The Mathematical Constant Pie

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Our preoccupation with the mathematical constant π goes back several thousand years. The number π has fascinated the Babylonians, the Egyptians, the Greek and countless mathematicians and amateurs from all cultures.

The number π (3.14159 26535 89793 23846 26435 83279 50288 41972 . . .) is an example of an irrational number and a transcendental number. Irrational numbers are real numbers not expressible as an integer or quotient of integers, whereas transcendental numbers cannot be the solution of an algebraic equation with rational coefficients and only a finite number of terms—that is to say that a transcendental number cannot satisfy an equation of the type $x^2 = 10$, $22x^4 - 3x = 2{,}149$ or $9x^4 - 240x^2 + 1{,}492 = 0$.

This interesting number is represented by the 16th letter of the Greek alphabet in lower case, π, and is often referred to as the circle number because its value is the ratio of the circumference of a circle to its diameter. The circle and everything pertaining to it has held captive humankind's interest, sometimes amounting to fascination, for several thousand years—longer than any other single feature in mathematics.

The early calculations were often crude approximations of its value and it was not uncommon to equate π with such values as $\sqrt{10}$ (Ch'ang Höng [78–139]), (Smith 1951, 141); $\frac{142}{45}$ (Wang Fan [229–67]), (Smith 1951, 142); $\sqrt{8}$ (Michael Constantine Psellus [1020–1110]), (Smith 1951, 197); or $\frac{355}{113}$ (Adriaen van Roomen [1561–1615]), (Smith 1951, 340). In 1769, Arima Raidō (1714–1783) provided a value of π given by $\frac{428224593349304}{136308121570117}$, which is correct to 29 decimal places.

Around 2000 B.C., the Babylonians assumed the value of π to be either 3 or $3\frac{1}{8}$. The Egyptian scribe Ahnes, in the Rhind papyrus (1500 B.C.), implied that π was 3.16049. To the Greeks, π was intimately connected to the problem of squaring the circle, that is, of finding by geometric construction, using only a compass and a ruler in a finite number of steps, a square whose area was exactly equal to the area of a given circle. It wasn't until 1882 that the German mathematician Carl Louis Ferdinand von Lindemann proved that π was a transcendental number and that the squaring of the circle was impossible. The essential argument proving that the circle cannot be squared can be stated something like this: A circle can be squared using only a compass and a ruler in a finite number of steps if a line segment of length π can be constructed (assuming the circle has diameter of 1 unit). With the event of the discovery of analytic geometry, it became possible to translate this construction problem into an algebraic problem whose solution would require π to be a root of an algebraic equation with rational coefficients of, at most, degree 8. Therefore, if the circle could be squared, then π would have to be an algebraic number. This brought closure to von Lindemann's proof that π is a transcendental number.

Archimedes (about 287–212 B.C.) calculated π to be between $3\frac{10}{71}$ and $3\frac{10}{70}$. This last value, $3\frac{1}{7}$, or $\frac{22}{7}$, is considered to be the best approximation of π using the ratio of two numbers less than 100. Both the Bible and the Talmud give π the value of 3. During the third century, π was calculated in China to six digits by using a polygon with 3,072 sides to approximate a circle. The most precise value of π computed by the Greeks was 3.1416, achieved by Ptolemy, the Greek astronomer, in around 150 A.D. There is evidence of others achieving close approximations to π. In China, Tsu Ch'ung-Chi and his son calculated π correct to seven decimal places. Although known for many other mathematical achievements, Fibonacci calculated π to 3 decimal places. In 1610, Ludolph von Ceulen, a German, calculated π correct to 35 decimal places.

The calculation of π was soon subjected to using infinite series and thus it was John Machine in 1706 who calculated π correctly to 100 decimal places. Euler, too, used infinite series and calculated π correct to 20 decimal places in just one hour. The infinite series used by Euler is: $\frac{\pi}{2} = \frac{3}{2} \bullet \frac{5}{6} \bullet \frac{7}{6} \bullet \frac{11}{10} \bullet \frac{13}{14} \bullet \frac{17}{18} \bullet \frac{19}{18} \bullet \frac{23}{22} \ldots$. Euler also popularized the use of the Greek letter π to denote this number in a 1748 publication.

Subsequent calculations seemed to surpass previous ones and, in 1949, the ENLAC computer calculated π to 2,037 places in 70 hours. In 1961, 100,000 decimal places of π were calculated. The millionth digit of π as computed by the French CDC 7600 computer in 1974. In 1983, a Japanese computer calculated π to 16,777,216 places using $\pi = 32\tan^{-1}(\frac{1}{10}) - 4\tan^{-1}(\frac{1}{239}) - 16\tan^{-1}(\frac{1}{515})$. This was further surpassed in 1988 by the Japanese, when their computer calculated π to 201,326,000 places. In 1991, two Russian émigrés to the United States, Gregory and David Chudnovsky, computed π to 2,260,321,336 digits using a computer they built themselves in their New York apartment from mail order parts.

Modern calculations are based on foundations laid by Ramanujan. We know now that π is not a Liouville number and that e^{π} is transcendental, but what is not known yet is whether $\pi + e^{\frac{\pi}{e}}$ or $\log \pi$ are irrational, although $e^{\pi i} = -1$. A Liouville number is an irrational number X such that, for any integer n, there is a rational number $\frac{p}{q}$ such that $q < 1$ and $|X - \frac{p}{q}| < 1 \frac{p}{q^n}$. All Liouville numbers are transcendental. There is a Liouville number between any two real numbers. In fact, the set of Liouville numbers is a set of second category (although it is of measure zero).

References

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Georg Cantor (1845–1918)

Cantor was a German set theorist. In his day, his theories concerning infinite sets seemed revolutionary and created much controversy. Cantor is known for his definition of real numbers, work on number theory and, with Schwarz, completing Riemann's memoir on uniqueness of trigonometrical series. His most original achievement is the theory of aggregates and transfinite numbers, created almost from scratch. Some regard the theory as one of the most beautiful of all mathematical creations. Others, disturbed by the paradoxes inherent in Cantor's reasoning, utterly rejected his conclusions.