A Dynamic Way to Teach Angle and Angle Measure

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What is missing from the following definitions of a rectangle?

Grade 6 Students' Definitions of a Rectangle

- A four-sided quadrilateral
- An object with four sides
- A figure with two sets of parallel, equal lines
- Has four long sides

Grade 8 Students' Definitions of a Rectangle

- A figure with four sides that are parallel. Two sides are equal to each other, and the other two are equal to each other.
- A figure that has four straight lines. Both pairs are parallel.
- A four-sided figure having two sets of parallel lines.
- A closed figure with four sides.
- A four-sided figure with two sides the same length.

These quotations reflect the properties of rectangles that students thought were important when they were directed, "In your own words, define a rectangle." Although most students stipulated that a rectangle needed four sides and some struggled with the concept of parallel opposite sides, no mention was made of angles! The foregoing definitions are from a study involving 145 Grade 8 students and 143 Grade 6 students. Of these students, only 2 per cent of the Grade 8 students and 1 per cent of the Grade 6 students mentioned angles or square corners in their definitions of a rectangle (Wilson 1988). These findings may reflect a lack of attention to the concept of angle until middle school, where it is generally given an abstract treatment.

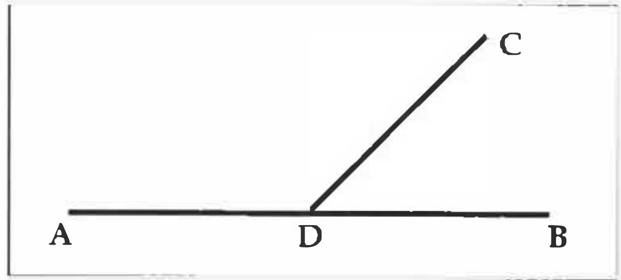
Getting students to think about angles and angle measure long before they are handed protractors and asked to verify a given theorem was addressed by classroom teachers at an elementary school in Georgia. In this article, we draw from their experiences to show that the development of the concept of angle through a dynamic interpretation as a turn can begin in Kindergarten and be part of the mathematics curriculum at every grade level. First, we discuss how students learn angle concepts and outline a basic teaching strategy. The main part of the article offers a sequence of sample activities for developing the concept of angle in K–6.

What We Know About Children's Concepts of Angle

Piagetian research indicates that children's concepts of angle develop slowly (Piaget, Inhelder and Szeminska 1960, 411). Students were shown two supplementary angles (Figure 1) and asked to make another drawing exactly like it without looking at the model while drawing. (The figure shown to the students did not have arrowheads to indicate rays.) The student could suspend drawing and refer to the model at any time. Preschool and primary-level students (4–7 years) used only visual estimates to record the slant of $\overline{CD}$ and did not attempt to devise any way to measure the slant. Middle grades students (7–9 years) tried to copy the slope of the slanted segment but could not devise a measure that would help them. Upper-level elementary school students (9–11 years) began to identify the angle and devised some ways to compare their angles. Some older students used the lengths of $\overline{AC}$, $\overline{CB}$ and $\overline{AD}$ to locate points D and C, creating an accurate slope for $\overline{DC}$. Others drew various lines perpendicular to $\overline{AB}$ to help them approximate the slope of $\overline{DC}$.

Figure 1

Piagetian Task: Make a Drawing Like This One



Piagetian research raises two important points:

1. The concept of angle develops over an extended period of time.
2. A static view of an angle (line segments on a piece of paper) is difficult for young students to perceive.

We also know that students progress through different levels of understanding about geometric figures (Van Hiele 1986). At first, students look at angles holistically. As they begin to recognize angles, students may notice that a triangle always has three angles, or corners, but they do not focus on any particular properties of those angles. Later, they understand that the measure of an angle may be smaller than the measure of a right angle (acute) or larger than the measure of a right angle (obtuse) and begin to identify properties and relationships of angles. The next step of development is to operate with such relationships as "a triangle cannot have more than one obtuse angle because the three sides must form a closed figure." From a developmental perspective, these ideas are not rules; they are understandings that students have developed from working with angles and triangles. Van Hiele's research leads to a third important point:

3. Students need good activities designed to help them explore angles and their properties and relationships.

These three ideas lend insight into how students learn about angles and have implications for curriculum and instruction. Among the many ways to perceive of the concept, angle as a rotation, or turn, seems to be especially appropriate for instruction at the elementary school level. This way of thinking about angle allows the student to anchor the concept of angle on the concrete experience of turning her or his own body. An advantage of perceiving angle as a turn is that it counteracts students' common misconceptions that the size of the angle is determined by the length of the pencil marks used to represent the angle and that one side of the angle must be horizontal.

A Basic Strategy for Teaching Measurement

One of the most important influences on students' achievement is the opportunity to learn. Affording the opportunity to develop conceptual understanding requires a greater depth of coverage than is accorded many topics in the elementary school classroom. Geometry and measurement often receive attention only when extra time is available. To add to the bleak picture, the topic of angle and angle measurement is not usually addressed directly. If it is addressed at all, angle measurement is usually a part of a discussion focusing on triangles or polygons. We suggest that the following basic strategy for teaching measurement be used as a guideline for planning instruction and increasing the opportunities for elementary school students to learn about angle.

Several authors (Hiebert 1984; Wilson and Osborne 1988) have suggested strategies for teaching measurement that share the common elements of exploring, comparing, developing a unit and creating formulas. These elements form the basis for the following teaching strategy, with each step extending the previous step and introducing the next step. Together the steps help students build useful concepts. It is important to cycle through the sequence several times using practical applications and activities that actively involve the students. Each cycle should give students an opportunity to learn at a deeper level. The following list summarizes the four steps as they relate to measuring angles:

1. Explore the concept of angle. What do we want to measure when we measure an angle? Measuring length, measuring volume and measuring angles are extremely different tasks. Students are most familiar with length and often overgeneralize ideas they have learned in measuring length. Students commonly think that they can measure an angle using a ruler; they may try to measure the length of a ray or a distance between rays. We suggest that the amount of turning can be used as the attribute being measured for angles.
2. Compare angles. How can we tell if one angle measure is greater than another angle measure? The act of comparing helps students focus on the attribute being measured and should initially be done without using any particular units. This activity promotes a new perspective on the work done in step 1. When students want to know how much larger one angle measure is than another, they are motivated to move to the next step.
3. Develop a unit that can be used to measure angles. How can we compare angles using units and instruments? Develop a nonstandard unit and a tool, such as a wedge, to help make comparisons. Using several wedges, create an instrument that can be used to count units (a protractor). After the pros and cons of various units are discussed, move to degrees (the standard unit) and the traditional protractor. Students are motivated to move to the fourth step so as to make the process more efficient.

4. Observe relationships and invent rules. What rules help us count the units and apply our knowledge of angles quickly? Formulas or rules should grow out of counting strategies that students develop for themselves. Students might develop such rules as "The sum of the angle measures of a triangle is 180°" or "Supplementary angle measures total 180°."

These four steps were the basis for the development of activities in a three-year project in which researchers at the University of Georgia and classroom teachers at South Jackson (Georgia) Elementary School worked on a K–6 geometry-and-measurement curriculum that would take advantage of what we know about students' learning and effective classroom teaching. Several cycles of writing, piloting and revising furnished insight into what is helpful in teaching about angles. The following activities focus on helping elementary school students progress through the first three steps of the teaching strategy outlined previously.

Activities for Learning About Angle

The activities described in this section illustrate how the perception of angle as a turn can be used to develop the concept of angle. Teachers can adapt the activities for use at different grade levels.

Exploring the Concept of Angle

Prior to the introduction of the concept of angle, students need experiences that will help them understand the concept. Ask the students to face the front of the room. Then ask them to try to stay on the same spot on the floor while they turn until they face the front of the room again. Identify this movement as a full turn. Discuss things that turn, such as doorknobs, wheels on a car and hands on a clock. Ask the students to make a turn that is more than a full turn, then discuss how they knew that the turn was more than a full turn. Have the students face the front of the room and then turn until they face the back of the room. Ask them if they turned less than or more than a full turn. Allow them to describe the size of turn (for example, "I went halfway around"). Then introduce the term *half-turn*. Using different walls or objects to mark the beginning of each turn, have students make other half-turns and full turns. Ask, "Do you turn more if you make a full turn or a half-turn? How many half-turns make a full turn?" Also introduce quarter-turns through similar activities and questions and compare them with half- and full turns.

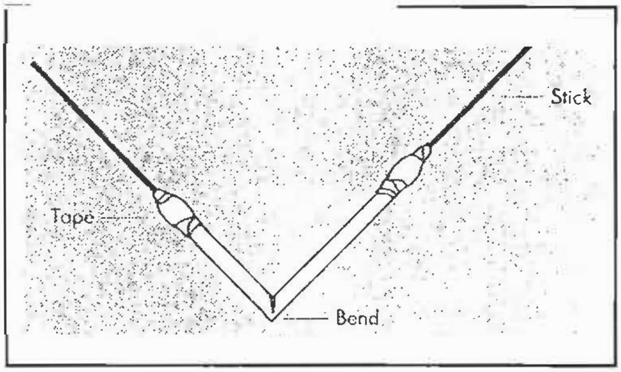
After students understand the language and relationships involved in making quarter-, half- and full turns, introduce the idea of angle as a sweeping motion. In one Grade 2 classroom, for example, a student pretended to be a robot following the directions given by the other students as it searched for an object hidden somewhere in the room. The robot held an arm straight out in front to emphasize the turns. Give a quarter-turn the special name *right angle*, being alert to helping students become aware that a right angle does not necessarily turn to the right. Students often generalize that there must be a "left angle." It is important that they realize that *right* describes the amount of turn, not the direction of turn.

Include activities that involve students in forming angles with their arms. They might, for example, use their arms to represent the hands on a clock. Have them find different ways to make right angles with their arms. They could, for example, start with both arms held straight out in front and sweep one arm to the left. One way to identify a right angle is to have a student stand at the corner of a table that has a square corner, extend both arms along one edge and then sweep out the corner with one arm.

Turns using a flexible hand-held angle can be used to make the connection between turns made with the body and turns represented on paper. A flexible hand-held angle (Figure 2) can be made by inserting sticks into a bent drinking straw and taping them securely in place. By using sticks of different lengths or by bending the straw at various points, the teacher helps students realize that the lengths of the sticks are not relevant.

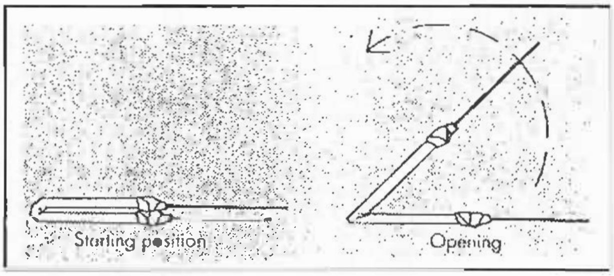
Figure 2

A Flexible Angle Made from a Drinking Straw



Have the students use the angle to demonstrate turns by starting with both sides touching and then rotating one side (Figure 3). Holding the flexible angle in the air, Grade 4 students easily showed right angles, angles greater than and less than a right angle, and straight angles. As students use the flexible angle, they will be constantly changing its orientation. Be sure that they notice that they have created a new angle only when they open or close the straws.

Figure 3
A Turn Using the Flexible Angle

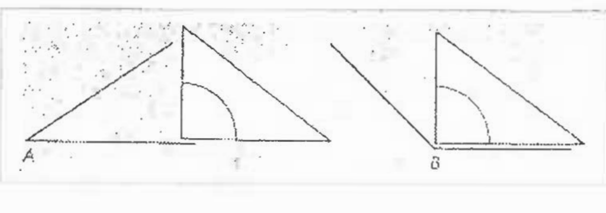


Comparing Angles

Comparisons of sizes of angles can initially focus on comparisons with right angles. Have students make an angle using their arms or the flexible angle and then describe it using language appropriate to their understanding. For example, a student might say, "My angle is more than a half-turn but less than a three-quarter turn" or "My angle is less than a right angle."

Comparing angles that are close in size necessitates creating static angles so that one angle can be placed on top of the other. The flexible straw is not very useful for making such comparisons because of the difficulty of keeping it from opening or closing as it is moved. It is important to note that, perceptually, dynamic and static angles are different. Students need to integrate the two types of angles as they build their concept of angle. Begin by looking at static angles that are right angles and creating a tool (for example, the corner of an index card) for comparing angles to a right angle. The tool can be used to identify corners that have right angles, such as the corner of the room or the corner of a book. After the students have a tool for measuring a right angle, ask them to use the tool to find right angles and angles that are greater than a right angle or less than a right angle. Existing language needs to be developed to describe the situation. For example, Kindergarten students said that a wedge placed in angles like those in Figure 4 did not fit in $\angle A$ but did fit in $\angle B$. Using examples, let students develop appropriate language to describe what it means to fit. Teachers may want to talk about fitting exactly.

Figure 4
Comparing a Right-Angled Wedge to Other Angles

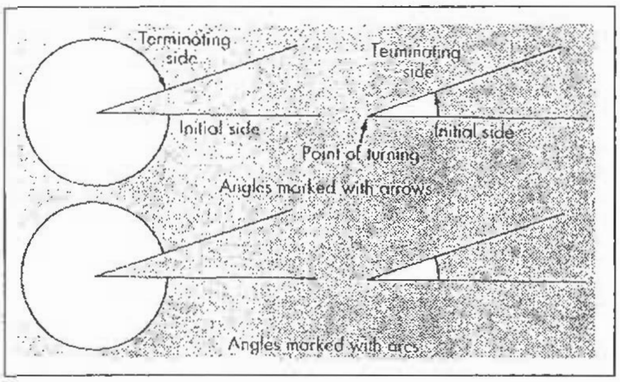


Static angles are created by drawing on paper or cutting wedges of paper. A transition from the dynamic way of thinking about angles to the static way of thinking about angles can be made by asking students to represent the turns on paper. Recording turns requires that students gain a deeper understanding of the nature of angles than they have been using. Teachers should help students focus on all four components of turning that are inherent in the dynamic situation:

- The point of turning
- The initial side of the angle
- The direction of the turn
- The terminating side of the angle

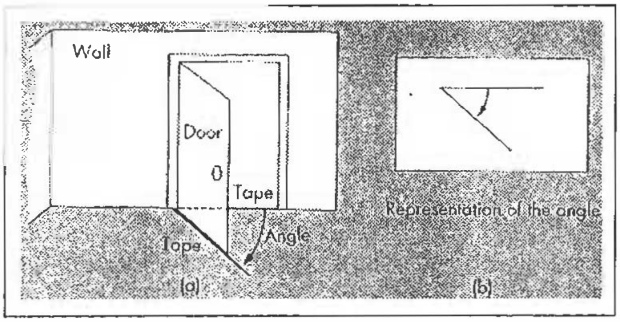
As students complete activities involving turns, these components need to be made explicit. Figure 5 shows an example of how these features of dynamic angles can be represented in static angles. The point of turning is called the vertex of the angle. Teachers should decide whether they want to introduce that terminology. An arrow is used for the purpose of identifying the initial side, terminating side and direction of turn. If the initial and terminating sides are not of interest, the angle can be indicated using arcs.

Figure 5
Representing Angles



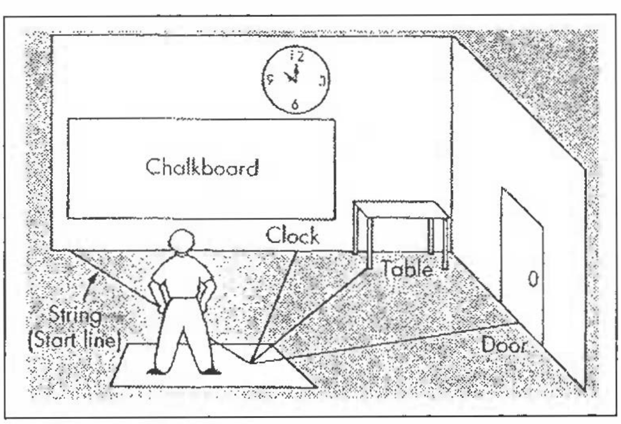
The first step in an activity that makes the transition to representing angles on paper is to put a line of tape in the doorway to show the closed position of the bottom of the door. Move the door to demonstrate the swing arm of an angle. Stop the door at some point of its motion, bringing attention to the line along the bottom of the door. Use tape to mark this line on the floor (Figure 6a). Close the door and repeat the swing from the start line to the end line. Identify the point of turning. Next, open the door well past the end line of the first angle so that the students can see the angle marked on the floor. Copy the angle onto a large sheet of paper at the doorway and tape it to the chalkboard (Figure 6b). Have students visually estimate the size of the angle by drawing it on smaller-sized paper at their desks. Have them compare their angle to the angle marked on the floor and the representation on the chalkboard. To do this task, they can also cut out their angle and fit it in the angle made by the door.

Figure 6
An Angle Made by a Swinging Door



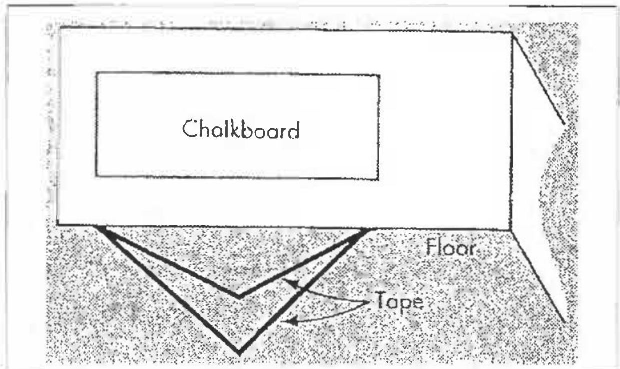
Have students sweep out angles and represent them on paper. Begin by taping a large piece of paper on the floor. Then tape a string from a point marked on the paper to a point on the floor below the corner of the chalkboard. Tape strings to other objects as shown in Figure 7. Each student should stand on the paper, point along a string and sweep out an angle from the start line to another object in the room. Have students make a drawing of the angle by copying the angle marked on the floor. Label it with a description, such as "Angle from the clock to the table." After they have made several drawings, instruct the students to order the angles from smallest to largest. Ask students to identify angles that have the same amount of turn; for example, the angle from the corner of the chalkboard to the clock has the same amount of turn as the angle from the clock to the corner of the chalkboard. Have students compare their drawings to the angles represented on the floor by the strings.

Figure 7
Angles Found in the Classroom



To emphasize the importance of the point of turning, mark two angles whose sides point at the same objects in the room, as shown in Figure 8. Using the flexible angle, have a student stand at the vertex of the angle that is farther from the objects. Have the student position the flexible angle so that the sides of the flexible angle point to the objects, matching the angle marked on the floor. Keeping the endpoints of the flexible angle aimed at the objects, have the student walk toward the second angle's vertex. Ask students to think about the two angles. Did the angle increase in size or decrease? If the student were to walk even closer to the objects, would the size of the angle increase or decrease?

Figure 8
Two Angles That Point to the Same Objects



Students' experiences in exploring the properties of polygons furnish opportunities to deepen their understandings of angle. As students work with the static angles of geometric figures, useful comparisons can be made by tracing the angles. The language and perceptions developed in thinking about angle as a turn, however, can still be applied. In a Grade 4 classroom, for example, finding the sum of the angle measures in a triangle was posed in terms of turning. The lesson began with the teacher's first reviewing the turning concept of angle, using the flexible angle described earlier and an overhead projector, demonstrating how turning relates to the three angles in a triangle. The students were asked, "How much turning is involved altogether?" After students made several conjectures and checked their conjectures by representing the angles as wedges and putting them together, they identified the sum as a half-turn. This experience lays the foundation for middle school lessons in which students develop the idea that the sum of the angle measures in a triangle is 180°.

Developing a Unit for Measuring Angles

Measuring an angle involves a comparison between the angle and an iteration of a unit angle. The angle in Figure 9, for example, has a measure of 3 units.

Conceptualizing angle as a rotation, or turn, allows the use of the circle as a basis for forming unit angles. The circle, representing a full turn, can be partitioned into wedges in the following way. Start by giving students a cutout circle marked in sixths and have them fold it in half along a diameter and then cut along the crease with scissors. Then using one of the halves (Figure 10), have them cut along the lines, dividing the half-circle into three equal parts. Have them fold each of the pieces in half and cut on the crease. They should have six congruent wedges that can be used to measure and draw angles.

Figure 9
Nonstandard Units

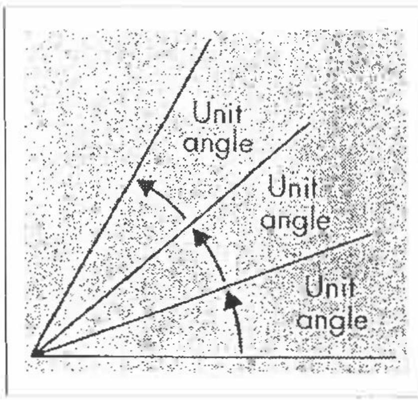
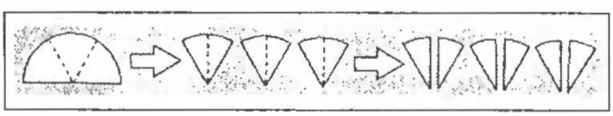


Figure 10
Creating Wedges



After students have experience using the wedges to measure angles, a protractor can be created from the other half-circle (Figure 11). Have the students fold the half-circle into three equal parts and then fold again in half. When they open the half-circle, the creases will mark the wedges. Have students use the protractor to measure angles by placing the edge of the protractor along one side of an angle with the centre on the vertex of the angle (Figure 12). Later, students may want to add scales to the protractor to make it easier to count the number of wedges.

Have students estimate the measures of angles that are not a whole number of wedges (for example, 1½ wedges). Discuss the difficulty of determining the measures of angles that are fractional parts of a wedge. Encourage students to offer suggestions for more precise measures.

Figure 11
Creating a Protractor

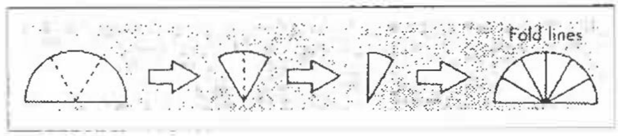
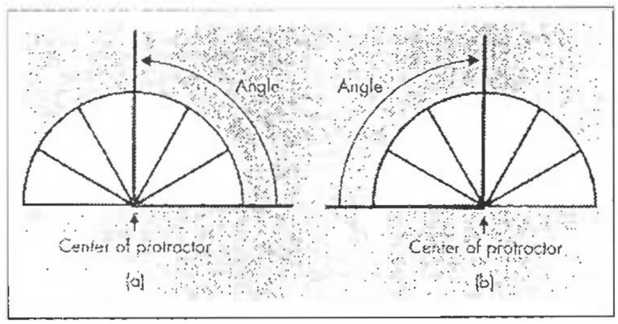


Figure 12
Using a Protractor



At this point, students have a foundation for moving to the study of degrees and the standard protractor. Although the Babylonians are credited with developing the standard unit of degrees by dividing the circle into 360 parts, more than one theory has been advanced about why this number of parts was used (Eves 1969). Encourage students to offer reasons why 360 parts might be a good choice. Relate the degree to relationships that the students have discovered. Ask, "If a whole circle has 360° (1 turn), how many degrees does a semicircle (½ turn) measure? How many degrees does a right angle (¼ turn) have?" Have students determine the number of degrees in each wedge of their wedge protractor. Teachers may want to have them label the wedges in degrees before giving them a standard protractor.

After transition activities, Grades 5 and 6 students are better able to learn to use a protractor. Development of the dynamic concept of angle helps students make judgments about the use of the protractor. For example, because the angle measures of 30° and 150° both appear on the protractor at the same mark, students need to make a judgment about whether the angle is greater than or less than a right angle. For more ideas on using a standard protractor, consult your textbook; Edwards, Bitter and Hatfield (1990); or Wilson (1990).

Summary

Students' lack of understanding of and attention to the concept of angle may be a result of neglecting the topics of angle and angle measurement in the early elementary school curriculum. Often the study of angle is limited to the study of static angles and the use of a protractor. The alternative of studying angles as both dynamic and static allows students to begin developing the concept of angle in Kindergarten and refine it at each grade level.

We have identified a four-step teaching strategy for planning instruction about angle and angle measure and have described activities for the first three steps: exploring the concept of angle, comparing angles and developing a unit to measure angles. The extended time needed for this development requires that students actively explore angles and their properties and applications throughout elementary school. The suggested sequence lays the foundation for introducing standard angle measure, which is not established if a traditional protractor is introduced too early.

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Ryan can drill the holes he needs in five minutes with a power drill or in 20 minutes with a hand drill. He starts with the power drill, but after two minutes it stops working and he finishes with the hand drill. How long does he work with the hand drill?