Edmonton Junior High Mathematics Contest 2000

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The Edmonton Junior High Mathematics Contest is designed to challenge the top 5 per cent of Grade 9 math students in Edmonton. The annual contest is run by a group of mathematics teachers and is written by Andy Liu.The main sponsors are the Association of Professional Engineers, Geologists and Geophysicists of Alberta (APEGGA), IBM, MCATA, Edmonton Public Schools and Edmonton Catholic Schools. After the contest, the top 50 students are recognized at a dinner banquet also attended by parents and teachers.

Part 1: Multiple Choice

1. What is the value of $\sqrt{2} - \sqrt{8} + \sqrt{18}$?

   (a) 0 (b) $\sqrt{2}$ (c) $2\sqrt{2}$ (d) $3\sqrt{2}$

   Solution: We have $\sqrt{2} - \sqrt{8} + \sqrt{18} = \sqrt{2} - 2\sqrt{2} + 3\sqrt{2} = 2\sqrt{2}$. **Answer: (c)**

2. Donald mixed x kg of peanuts and y kg of raisins to make a mixture. The peanuts cost \$5/kg and the raisins cost \$4/kg. When the cost of the peanuts rose by 10 per cent and the cost of the raisins dropped by 15 per cent, Donald's total cost for making the mixture remained the same. What was the ratio $x{:}y$?

   (a) 2:3 (b) 5:6 (c) 6:5 (d) 3:2

   Solution: The original cost (in dollars) for Donald was $50x + 40y$, and the new cost was $55x + 34y$. Equating the two, we have $6y = 5x$, so that $x{:}y = 6{:}5$. **Answer: (c)**

3. Alice is twice as old as Betty, and Betty is seven years younger than Cecilia. The total of their ages is 67. How old is Betty?

   (a) 10 (b) 12 (c) 15 (d) 20

   Solution: Let Betty's age be x. Then Alice's age is $2x$, and Cecilia's age is $x + 7$. From $2x + x + x + 7 = 67$, we have $4x = 60$, or $x = 15$. **Answer: (c)**

4. What is the solution to the equation $\sqrt{3x + 10} = -x$?

   (a) –2 (b) 5 (c) –2 and 5 (d) no solution

   Solution: Squaring the equation, we have $3x + 10 = x^2$ or $(x - 5)(x + 2) = 0$. Hence, the solutions are –2 and 5. **Answer: (c)**

5. Let x be any number; $a = 2{,}000x + 1{,}999$; $b = 2{,}000x + 2{,}000$; and $c = 2{,}000x + 2{,}001$. What is the value of $a^2 + b^2 + c^2 - ab - bc - ca$?

   (a) 3 (b) 4 (c) 6 (d) dependent on x

   Solution: Note that $a = b - 1$ and $c = b + 1$. Hence $a^2 + b^2 + c^2 = a(a - b) + b(b - c) + c(c - a) = -a - b + 2c = 3$. **Answer: (a)**

6. The numbers a and b satisfy $1 > -b > a > 0 > b$. Which of the following inequalities must be correct?

   (a) $1 - b > 1 + a > -b > a$ (b) $1 + a > 1 - b > -b > a$
   (c) $1 + a > a > 1 - b > -b$ (d) $1 - b > -b > 1 + a > a$

   Solution: In (b), $1 + a > 1 - b$ is wrong. In (c), $a > 1 - b$ is wrong. In (d), $-b > 1 + a$ is wrong. On the other hand, since $-b > a$, we have $1 - b > 1 + a$. Since $1 > -b$ and $a > 0$, we have $1 + a > -b$. **Answer: (a)**

7. No two of the numbers a, b and c are equal, and none of the numbers x, y and z is equal to 0. Which of the following statements must be correct if $\frac{x}{b-c} = \frac{y}{c-a} = \frac{z}{a-b}$?

   (a) $ax + by + cz = 0$ (b) $ax = by = cz$
   (c) $x + y + z = 0$ (d) $x = y = z$

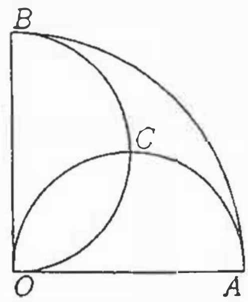
   Solution: Let the common value of the three given fractions be r. Then $x = br - cr$, $y = cr - ar$ and $z = ar - br$. Hence $x + y + z = 0$. **Answer: (c)**

8. In ΔABC, $3\angle A > 5\angle B$ and $2\angle B \geq 3\angle C$. What kind of triangle is ABC?

   (a) acute triangle (b) right triangle
   (c) obtuse triangle (d) inconclusive

   Solution: Let $\angle A = 25x°$. Then $\angle B < 15x°$ and $\angle C \leq 25x°$. Hence $180° = \angle A + \angle B + \angle C < 50x°$ so that $\angle A = 25x° > 90°$. **Answer: (c)**

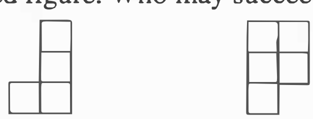
9. The diagram below shows a circular arc AB with centre O, such that OA is perpendicular to OB. Two semicircles with respective diameters OA and OB meet at C. Which has the larger area—the football-shaped region OC or the scythe-shaped region ABC?



(a) OC (c) equal
(b) ABC (d) inconclusive

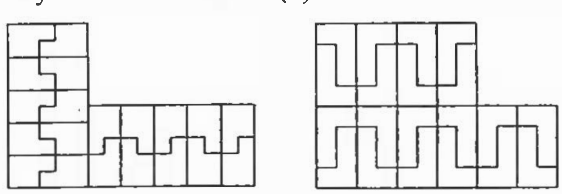
Solution: Let the area of OC be x and the area of ABC be y. The other two regions obviously have the same area, and let their common value be z. Let $OA = 2$. Then $x + y + 2z = \frac{1}{4}\pi(2)^2 = \pi$ while $x + z = \frac{1}{2}\pi(1)^2$. Hence $x + y + 2z = 2(x + z)$ or $x = y$. **Answer: (c)**

10. The diagrams below show an L-shaped figure and a P-shaped figure. Leon wants to build a similar larger P-shaped figure using several identical smaller copies of the L-shaped figure, while Paul wants to build a similar larger L-shaped figure using several identical smaller copies of the P-shaped figure. Who may succeed?



(a) both (b) only Leon (c) only Paul (d) neither

Solution: The diagram below shows that both may succeed. **Answer: (a)**



Part 2: Numeric Response

1. If $2^8 + 4^4 + 16^2 + 256 = 2^x$, what is the value of x?

Solution: We have $2^8 + 4^4 + 16^2 + 256 = 2^8 + 2^8 + 2^8 + 2^8 = 2^2 \times 2^8 = 2^{10}$. **Answer: 10**

2. In a contest, there are 80 per cent more boys than girls, and the average of the girls is 20 per cent higher than that of the boys. If the overall average is 75, what is the average of the girls?

Solution: Let the girls' average be $6x$ so that the boys' average is $5x$. Now the ratio of girls to boys is 5:9. Hence, $5(6x) + 9(5x) = (5 + 9)75$ or $x = 14$. Then, $6x = 6(14) = 84$. **Answer: 84**

3. The number of red balls is more than the number of green balls but less than twice the number of green balls. Each green ball costs \$2, and each red ball costs \$3. If the total cost of the balls is \$60, what is the total number of balls?

Solution: Let there be x red balls and y green balls. Then $y < x < 2y$ and $3x + 2y = 60$. Note that x must be divisible by 2 and y by 3. Let $x = 2u$ and $y = 3v$. Then $3v < 2u < 6v$ and $u + v = 10$. If $u \le 6$, then $v \ge 4$ and $3v \ge 2u$. If $u \ge 8$, then $v \le 2$ and $2u \ge 6v$. Hence, $u = 7$, $v = 3$, $x = 14$ and $y = 9$. Then, $x + y = 14 + 9 = 23$. **Answer: 23**

4. If both p and $p^5 + 5$ are prime numbers, what is the smallest value of $n > 5$ such that $p^n + n$ is also a prime number?

Solution: Since $p^5 + 5$ is a prime number, p must be even. Since p is also a prime number, we must have $p = 2$ so that $p^5 + 5 = 37$. In order for $p^n + n$ to be a prime number also, n must be odd. When $n = 7$, we get 135, which is divisible by 5. When $n = 9$, we get 521, which is less than the square of 23. Since 521 is not divisible by any of the prime numbers 2, 3, 5, 7, 11, 13, 17 and 19, it is a prime number itself. **Answer: 9**

5. If $x + y = 5$, $y + z = 8$ and $z + x = 7$, what is the value of $2x + 3y + 5z$?

Solution: Adding the given equations, we have $2(x + y + z) = 20$ or $x + y + z = 10$. Subtracting from this the given equations one at a time, we have $z = 5$, $x = 2$ and $y = 3$. Hence, $2x + 3y + 5z = 2(2) + 3(3) + 5(5) = 38$. **Answer: 38**

6. If $x + \frac{1}{x} = 3$, what is the value of $\frac{x^2}{x^4 + x^2 + 1}$?

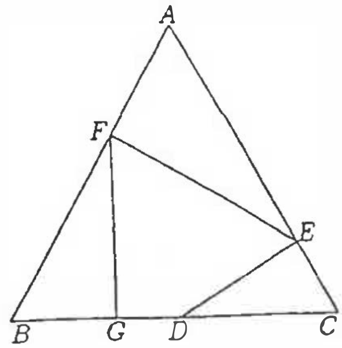
Solution: Squaring the given equation, we have $x^2 + 2 + \frac{1}{x^2} = 9$ so that $x^2 + 1 + \frac{1}{x^2} = 8$. Dividing the denominator of $\frac{x^2}{x^4 + x^2 + 1}$ by the numerator, we obtain $x^2 + 1 + \frac{1}{x^2}$. **Answer: 1/8**

7. How many integers x satisfy the inequality $\frac{4}{\sqrt{3} + \sqrt{2}} < x < \frac{60}{\sqrt{170} - \sqrt{2}}$?

Solution: We have $1 = \frac{4}{2+2} < \frac{4}{\sqrt{3}+\sqrt{2}} < \frac{4}{1+1} = 2 < 5 = \frac{60}{13-1} < \frac{60}{\sqrt{170}-\sqrt{2}} < \frac{60}{12-2} = 6$. Thus the acceptable values are 2, 3, 4 and 5. **Answer: 4 integers**

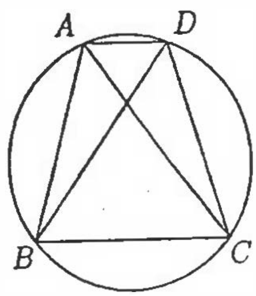
8. The diagram below shows an equilateral triangle ABC of area 128. D is the midpoint of BC. E is the point on CA such that DE is perpendicular to CA. F is the point on AB such that EF is perpendicular to AB. G is the point on BC such that FG is perpendicular to BC. What is the area of the quadrilateral $DEFG$?

Solution: Note that triangles *CDE, AEF* and *BFG* are similar to one another. We also have $CD = 2CE$ and $AE = 2AF$. Let $BC = 16x$. Then $CD = 8x$, $CE = 4x$, $AE = 12x$, $AF = 6x$ and $BF = 10x$.



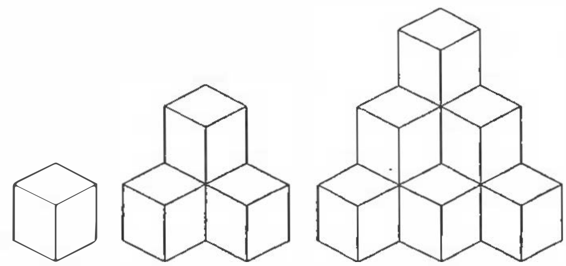
Since $BC = CD$, the area of *ACD* is 64. Since $AE = 3CE$, the area of *CDE* is 16. Since $CD{:}AE = 8{:}12$, the area of *AEF* is 36. Since $AE{:}BF = 12{:}10$, the area of *BFG* is 25. Hence, the area of *DEFG* is $128 - 16 - 36 - 25 = 51$. **Answer: 51**

9. The diagram below is a rough sketch of a quadrilateral *ABCD* inscribed in a circle. The arcs *AB*, *BC* and *CD* have the same length. The diagonals *AC* and *BD* meet at *E*, and $\angle AED = 70°$. What is the measure of $\angle ABD$?



Solution: From the equal arcs, we have $\angle ACB = \angle CAB = \angle DBC$. Since $\angle CEB = \angle AED = 70°$ and $\angle CEB + \angle ACB + \angle DBC = 180°$, each of those three equal angles has measure 55°. Now $\angle ABD = \angle AED - \angle CAB$. **Answer: 15°**

10. The diagram below shows the first three of a sequence of towers. After the first, the next tower is obtained from the preceding one by adding a unit cube on top of each stack of cubes plus a row of unit cubes at the base in front. There are 1, 4 and 10 unit cubes in the first three towers, respectively. How many unit cubes are needed to build the sixth tower?



Solution: To go from the $(n - 1)$st tower to the nth, we add $1 + 2 + \ldots + n = \frac{1}{2}n(n + 1)$ unit cubes. Since the third tower requires 10 unit cubes, the number of unit cubes needed to build the sixth tower is $10 + \frac{1}{2}[(4 \times 5) + (5 \times 6) + (6 \times 7)]$. **Answer: 56**

Each week Susan's class takes a 30-item spelling test. If Susan scored 20 on the first test, what is the lowest she can score on the second test in order for the average of her first three tests to be 26?