems are the Babylonian base 60 and the absence of the equivalent of the decimal point in the sexagesimal system.

There is nothing especially outstanding about the numbers 10 and 60. Our predecessors' choice of 10 is just a matter of coincidence, and though the Babylonians were not above counting on their fingers, as we can conclude from their special sign for 10, their choice of 60 as a base also had its motivation outside mathematics. It is not hard to prove that any integer n greater than 1 can serve as the base of a positional or place-value number system (as we call a number system with the common characteristics of the decimal and the sexagesimal number systems).

In such a system, we will need n different symbols or digits whose principal values are 0, 1, 2, . . . , $n - 1$. To move a digit one place to the left will mean to multiply its value by n and to move it one place to the right, even beyond the units' place, will mean to divide its value by n.

We show this by using binary system, as an example. We then have two digits, 0 and 1. The first 10 numbers are written in this system. Thus:

1, 10, 11, 100, 101, 11, 1000, 1001, 1010

In order to translate the binary number 1001011 into decimal notation, we observe that

$1001011 = 1 \cdot 2^6 + 0 \cdot 2^5 + 0 \cdot 2^4 + 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2 + 1 = 75$

If we are to write, say, the number 308 (base 10) in binary form, we see that 308 lies between the following two consecutive powers of 2:

$2^8 = 256$ and $2^9 = 512$

and so

$308 = 2^8 + 52.$

Now, 52 is between

$2^5 = 32$ and $2^6 = 64$

and so

$52 = 2^5 + 20.$

Similarly,

$20 = 2^4 + 4 = 2^4 + 2^2,$

and so

$$308 = 2^8 + 2^5 + 2^4 + 2^2$$
$$= 1 \cdot 2^8 + 0 \cdot 2^7 + 0 \cdot 2^6 + 1 \cdot 2^5 + 1 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2 + 0$$

This, written in binary form becomes 100110100.

In our binary example, the multiplication and addition tables are as simple as can be:

·	0	1
0	0	0
1	0	1

+	0	1
0	0	1
1	1	10

Accordingly, a binary multiplication is carried out thus:

```
   1101
    110
   ----
   0000
  1101
 1101
-------
1001110
```

We can now return to the easily seen differences between the sexagesimal and the decimal systems. It should be clear that the base 60, though unfamiliar, does as well as 10. Each base has its advantages and disadvantages. An obvious disadvantage of the larger base 60 is that a multiplication table has a size (59 by 59) that practically prohibits memorization. On the other hand, it is possible to write large numbers with few sexagesimal digits.

One further advantage of the Babylonian base is that more fractions can be written as finite sexagesimal fractions than can be written as finite decimal fractions. I have already described such fractions in my previous article, "Babylonian Mathematics in Cuneiform I" (the section on the reciprocal table) but it is natural to ask the more general question: When does a reduced fraction (namely, in lowest terms) p/q have a finite expansion in a number system with the base n?

A finite decimal fraction can be thought of as a fraction whose denominator is a power of 10, and a finite sexagesimal fraction as one whose denominator is a power of 60. Similarly, a finite fraction in any other number system with the base n is a fraction whose denominator is a power of n. Our question is then: When can a reduced fraction p/q be turned into a fraction with denominator n^x? Since we can only change the denominator of a reduced fraction by multiplying both the numerator and denominator of the fraction by some integer, the answer is that p/q can be turned into a fraction p/n^x precisely if the denominator q contains only prime factors that are also in n^x, and therefore in n.

So, since 2 is itself a prime, the only reduced fractions that can be written as finite binary fractions are those whose denominators are already powers of 2. Those that can be turned into finite decimal fractions are the ones whose denominators have no other prime factors than 2 and 5, since $10 = 2 \cdot 5$. But since $60 = 2^2 \cdot 3 \cdot 5$, the allowable prime factors for finite sexagesimal expansions are 2, 3, and 5. Thus, if we consider the denominators 2, 3, 4,...20, only four of them will produce finite binary fractions and seven will give finite decimal equivalents, while 13 have finite sexagesimal expansions.

The other major difference—namely, the absence of the equivalent of the decimal point—is, to be sure, a flaw in the sexagesimal system. Yet it is not as serious as one might think at first. We only need to remember that when we are concerned with multiplication and division of decimal fractions, we can forget about the decimal points. After all, they have no influence on the sequence of digits in the result, but control only its size. In fact, when we use a slide rule or look up the logarithm of a number (the power to which a fixed number, usually10, must be raised in order to produce a given number), we are in a situation not too different from that of the Babylonians. We only get the digits of the answer and then have to decide the position of the decimal point. At any rate, this deficiency is a small price to pay for the enormous advantage that operations with fractions are usually no more complicated than those with integers.

The origin of the sexagesimal system is not certain. We know, however, that in early times there was a system of weights and measures whereby the larger unit was 60 times smaller. It was customary to write a measure of, say, 72 smaller units as a large 1 followed by a small 12; this represented 1 large and 12 small units. The number 60 may have become important because the principal unit of weight for silver—the mana—was subdivided into 60 shekels. This may have given way to the consideration of sixtieths as a natural subdivision of units, and the preference for the base 60 in general.

Babylonian Arithmetic

One disadvantage of a base as large as 60 is the large size of the multiplication table showing the products of any two one-digit numbers. One may tremble to imagine Babylonian schoolboys trying to memorize a 59 by 59 multiplication table. But we have found quantities of tables of various kinds, including multiplication tables, so it is clear that such memorization was unnecessary.

This is not to say that we have tablets containing the 59 times 59 products, for we do not. What we find are many 9-table type tables arranged according to multiples of p:

1	p
2	$2p$
3	$3p$
.	.
.	.
.	.
19	$19p$
20	$20p$
30	$30p$
40	$40p$
50	$50p$

and sometimes ending with p. We call p the principal number of the multiplication table.

From this table, any multiple of p can readily be found—$47p$ is simply the sum of $40p$ and $7p$. One might think that there were 59 such tables with $p = 1, 2, 3,...59$. But what we actually find is a selection of principal numbers, which at first is quite puzzling. We have, for example, a multiplication table with $p = 44, 26, 40$, an enormous number, but none for $p = 17$. It is the presence of this curious principal number 44, 26, 40 that makes the puzzle pieces fall into place, for 44, 26, 40 is the last number in the standard reciprocal table in Figure 2 in Babylonian Mathematics I, which we have discussed before. It seems that the principal numbers are essentially the numbers we find in a standard reciprocal table. One exception is 7 which quite naturally appears as a principal number even though it is absent from the reciprocal tables, and so the product of any two numbers can quite easily be found. The coincidence of principal numbers and the numbers in the reciprocal table gives a clue as to how the Babylonians actually computed. It is quite clear that the reciprocal table, combined with these multiplication tables, also served for divisions, since r divided by n is r multiplied by the reciprocal of n, or $(\frac{r}{n}) = r \cdot (\frac{1}{n})$.

In our decimal system, we have a variety of rules and shortcuts that make computation easier: to multiply by 5 we divide by 2 and multiply by 10; a number is divisible by 3 (or 9) if the sum of the digits is divisible by 3 (or 9). If one works consistently with the sexagesimal system, one soon finds many such simple devices. Many more rules are possible in the sexagesimal than in the decimal system since the base 60 has so many divisors.

Sexagesimal calculations were further assisted by quite a large variety of tables. We find extended reciprocal tables—even giving the reciprocals to several places—of numbers such as 7 and 11 whose reciprocals do not have finite sexagesimal expansions. There are tables for the computation of compound interest, of squares and square roots, of cubes and several complicated tables that indicate an interest in numerical procedures far beyond the requirements of simple arithmetic.

Thus, it is perfectly clear that the Babylonians found no more difficulties in arithmetical computation than we do today. In this respect they were unique in the classical world, and it is therefore not surprising that when Greek astronomy had reached the stage where extensive calculations were called for, the Greek astronomers turned to the sexagesimal number system for a sensible way of expressing fractions.

This is the reason Babylonian fractions are used even now, for example, in the subdivision of degrees and hours—the units for measuring angle and time, the two basic quantities observed in classical astronomy. The Greeks wrote the measure of angles using the Babylonian system, and so do we when we write 120°12' 20". When we say the time is 2 hours, 30 minutes and 10 seconds, we are actually using the terminology of the Babylonians of 4,000 years ago who would have said, somewhat more simply, that 2, 30, 10 hours have passed since noon.

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Leonhard Euler (1707–1783)

A farmer wanted to buy horses and oxen for a total of 1,770 coins. For a horse he paid 31 coins and for an ox 21 coins. How many horses and oxen did he buy? (Is there more than one solution?)

A Pitfall in Using Newton's Method

David E. Dobbs

Introduction

Newton's Method (NM) is the most familiar numerical method for root approximation in a first course on calculus (Stewart 2001, 324–29). Not only is NM rather straightforward to implement, it is also often effective, yielding quadratic convergence to a root of a given function f under reasonably mild assumptions (Wade 1995, 206). In addition to providing practice with NM, calculus texts often mention pitfalls that may arise in using NM. For instance, Stewart (2001, 325) indicates that a sequence generated via NM may not converge (let alone, to a root). Additional pitfalls associated with NM appear in *Calculus-Concepts and Contexts*, Exercises 21–23 (Stewart 2001, 328), such as the "sequence" $\{x_n\}$ generated via NM stopping with its m-th term, provided that $f'(x_m) = 0$. Our main purpose here is to make explicit another pitfall that affects NM due to the fact that any computing device has an upper bound for the decimal place accuracy that it can calculate or report.

The pitfall in question concerns the following advice offered by Stewart (2001, 326) if one wishes to use NM to approximate a root of f to k decimal place accuracy: "The rule of thumb that is generally used is (to use x_n or x_{n+1} to approximate a root of f) if x_n and x_{n+1} agree to (at least) k decimal places." In this regard, we show in Examples 2.1 and 2.2 that the user of any computing device, working in conjunction with NM, can be utterly misled and defeated when following the above "rule of thumb," in the following sense. Suppose that we are given a differentiable function f, positive integers m and N, and a real number x_1 such that $f(x_1)$ is "small" in the sense that $|f(x_1)| \le 10^{-m}$; and suppose that NM responds to input x_1 with output $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$ satisfying $x_2 \approx x_1$ in the sense that $|x_2 - x_1| \le 10^{-N}$. (To see how these assumptions relate to the above rule of thumb, consider $N > k$; consider also N such that the computing device at hand can report at most the first $N - 1$ decimal places, or nonzero significant digits, of calculated numbers.) Under these assumptions, is it certain that x_1 (or x_2) is "near" a root of f? Absolutely not! Indeed, given m and N as above, Examples 2.1 and 2.2 each produce a differentiable (in fact, polynomial) function f and a real number x_1 as above such that x_1 is arbitrarily far (say, at least 1 unit away) from any real root of f.

Rather than casting doubt on the effectiveness of NM (or any other numerical method for root approximation), results of the kind in Examples 2.1 and 2.2 are intended to make for better-informed users of technology, as one seeks to understand both the strengths and the inherent limitations of a particular numerical method. The material in this note could be used to enrich courses on calculus, real analysis, advanced calculus or numerical analysis.

Results

We begin by describing a simple construction that has all the desired properties.

Example 2.1. As in the introduction, let m and N be positive integers. Observe that if n and r are positive integers and c is a nonzero real number, then the n-th degree polynomial function $f(x) = c(x - r)^n$ has r as its only root (and r has multiplicity n as a root of f). Begin an application of NM by choosing $x_1 = 0$; this ensures (as desired) that $|x_1 - r| = r \ge 1$. Consider

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = -\frac{f(0)}{f'(0)} = -\frac{c(-r)^n}{cn(-r)^{n-1}} = \frac{r}{n}.$$

To arrange that $x_2 \approx x_1$ in the sense that $|x_2 - x_1| \le 10^{-N}$, we need n such that $\frac{r}{n} \le 10^{-N}$; that is, such that $n \ge 10^N r$. Does some such choice of n lead to $f(x_1)$ being "small" in the sense that $|f(x_1)| \le 10^{-m}$, that is, such that $r^n c \le 10^{-m}$? Certainly: given $(m, N,)$ r and $n \ge 10^N r$ as above, it suffices to take $c = r^{-n} 10^{-m}$.

It could be argued that the n-th degree polynomial in Example 2.1, having just one root (with multiplicity n), is somewhat pathological. Thus, one could ask for a polynomial f, which not only has the above properties but also only simple (that is, multiplicity 1) roots. Such an example is constructed in Example 2.2. As background, we recall two facts. The first of these is the Linear Factor Theorem (Dobbs and Hanks 1992, 39–40): if $r_1,\ldots,r_n$ are all the roots (counted with multiplicity) of an n-th degree polynomial and g and c is the leading coefficient of g, then $g(x) = c\prod_{i=1}^{n}(x-r_i)$. The second fact needed for Example 2.2 is the divergence of the harmonic series (see, for instance, [Stewart 2001, 577, Example 7] for a particularly accessible proof of this fact).

Example 2.2. Once again, let m and N be positive integers. Fix any positive number E (for "error"), with the requirement that our example must satisfy $|x_1 - s| \geq E$ for each root s of f. There is no loss of generality in also supposing that E is an integer. For convenience of notation, we once again choose $x_1 = 0$. Next come the three key steps. Since the harmonic series diverges, we can pick a positive integer n such that $\sum_{j=E}^{E+n-1}\frac{1}{j} \geq 10^N$.

Further, define

$$c = \frac{10^{-m}}{E(E+1)(E+2)\cdots(E+n-1)} = \frac{(E-1)!}{(E+n-1)!\,10^m}$$

and

$$f(x) = c(x-E)(x-E-1)\cdots(x-E-n+1) = c\prod_{j=E}^{E+n-1}(x-j).$$

We shall prove that the above construction has produced data x_1 and f with the desired properties.

By the Linear Factor Theorem, we can label the roots of f as $r_1 = E, r_2 = E+1,\ldots, r_j = E+j-1,\ldots, r_n = E+n-1$. Notice that each r_j is a simple root of f and $\min\{|x_1 - r_j|: 1 \leq j \leq n\} = \min|-r_j| = E$. Moreover, $f(x_1)$ is approximately "small," since $|f(x_1)| = |f(0)| = |c(-E)(-E-1)\cdots(-E-n+1)| = |(-1)^n 10^{-m}| = 10^{-m}$.

It remains to explain why $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx x_1$ in the sense that $|x_2 - x_1| \leq 10^{-N}$. As in Example 2.1, our task is to verify that $\left|\frac{f(0)}{f'(0)}\right| \leq 10^{-N}$. It will be notationally easier to work with the polynomial

$$g(x) = c^{-1}f(x) = (x-E)(x-E-1)\cdots(x-E-n+1) = \prod_{j=E}^{E+n-1}(x-j).$$

As $g'(x) = c^{-1}f'(x)$, it follows that $\frac{g(x)}{g'(x)} = \frac{f(x)}{f'(x)}$, and so it will suffice to show that $\left|\frac{g(0)}{g'(0)}\right| \leq 10^{-N}$.

Since $g(x) = \prod_{j=1}^{n}(x - r_j)$, we can find $g'(x)$ by the Product Rule:

$$g'(x) = \sum_{i=1}^{n}\prod_{j\neq i}(x - r_j)$$

and so

$$\frac{g'(x)}{g(x)} = \sum_{i=1}^{n}\frac{1}{x-r_i} \text{ for } x \neq r_1,\ldots,r_n.$$

More to the point, we have that

$$\frac{g(x)}{g'(x)} = \frac{1}{\sum_{i=1}^{n}\frac{1}{x-r_i}} \text{ for } x \neq r_1,\ldots,r_n \text{ such that } g'(x) \neq 0.$$

In particular,

$$\left|\frac{g(0)}{g'(0)}\right| = \left|\frac{1}{\sum_{i=1}^{n}\frac{1}{-r_i}}\right| = \frac{1}{\sum_{j=E}^{E+n-1}\frac{1}{j}} \leq \frac{1}{10^N} = 10^{-N}$$

to complete the proof.

Remark 2.3. (a) The simplest example of a polynomial resulting from the construction in Example 2.1 is $f(x) = 10^{-m}(x-1)^{10N} = \frac{(x-1)^{10^N}}{10^m}$. Is it realistic to expect to be able to compute $f(x)$? Certainly, $f(0)$ can be computed, for we have supposed that it is known that $|f(0)| \leq 10^{-m}$. For any fixed x such that $0 < x < 2$, does $f(x)$ "look like" 0 to your computing device? The answer is yes if N is much larger than m, since $\lim_{N\to\infty}(x-1)^{10^N} = 0$. However, the answer is no if m is much larger than N (provided that your computing device can calculate the ratio of the two typically "small" numbers $(x-1)^{10^N}$ and 10^m). Thus, it is important to understand the relative sizes of N and m (as well as the relevant domain of x values) if one is to make practical use of Example 2.1.

(b) We next address similar practicality issues in regard to Example 2.2. The simplest example of a polynomial resulting from the construction in Example 2.2 is

$$f(x) = \frac{(x-1)(x-2)\cdots(x-n)}{n!\,10^m}.$$

The practicality of this construction is somewhat compromised by the fact that the harmonic series diverges notoriously slowly. For instance, most, if not all, of today's graphing calculators would fail to determine a suitable value of n in case $N = 2$.

What about the coefficients of $f(x)$? Is it realistic to expect your computing device to calculate these coefficients? Certainly, the constant coefficient, say k, of $f(x)$ presents no problem, since $k = (-1)^n 10^{-m}$ and we have supposed that 10^{-m} is known (as an upper bound for $|f(0)|$). Similarly, it is not difficult to show that d, the coefficient of x in $f(x)$, is given by

$$d = \frac{(-1)^{n-1}(1 + \frac{1}{2} + \ldots + \frac{1}{n})}{10^m}$$

and so $|d| \geq \frac{10^N}{10^m}$. Bounding the general coefficient of $f(x)$ would take us too far afield, but it is already clear from the above bound on $|d|$ that the practicality of Example 2.2, just like that of

Example 2.1, is impacted by the relative sizes of N and m.

In closing, we raise two questions. Is it possible to construct more effectively computable functions having the properties of the constructions in Examples 2.1 and 2.2? Are all numerical root approximation methods subject to the sort of pitfall that we have identified for NM?

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Two Problems from Isaac Newton (1642–1727)

A geometric sequence has three terms. The sum of these terms is 19, and the sum of their squares is 133. What are the terms of the sequence?

A geometric sequence has four terms. The sum of the outer terms (that is, the first and fourth) is 13 and the sum of the two middle terms (that is, the second and third) is 4. What are the terms of the sequence?

Logical Reasoning Through Classification

Lynn M. Gordon Calvert

Logical reasoning skills developed through sorting, classifying and ordering activities are essential precursors to recognizing, extending and generalizing patterns and provide a foundation for understanding formal and abstract algebra in later years. While students may succeed at classifying data and objects that involve specific topics in mathematics (for example, pyramids, prime numbers and algebraic expressions), it is valuable for them to recognize that classification schemes are all based on similar forms of logical structures, whether in mathematics, science, biology or everyday life.

Adults often take for granted the abilities and experiences required for using and creating classification systems. How might an elementary teacher set up a filing system for activities? According to grade level? According to subject? Difficulties arise for the odd activities that span grade levels and/or subjects. What reasoning is made to determine where to file these activities that cross categories? Similarly, how might the materials in a classroom be organized and arranged for efficiency and ease of use? Should children's literature books be arranged alphabetically by title or author, or should they be sorted by other attributes, such as reading level or subject? Understanding classification requires an ability to recognize and identify similarities and differences and decide which attributes are most appropriate for arranging or sorting into sets or categories.

The ability to classify and sort objects by various characteristics or attributes into logical categories is a skill we use and are exposed to every day, often without being aware of its mathematical significance. For example, in our households we sort our laundry by color, such as light and dark, and perhaps further into subcategories of delicate and regular wash items. We arrange our kitchen cupboards according to a variety of attributes, such as cans, dried foods, dishes, plastic containers and so on. We usually place these groups of items according to a logical scheme related to size and/or function: pots and pans are assembled from smallest to largest near the oven; glasses are by the sink; plastic containers are stacked according to size and shape in the pantry; and the small odds and ends that may come in handy some day are put into the junk drawer. A place for everything and everything in its place (most of the time).

Not only do we create our own classification systems in our homes and offices, but we also use schemes created by other people to help us sort through information or to identify objects. Imagine a person surfing the Internet looking for teaching sites related to mathematics; she could use one of the search engine directories such as Yahoo or Canada.com. A directory is a means for classifying and sorting webpages by starting with general categories and following links to more and more specific categories. For example, the person will find the same or similar websites on teaching mathematics by following links such as these:

Science >Mathematics >Education >Teaching Education >K12 >Mathematics
Canada > Science > Mathematics Education > Primary and Secondary Teaching

If a person was trying to find text resources for teaching geometry to children, he could do a search for articles and books on a library's computer system. Depending on his familiarity with the database's subject headings, he may need to experiment with key words to locate a reasonable number of entries relevant to his topic. For example, he might enter the key words geometry, children and activities. Depending on the database used, he may be able to narrow his search further using logical connectives such as AND, OR and NOT, and wild cards (for example, * or ?) for searching for multiple forms of a word (for example, geomet* for geometric, geometry, geometrically). The search may require more and more specificity depending on the number of hits or library resources identified. Perhaps he will have more luck with the following: geometr* AND (children OR elementary) AND activit* NOT video. Once a reasonable and relevant list of library resources is created, locating the references requires familiarity with yet another classification system—either the Dewey Decimal System or the Library of Congress Classification.

Not only do we create and use classification schemes, but we, ourselves, fall into various classifications systems—biological, demographical and geographical.

Examples of Classification Systems

Taxonomy of Living Organisms

Kingdom:	*Animalia*
Phylum:	*Chordata*
Class:	*Mammalia*
Order:	*Primates*
Family:	*Hominidae*
Genus:	*Homo*
Species:	*Sapiens*

Blood Types

The four blood types—A, B, AB and O—are determined by the presence or absence of two types of molecules called A and B: A (has A not B), B (has B not A), AB (has A and B), O (has neither A nor B).

Fingerprint Classification System

The system used by the Federal Bureau of Investigation (FBI) in the United States recognizes eight different types of patterns: radial loop, ulnar loop, double loop, central pocket loop, plain arch, tented arch, plain whorl and accidental.

Postal Codes in Canada, such as T5J 4J4:

The first character represents a particular province or territory (or major area within the province); T is for all locations in Alberta.

The second character: 1–9 for urban locations, 0 for rural; 5 represents a major area within the City of Edmonton.

The third character represents a postal station or city post office in urban places or a set of post offices in a rural geographic area; J represents an area in downtown Edmonton.

In urban centres, the fourth character refers to one side of the street on a block face, a large business, an office building, or another destination. In rural areas it refers to a particular community.

Fifth and sixth characters continue to pinpoint the location further to a particular mail drop.

The last three characters of the postal code narrow in on the location of Canada Post Corporation in Edmonton.

These classification schemes—from organizing our kitchens to sorting mail—have a similar underlying logic. To create and make use of the schemes requires (1) the ability to think flexibly about the characteristics of the data or objects in the collection; (2) an understanding of the differences and similarities in attributes; (3) the ability to construct categories that sort data distinctly; (4) decision making for odd data to further clarify the description of a category; (5) a means for labeling or representing the categories; and (6) the (short-term) adjustments to the representations or descriptions as further data is collected and classified.

The examples of sorting and classifying provided above require logical reasoning skills to make sense of and make use of classification models essential in mathematics as well as in other subject areas. Sorting and classifying activities specific to mathematical attributes, such as size, shape and quantity, are addressed in every strand of the curriculum. For example, we sort numerical and non-numerical data for statistical purposes; we classify geometric shapes such as polygons into quadrilaterals, which can be classified further as rectangles and even further as squares; we distinguish between different types of numbers, such as even and odd, prime and composite, and whole numbers, integers and rational numbers; and we arrange and order objects according to their measurements, including length, volume and mass.

Pointing to the variety of classification schemes used and created daily begins to make explicit the underlying logic, principle methods and attributes by which data may be classified. Such knowing provides a foundation for enhancing generalization skills and logical reasoning in mathematics.

The Age of the Young Woman

Klaus Puhlmann

Three friends, Alvin, Peter and Leigh, met for coffee in a sidewalk café. They were enjoying their coffee, cake and company. The weather was beautiful and many people were out for a walk. As they were reminiscing about the years gone by, a young, beautiful woman walked past their table. All three had their eyes fixed on her.

Leigh said, "How old do you think she is? I believe she is 23 years old."

"No," Alvin said, "she is younger. She is 22 years old."

Peter shook his head and said, "You are both wrong. She is older than that. I say she is 26 years old."

Unfortunately, they were in no position to determine her age. Neither of them dared to follow her and ask for her age. One week later, Leigh attended a party at one of his colleague's place where he noticed that same young, beautiful woman in attendance. He recognized her at once. In the course of the evening, he asked several guests if they knew how old she was. One of the guests in attendance knew her age and responded to Leigh's question with a number. "Just imagine," Leigh said, "none of us had guessed her age correctly. One of us was out by one year, another one was out by two years and the third one was out by three years."

Do you know the age of this young woman?

Michael Stifel (1487–1567)

The sum of two numbers is 19; the sum of their squares is 205. What are the two numbers?

Leonardo Fibonacci (c. 1175–c. 1250)

Klaus Puhlmann

Fibonacci is actually the nickname of the Italian number theorist and algebraist Leonardo da Pisa or Leonardo Pisano. Fibonacci is a short form of Filius Bonacci, meaning son of Bonacci. He is the greatest and most productive mathematician of the Middle Ages and his work is still prevalent in mathematics courses today. Fibonacci's publication of *Liber Abaci* (*Book of the Abacus*) in 1202 greatly influenced the replacement of the Roman numerals by the Hindu–Arabic system of numbers in Europe. His second version of this book was written in 1228 and still exists today. It is a comprehensive work containing almost all the arithmetic and algebraic knowledge of that day. Fibonacci, having received his early mathematical training from Muslim tutors, quickly recognized the superiority of the Hindu–Arabic decimal system, with its positional notation and zero symbol, over Roman numerals. The merits of the Hindu-Arabic decimal system were obvious to Fibonacci and he defended them in this book. While having little influence on the merchants in his native Italy, the book played an important role in the development of mathematics in western Europe in the course of the several centuries that followed. His book *Liber Abaci* contained 15 chapters dealing with the following content:

1. Reading and writing of numbers in the Hindu–Arabic system
2. Multiplication of integers
3. Addition of integers
4. Subtraction of integers
5. Division of integers
6. Multiplication of integers by fractions
7. Further work with fractions
8. Prices of goods
9. Barter
10. Partnerships
11. Allegation
12. Solutions of problems
13. Rule of False Position
14. Square and cube roots
15. Geometry and algebra, the former being devoted to problems in mensuration

Fibonacci also wrote three other works, the *Practica geometriae* (1220), the *Liber quadratorum* (1225) and the *Flos* (meaning blossom or flower)

It was not until the 19th century that the number sequence that appeared in *Liber Abaci*, was named the Fibonacci sequence by the French number theorist Edourd Lucas. The sequence of numbers 1, 1, 2, 3, 5, 8, 13, 21, ..., each of which, after the second, is the sum of the two previous ones. Or, putting it another way, these numbers are such that, after the first two, $F_n = F_{n-1} + F_{n-2}$. These numbers are called Fibonacci numbers and play an important role in mathematics and nature. The ratio of one Fibonacci number to the previous one is a convergent of the continued fraction

$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\dots}}}}$$

which satisfies the equation $x = 1 + \frac{1}{x}$ and is equal to $\frac{1}{2}(\sqrt{5} + 1)$. Thus the sequence $\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \dots$ has the limit $\frac{1}{2}(\sqrt{5} + 1)$ and the ratio of successive terms approximates the golden mean. That is, $F_n/F_{n-1} = \varphi$ (Greek letter phi). Here φ is known as the golden ratio and is equal to 1.61803.... It appears in the most surprising places in nature, art and mathematics.

In nature, the number of spiral floret formations visible in many sunflowers, spiraled scales on pinecones and segments on the surface of a pineapple have been found to match Fibonacci numbers. The arrangements of leaves, buds and branches on the stalk of a plant correspond to the numbers in the Fibonacci sequence.

Fibonacci numbers, besides bearing a curious relationship to botany, also appear to exert a strange influence on art and architecture. The ratio between any two adjacent Fibonacci numbers after 3 is about 1:1.6. This is the so-called Golden Ratio, or Golden Section, which has fascinated experts for centuries because of its connection with esthetics. The ratio, more precisely expressed as 1:1.618, occurs in pentagons, circles and decagons, but most notably in the Golden Rectangle, a figure whose two sides bear the magic relationship to each other. Because it presents such a visually pleasing form, it was present in all edifices of ancient Greece and in the art of the masters. The presence of the Fibonacci numbers is also felt in the area of music. Some musicians, like Bartok, based the entire structure of their music on the golden mean and the Fibonacci sequence.

The Fibonacci numbers and sequences are truly a rich area for study because they give rise to a vast amount of substantial mathematics. Be it in the area of art, music, architecture, science, economics or engineering, physical applications and connections with various branches of mathematics abound. Students and teachers willing to explore the Fibonacci numbers and sequences will make exciting discoveries and experience wonder and amazement.

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Change Sides and Keep the Area the Same

One side of a rectangle is increased by 25 percent. By what percentage must the other side be reduced if the area of the rectangle is to remain the same?

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