"Aha!" Problem Solving: Solution Begging for Meaning

Jerry Ameis

A key recommendation of the National Council of Teachers of Mathematics (NCTM 2000) *Principles and Standards for School Mathematics* concerns problem solving. In relation to classroom practice, a useful way to conceptualize problem solving is (1) teachers creating a classroom climate that facilitates learning in a problem-solving way and (2) students participating in two types of mathematical problem solving—"Aha!" problem solving and "Eureka!" problem solving.

A classroom climate of problem solving can be associated with the notion of teacher as facilitator. Facilitating is generally taken to mean setting up a classroom environment where children can ask and answer their own questions and building openness, reflective thinking, risk taking and investigation into lessons and activities. In such a climate, learning itself is thought of as a problem to be solved.

In the second aspect of problem solving, students solve mathematical problems using "Aha!" problem solving or "Eureka!" problem solving. Other labels for these two types of problem solving are used. For example, some curriculum documents refer to them, respectively, as routine problem solving and nonroutine problem solving.

"Aha!" problem solving involves an "Aha! I know how to do this!" reaction to a problem. This occurs when someone has seen a similar problem before and recognizes that its solution can be readily applied to the current problem. In other words, the problem solver almost automatically knows how to solve a problem because he or she has solved something like it before.

"Eureka!" problem solving involves a "Eureka! I finally figured it out!" reaction to a problem. This occurs when someone finally solves a problem after searching for and trying strategies, such as guess-and-check or think-backward strategies, that might be useful in solving the problem. In this case, a search for strategies is necessary because no convenient model or solution path is readily available to apply to the problem. This kind of problem solving involves much sweat and tears.

Both types of mathematical problem solving are useful and equally valuable. An analogy of going to see a doctor about an illness may help to illustrate the point. When you go to the doctor, you do not want the doctor to say, "Oh dear, that's an interesting illness. I wonder what it is and how to cure it." Rather, you would prefer that the doctor say, "Aha! I saw that illness just yesterday. Here is what is needed to cure it." However, there are times when an illness cannot be readily diagnosed and, therefore, investigation is required. You want the doctor to be able to deal with that circumstance as well.

"Aha!" Problem Solving

In the curricular sense, "Aha!" problem solving involves knowing whether to add, subtract, multiply, divide or use ratio to solve a problem. In the curricular view, problem solving must serve a socially useful function and have immediate and future payoff. For this reason, "Aha!" problem solving is not old-fashioned. Rather, it is a vital part of our lives. Children do "Aha!" problem solving as early as age five or six. They combine and separate toys or money in their normal activities. Adults are regularly called upon to do simple and complex "Aha!" problem solving. For example, a sales promotion in a store advertises a jacket selling for 20 per cent off the regular price of $125.98. If you are interested in buying the jacket, you might think, *Aha! I know that .20 × $125.98 is about $25. This means that the jacket will cost me about $100. Yup, I can afford it!*

This is a good opportunity to clarify an important matter: the ability to estimate .20 × $125.98 and subtract $25 from $125 is not the critical characteristic of the curricular sense of "Aha!" problem solving. Rather, the critical characteristic is realizing that you need to do those calculations to solve the problem. Actually doing the calculations is only the necessary final step in solving the problem. In other words, knowing what arithmetic to do, not actually doing it, is the critical characteristic of the curricular sense of "Aha!" problem solving.

Knowing what arithmetic to do is directly related to understanding the meanings of the four arithmetic operations and the concept of ratio. Although there are only four arithmetic operations—adding, subtracting, multiplying and dividing—these four operations have more than four distinct meanings (for more detail, refer to www.uwinnipeg.ca/~jameis/New%20Pages/EYR23.html). For example, *subtract* has at least three meanings: "take away," "compare two sets" and "find the change from one measurement to another." Consider the following situations. Each can be mathematically modelled by the number sentence 7 – 3 = 4, yet each involves a quite different meaning of *subtraction:*

- I had seven cookies in my lunch bag. I gave three away. Now I have four cookies.
- I have $7. Sam has $3. I have $4 more than Sam.
- The temperature at 9 a.m. was 3°C. At noon, the temperature was 7°C. The change in temperature was 4°C.

If students do not understand the meanings of the arithmetic operations and of ratio, "Aha!" problem solving can easily become a quagmire of confusion for them. Consider the following problem presented to a group of Grades 4 and 5 students:

> Harry went on a trip to the Nile River. His task was to completely fill his crocodile-egg incubator. On the first day of his journey, Harry came upon a crocodile nest. He dug up 23 unhatched crocodile eggs and placed them in the incubator. Harry walked five paces and dug up some more unhatched crocodile eggs, which he quickly placed in the incubator as well. Harry's task was now done: he had completely filled his egg incubator with 40 unhatched crocodile eggs. How many eggs did Harry find at the second crocodile nest?

The students had been taught two problem-solving strategies: (1) guess and check and (2) look for a pattern. The responses of three students typify the thinking of the other students in the class.

Student 1 read the problem carefully, underlined all the numbers and then said, "Easy numbers!" The student's answer of 44 was incorrect (although he strongly insisted that it was correct). His calculations were as follows: 40 + 5 = 45; 45 – 23 = 22; 22 × 2 = 44. To explain the work, the student stated that, because there were 40 eggs, it was necessary to add "the little number" (the 5) and then subtract 23. That answer then had to be multiplied by 2 because the problem had stated the word *second* somewhere.

Student 2 did not do the problem, explaining that the problem did not make any sense because it did not fit either of the problem-solving strategies learned in class.

Student 3 made a guess-and-check chart, but did not do any guessing. She obtained the correct answer of 17 by subtracting 23 from 40 but could not explain the subtraction strategy. When asked if she had done any guessing, she replied no. When asked why she had made a guess-and-check chart, she replied, "All word problems are done by guess and check."

The three student responses illustrate several things, including the inadequacy of the guess-and-check and look-for-a-pattern strategies in solving problems such as the one about the crocodile eggs. These strategies have little value in "Aha!" problem solving. They are, however, quite useful for "Eureka!" problem solving. The responses also illustrate the problem-solving difficulties that arise when students do not understand the meanings of the arithmetic operations.

The strategies that children often use for "Aha!" problem solving take a variety of forms. The following is a representative list (Sowder 1988):

1. Find the numbers and add (or multiply or . . .). The choice tends to be dictated by recent activities or by what the child feels comfortable with.
2. Guess the operation to be used and see what you get. This is a form of guess and check.
3. Look at the numbers; they will "tell" you which operation to use (for example, the numbers 63 and 59 suggest add or subtract, while 25 and 5 suggest divide).
4. Try all the operations and select the most reasonable answer.
5. Look for key words that tell you what to do.
6. Decide whether the answer should be larger or smaller than the given numbers. If larger, try both addition and multiplication, and select the more reasonable answer. If smaller, try both subtraction and division and select the more reasonable answer.
7. Choose the operation whose meaning fits what is happening in the problem.

The strategy of looking for key words warrants special comment because of its prevalence. Using this strategy, if the word *minus* appears in a problem, you subtract to get the answer; if the word *more* appears, you add to get the answer. Looking for key words is inappropriate for at least three reasons.

First, the strategy has little to do with understanding what the problem is actually about. Therefore, transfer to more complex problems—those that contain more than one step—tends to be difficult. For complex problems, merely picking out key words as indicators of what to do often leads to incorrect solutions.

Second, in contrast to the problems typically found in mathematics textbooks and resource books, many real-world problems do not contain key words. In life, there are rarely any flashing neon signs that signal what to do to solve a problem. What does the problem solver do, then, if all he or she knows is to look for key words?

Finally, key words can be misleading. For example, *more* is not necessarily a signal to add. Consider the following:

> Johnny had 139 candies. He ate 12 of them. Then he ate 15 more. How many candies does Johnny have now?

In this problem, adding would lead to an interesting (and incorrect) answer.

All key words can be misleading, even the word *total.* Is *total* always a signal to add? Consider the following two problems:

> Bucky the squirrel was storing acorns for the winter. Bucky had a large family. He hid 235 acorns in each of 985 holes. What was the total number of acorns Bucky hid?

> Sparks kept all his money in a piggy bank. When he counted it one day, he found that he had $345. He decided to put 2/3 of his money in a real bank. What was the total amount of money he put in the real bank?

To further appreciate the trap that key words can be for students, consider this example:

> Four out of five coaches recommend heavy-weight lifting as an important part of becoming an athlete. What percentage of coaches do not recommend it?

A student quickly answered the problem correctly by saying that the answer was 20 per cent. The teacher asked the student to explain how he had figured out the answer. The student replied, "I multiplied 4 by 5 because *of* means 'multiply.'"

Of the strategies in Sowder's (1988) list, only the seventh—"Choose the operation whose meaning fits what is happening in the problem"—will consistently lead to success with "Aha!" problem solving. That strategy involves identifying what is going on in the problem and then representing it mathematically. Mathematical forms such as number sentences can be used to represent what is going on in a problem. In the problem about the crocodile nest, eggs were combined in a "put together" sense, which can be modelled mathematically by the number sentence $23 + ? = 40$.

Research shows that good problem solvers identify what is going on in a problem rather than searching for the algorithm (the arithmetic process) to apply. This research strongly suggests that, to develop student proficiency in "Aha!" problem solving, teachers should use a teaching approach that involves modelling/representing the events or circumstances of problems.

A study found that modelling is a useful strategy for "Aha!" problem solving as early as Kindergarten (Carpenter et al. 1993). The Kindergarten students in the study had spent a year solving a variety of basic word problems related to addition, subtraction, multiplication and division. They used concrete materials to model the events of the stories. Overall, the children demonstrated remarkable ability in solving the word problems. Most of them solved most of the problems. They did as well as or better than children in Grades 1 and 3 who solved similar problems but did not consistently use modelling as the problem-solving strategy. The Kindergarten students' success can be explained by their consistent use of modelling to represent the actions of the problems in a way that reflected what was going on in the story.

The approach recommended here for developing proficiency in "Aha!" problem solving requires that students deeply learn the necessary conceptual tools—the meanings of the arithmetic operations and of ratio. The students use these tools to understand what is going on in a problem and, based on that understanding, to represent what is going on through a number sentence. Then, they decide which algorithm to use and perform that algorithm, using a calculator, paper and pencil, or mental arithmetic.

What is happening in the problem and the algorithm do not have to match. Consider the following problem:

> Sleepy the dwarf had some jewels. On the way home, he gave 123 away. Now he has 767 jewels. How many did he have to begin with?

The problem involves giving away jewels. The number sentence that best represents that action is $? - 123 = 767$, not $123 + 767 = ?$. Nothing has been put together or combined; rather, something (123 jewels) has been removed, and that involves the "take away" meaning of *subtraction.* However, the convenient algorithm to use for obtaining the answer involves addition.

Transforming the number sentence ? – 123 = 767 to ? = 123 + 767 turns the arithmetic operation from subtraction into addition. The transformation is justified by the inverse relationship between addition and subtraction.

At this point, the reader may be thinking, *I do the problem about Sleepy by simply adding the numbers. Why can't children?* That is a valid question that needs to be addressed. Once children deeply understand the meaning of an arithmetic operation, they should be encouraged and expected to take shortcuts with it. The meanings of the operations are best seen as models for viewing the world. They are not truths; rather, they are useful or not-useful tools. Once a tool is understood, it can be applied in a variety of ways. The problem about Sleepy could be conceptualized as "What I have left combined with what I gave away gives me what I started with." The number sentence for that would be 767 + 123 = ?. That way of thinking is a shortcut to solving the problem. Taking shortcuts should be encouraged once children deeply understand the meanings of the arithmetic operations. But shortcuts are not likely to be used consistently or reliably if children do not understand what is going on in the problem in the first place.

To conclude, here is a problem for the reader:

> George is opening a pizza restaurant. He must think small at first because his resources are limited. He decides to sell only two-topping pizzas, with one meat topping and one vegetable topping. He offers four meat toppings: pepperoni, sausage, ham and salami. Because George believes more is better, he offers five vegetable toppings: onions, mushrooms, green peppers, olives and tomatoes. George wants to make a menu listing all the combinations of two-topping pizzas he sells. What is the total number of pizzas he should list on the menu?

Write the number sentence that best represents what is going on in the problem. Explain why you wrote that number sentence, based on one of the meanings of the arithmetic operations (see www.uwinnipeg.ca/~jameis/New%20Pages/EYR23.html).

References

Carpenter, T. P., et al. "Models of Problem Solving: A Study of Kindergarten Children's Problem-Solving Processes." *Journal for Research in Mathematics Education* 24, no. 5 (1993): 428–41.

National Council of Teachers of Mathematics (NCTM). *Principles and Standards for School Mathematics.* Reston, Va.: NCTM, 2000. Available at http://standards.nctm.org [accessed May 29, 2003].

Sowder, L. *Concept-Driven Strategies for Solving Story Problems in Mathematics.* Washington, D.C.: National Science Foundation, 1988.

The average of five numbers is 20. If x and y are two of the numbers, what is the average of the other three numbers in terms of x and y?
