The Ancient Problem of Trisecting an Angle

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The problem of trisecting an angle dates back to the ancient Greeks, and as early as the fifth century B.C., Greek and Muslim geometers devoted much time to this puzzle. This problem is one of the three famous geometric problems of antiquity, which also include doubling the cube and squaring the circle. These three great construction problems of geometry could not be solved using an unmarked straightedge and compass stone, the only implements sanctioned by the ancient Greeks. But it was not until the 19th century that advances in the algebra of the real-number system allowed us to make instruments that made possible these constructions that were impossible with the straightedge and compass alone.

The problem of trisecting an angle is the simplest of the three famous problems to comprehend, and because the bisection of an angle presented no difficulty to the geometers of antiquity, there was no reason to suspect that its trisection might prove impossible.

The multisection of a line segment with Euclidean tools is a simple matter, and it may be that the ancient Greeks were led to the trisection problem in an effort to solve the analogous problem of multisection of an angle. Or, perhaps more likely, the problem arose in efforts to construct a regular nine-sided polygon, where the trisection of a 60° angle is required.

The angle trisection problem is not entirely unsolvable using the classical method of compass and straightedge. The Greeks knew this, but they were searching for a generalized construction (as in angle bisection) that could be used to trisect any angle.

Actually, an infinite number of angles can be trisected. Among this group are angles whose degree measure equals $360/n$, where n is an integer not evenly divisible by 3. For example, a 90° angle can be trisected because n in $360/n$ is 4, which is not evenly divisible by 3. Figure 1 shows a trisected 90° angle. The trisection of the 90° angle can be done quite simply using the following method:

> Construct a 90° angle, $\angle AOB$. Then, draw arc AB. Without changing the size of the compass opening, place the compass at point B and draw an arc intersecting arc AB at point C. Line OC is the line trisecting right angle $\angle AOB$. Line OD also trisects right angle $\angle AOB$ using the same method outlined above and placing the compass at point A. Line OD also bisects 60° angle $\angle COB$.

However, an infinite number of angles cannot be trisected by means of compass and straightedge. These are angles whose degree measures are equal to $360/n$, where n is an integer divisible by 3. For example, a 60° angle cannot be trisected because n, in $360/n$, would be 6, which is divisible by 3. To prove that general angle trisections are impossible with just an unmarked straightedge and compass, we use the special case of a 60° angle (see Figure 2). Suppose $\angle COA = 60°$ and $\angle BOA = 20°$. For the proof, we use the following trigonometric identity:

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

Let $3\theta = 60°$ and let $x = 2\cos\theta = 2\cos 20°$. Then,

$$\cos 60° = \frac{x^3}{2} - \frac{3x}{2}$$

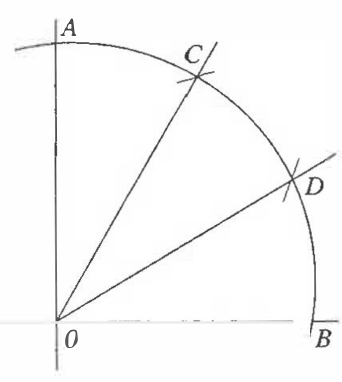
$$2\left(\frac{1}{2}\right) = \left(\frac{x^3}{2} - \frac{3x}{2}\right)2$$

$$1 = x^3 - 3x$$

$$x^3 - 3x - 1 = 0.$$

This cubic equation is irreducible. Thus, its roots cannot be constructed with a straightedge and compass. From this, we can conclude that the construction of trisecting the general angle cannot be performed with straightedge and compass alone.

Figure 1
Trisection of a 90° Angle, $\angle AOB$



An ingenious method for trisecting angles was presented by Archimedes, who used the marking of two points on a straightedge to mark off a line segment (not using the classical rules of compass and straightedge). Figure 3 shows a trisected 60° angle.

Archimedes' method is as follows. Draw $\angle AED$. Through the angle, draw a semicircle with a radius the same length as DE. Extend DE to the right. With the compass open to the length of radius DE, hold the legs of the compass against the straightedge and hold the straightedge so it passes through point A. Adjust the straightedge until the points marked by the compass intersect points B and C (where BC is equal to DE). Arc BF is 1/3 of arc AD. Because central angles are congruent in degree measure to their intercepted arcs, $\angle BEF$ is 1/3 of the degree measure of $\angle AED$.

Another method of trisecting an angle is using the conchoid of Nicomedes. Figure 4 helps to define *conchoid.*

Nicomedes took a fixed point O, which is d distance from a fixed line AB, and drew OX parallel to AB and OY perpendicular to OX. He then took any line OA through O and on OA made $AP = AP' = k$, a constant. Then the locus of points P and P' is a conchoid. The equation of the curve is

$$(x^2 + y^2)(x - d)^2 - k^2x^2 = 0.$$

To trisect a given angle, let $\angle YOA$ be the angle to be trisected. From point A, construct AB perpendicular to OY. From point O as pole, with AB as a fixed

Figure 2
Attempted Trisection of a 60° Angle, $\angle COA$

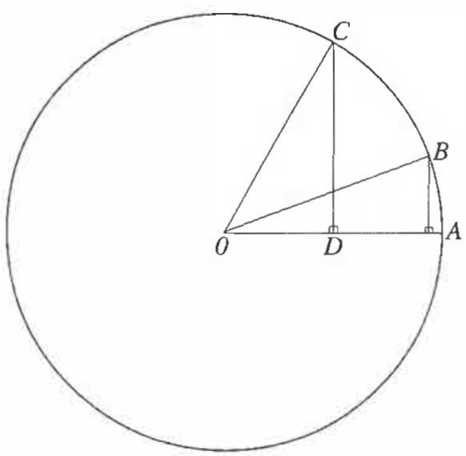


Figure 3
A Trisected 60° Angle, $\angle AED$

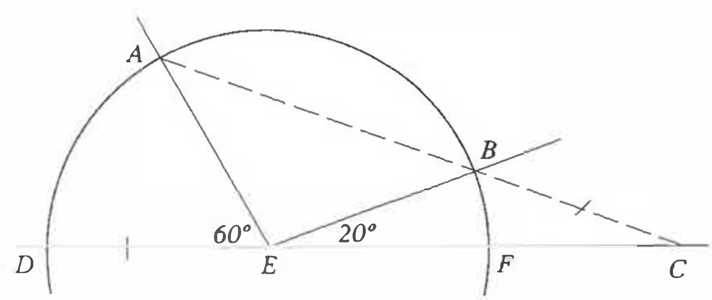
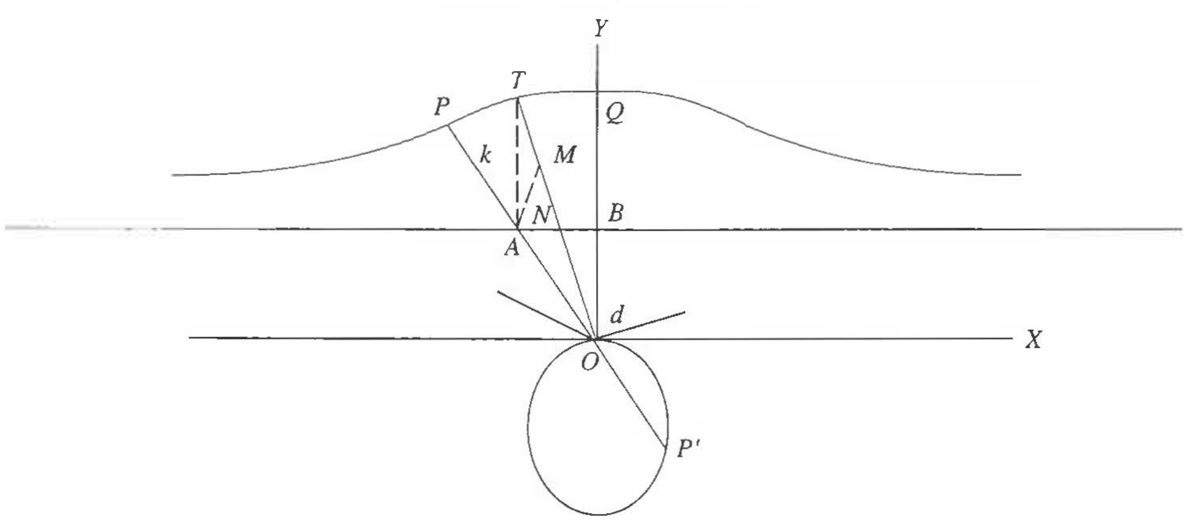


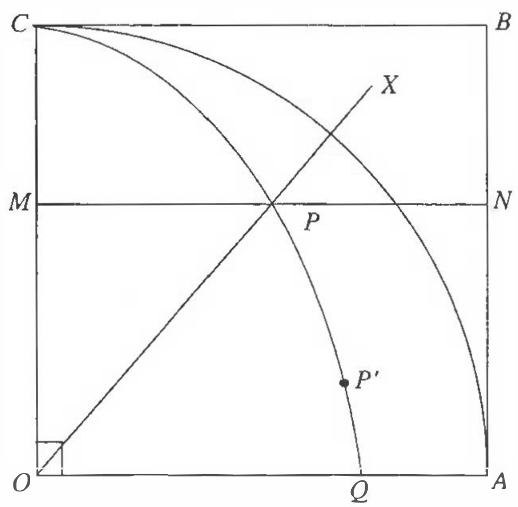
Figure 4
Conchoid of Nicomedes



straight line, $2(AO)$ as a constant distance, construct a conchoid to meet OA produced at P and to cut OY at Q. At A, construct a perpendicular to AB meeting the curve at T. Draw OT and let it cut AB at N. Let M be the midpoint of NT. Then $MT = MN = MA$. But $NT = 2(OA)$ by construction of the conchoid. Hence, $MA = OA$. Hence, $\angle AOM = \angle AMO = 2\ \angle ATM = 2\ \angle TOQ$. That is, $\angle AOM = \frac{2}{3}\ \angle YOA$, and $\angle TOQ = \frac{1}{3}\ \angle YOA$.

Hippias of Elis wrestled with this problem and, realizing the inadequacy of the ruler-and-compass method, resorted to other devices. These involved the use of curves other than the circle. The one employed by Hippias was the quadratrix, so called because it serves as well for the problem of quadrature (squaring the circle) as for dividing an angle into three or more equal parts. The quadratrix of Hippias may be defined as follows (see Figure 5). Let the radius OX of a circle rotate uniformly about the centre O from OC to OA, with $\angle COA$ forming a right angle. At the same time, let a line MN parallel to OA move uniformly parallel to itself from CB to OA. The locus of the intersection P of OX and MN is the quadratrix.

Figure 5
Quadratrix of Hippias



In the trisection of an angle, X is any point in the quadrant AC. As the radius OX revolves at a uniform rate from OC to OA, MN always remains parallel to OA. Then if MN is one nth of the way from CB to OA, the locus of point P (the intersection of OX and MN) is one nth of the way from OC around to OA. If, therefore, we make $CM = 1/3(CO)$, MN will cut CQ at a point P such that OP will trisect the right angle. In the same way, by trisecting OM, we can find a point P' on CQ such that OP' will trisect $\angle AOX$, and so for any other angle. This method evidently applies to the multisection as well as the trisection of an angle.

I have cited only three of the most ancient methods. There are many other techniques for trisecting an angle using tools such as a tomahawk or a mira.

Ingenious mathematicians of recent times have developed original methods to trisect angles. Leo Moser of the University of Alberta trisected angles with the use of an ordinary watch. He discovered that if the minute hand passed over an arc equal to four times the measure of the angle to be trisected, the hour hand would move through an arc exactly 1/3 the measure of the given angle to be trisected. Alfred Kempe, a London lawyer, developed a linkage method of folding parallelograms so that the two opposite sides cross.

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