On Circumscribed Circles

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My recent article (Dobbs 2003) proposed some enrichment material for the typical precalculus course by developing several methods to study angle bisectors and inscribed circles of triangles. It seems natural to ask if the subject of circumscribed circles of triangles can also provide enrichment material for precalculus. This article shows how that may be done. The insight that such coverage is possible at the precalculus level is not new. For instance, Smith's (1956, 101) classic treatise includes an exercise asking for an equation of the circumscribed circle of a given triangle. Because Smith's text does not suggest a method to work that exercise, it is of some interest to find several such methods, and I do so here.

One way to proceed is to develop methods for finding the perpendicular bisector of a line segment, because the centre K of the circumscribed circle of a triangle Δ is the intersection of the perpendicular bisectors of the sides of Δ. For this reason, I devote the next section to developing two methods for constructing perpendicular bisectors of segments. Of course, once we have found K, we can use the distance formula to find the radius r of the circumscribed circle, because r is the distance from K to any vertex of Δ. Then, given K and r, we can move directly to the standard form equation of the circumscribed circle. The associated discussion in this section reinforces several precalculus topics: standard form equations of circles, midpoint formula, equations of lines, solution of systems of two linear equations in two unknowns, slope and the "negative reciprocals" criterion for perpendicularity. Note that the last of these topics is so fundamental that it can be proved in at least four ways in a precalculus course (Dobbs and Peterson 1993, 39, 41, 338, 427). To make matters concrete and more user-friendly, I give a numerical illustration of the methods here and in the following section by applying them to a particular triangle Δ.

In the final section of the article, I turn matters around by giving two methods for directly finding an equation for the circumscribed circle of a given triangle. Of course, with such an equation in hand, we can recover the coordinates of the centre and the radius of this circle by completion of squares. Both methods in this section are necessarily more algebraic than the methods developed in the earlier section. The first of these algebraic methods involves solving a system of three linear equations for three unknowns and thus reinforces an important topic from precalculus/algebra. The second method can be approached through Cramer's rule, thus reinforcing the study of determinants (and, possibly, matrices). In a closing comment, I draw an analogy between the second algebraic method and an old, but often overlooked, method for finding an equation of the line through two given points.

Two Methods for Finding Perpendicular Bisectors

Suppose we are given two distinct points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ in the Euclidean plane. There is an obvious, direct way to find the perpendicular bisector of the line segment P_1P_2. For this method, first recall from the midpoint formula (Dobbs and Peterson 1993, 34–35), a nice application of the theory of proportion and similar triangles, that the midpoint of the segment is $Q[(x_1 + x_2)/2, (y_1 + y_2)/2)]$. Next, we need only write an equation for the line that passes through Q and is perpendicular to P_1P_2.

Let us illustrate the above method by finding an equation for the circumscribed circle of ΔABC, given the vertices $A(-9, 11)$, $B(2, -4)$ and $C(6, 8)$. (Readers of Dobbs [2003] will surely recognize this "random" triangle.) The midpoint of the segment AB is $(-7/2, 7/2)$, and the slope of AB is $(-4 - 11)/[2 - (-9)] = -15/11$. By the "negative reciprocals" result, the perpendicular bisector of AB has slope 11/15 and, thus, by the point-slope form of the equation of a line, has equation

$$y = \frac{11}{15}\left[x - \left(-\frac{7}{2}\right)\right] + \frac{7}{2}$$

or, equivalently,

$$11x - 15y + 91 = 0.$$

Similarly, one verifies that the midpoint of the segment CA is (–3/2, 19/2) and that the perpendicular bisector of this segment is $5x - y + 17 = 0$.

By solving the system of linear equations

$$\begin{cases} 11x - 15y + 91 = 0 \\ 5x - y + 17 = 0 \end{cases}$$

for the unknowns x and y, one finds the coordinates (h, k) of the centre K of the circumscribed circle of ΔABC to be (–164/64, 67/16) = (–41/16, 67/16). The radius r of this circle is the distance between K and any vertex, say C, and so, by the distance formula,

$$r^2 = \left[6 - \left(-\frac{41}{16}\right)\right]^2 + \left(8 - \frac{67}{16}\right)^2 = \frac{22{,}490}{256} = \frac{11{,}245}{128}.$$

Then the standard form equation of the circumscribed circle is

$$(x - h)^2 + (y - k)^2 = r^2,$$

that is,

$$\left[x - \left(-\frac{41}{16}\right)\right]^2 + \left(y - \frac{67}{16}\right)^2 = \frac{11{,}245}{128}$$

or, equivalently, $x^2 + y^2 + (41/8)x - (67/8)y - 255/4 = 0$.

I turn next to the second method promised in the title of this section. This method depends on the following fact from Euclidean plane geometry (a prerequisite for the typical precalculus course): given distinct points P_1 and P_2 in the plane, a point Q in that plane is on the perpendicular bisector of the segment P_1P_2 if and only if Q is equidistant from P_1 and P_2. (The proof of this fact is a familiar application of congruence criteria: use side–angle–side and either side–side–side or hypotenuse–side.) This fact justifies my earlier comment that the centre of the circumscribed circle of a triangle is the intersection of the perpendicular bisectors of the sides of that triangle. Next, we can use this fact to find the perpendicular bisector of P_1P_2, as follows. By the distance formula, a point $Q(x, y)$ is on this perpendicular bisector if and only if

$$(x - x_1)^2 + (y - y_1)^2 = (x - x_2)^2 + (y - y_2)^2.$$

It is clear that algebraic simplification of the preceding equation leads to the desired linear equation (because the terms in x^2 and y^2 cancel). Rather than write the general form of this linear equation, let us illustrate it by returning to the data examined above.

Consider the segment AB, given $A(-9, 11)$, $B(2, -4)$. The above method yields that the perpendicular bisector of AB is given by

$$[x - (-9)]^2 + (y - 11)^2 = (x - 2)^2 + [y - (-4)]^2$$

or, equivalently (after cancellation of the terms in x^2 and y^2), $22x - 30y + 182 = 0$. This equation is equivalent to $11x - 15y + 91 = 0$, thus agreeing with the equation found by the first method. (It is interesting that our foray into quadratic equations has led to an arguably faster way to find this linear equation!) I encourage the reader to practise the second method to recover the earlier equation for the perpendicular bisector of the segment CA. Of course, with these two equations in hand, one can proceed as above to find the centre, radius and standard form equation for the circumscribed circle of ΔABC.

Two Algebraic Methods

Suppose that we are given three noncollinear points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ and $P_3(x_3, y_3)$ in the Euclidean plane (that is, the vertices of some triangle Δ). One way to find an equation for the circle passing through these three points (that is, the circumscribed circle of Δ) is to solve for the coefficients A, B and C in an equation, $x^2 + y^2 + Ax + By + C = 0$, for this circle. (Recall that circles are characterized as the graphs of equations of this form such that $A^2 + B^2 > 4C$.) For this, we solve the system of linear equations

$$\begin{cases} x_1^2 + y_1^2 + Ax_1 + By_1 + C = 0 \\ x_2^2 + y_2^2 + Ax_2 + By_2 + C = 0 \\ x_3^2 + y_3^2 + Ax_3 + By_3 + C = 0 \end{cases}$$

for the unknowns A, B and C. By completion of squares, we then find the centre $K(h, k)$ and radius r of this circle. We can then easily find equations for the perpendicular bisectors of the sides of Δ.

I next illustrate the procedure for the usual data, the triangle with vertices (–9, 11), (2, –4) and (6, 8). For this example, the above system of equations is

$$\begin{cases} (-9)^2 + 11^2 - 9A + 11B + C = 0 \\ 2^2 + (-4)^2 + 2A - 4B + C = 0 \\ 6^2 + 8^2 + 6A + 8B + C = 0 \end{cases}.$$

Any of the standard methods for solving such a linear system leads to the unique solution $A = 41/8$, $B = -67/8$ and $C = -255/4$, thus agreeing with the result of the methods in the preceding section. By completing squares, we can rewrite this equation as

$$\left[x - \left(-\frac{41}{16}\right)\right]^2 + \left(y - \frac{67}{16}\right)^2 = \frac{11{,}245}{128}$$

from which we recover the facts that K is (–41/16, 67/16) and r is determined as the principal square root of $r^2 = 11{,}245/128$. Finally, to identify the perpendicular bisector of one of the sides of the given triangle, say of AB, one of many possible ways to proceed would be to write the two-point form of the equation of the line through K and the midpoint of AB. We then obtain

$$y = \frac{\frac{67}{16} - \frac{7}{2}}{\frac{-41}{16} - \frac{-7}{2}}\left(x - \frac{-41}{16}\right) + \frac{67}{16}.$$

This equation simplifies to $352x - 480y + 2{,}912 = 0$, or equivalently, $11x - 15y + 91 = 0$, thus agreeing with a calculation in the preceding section.

In presenting the above algorithm, I glossed over one theoretical point—namely, how we can be sure, once we have solved for the unknowns A, B and C, that the equation $x^2 + y^2 + Ax + By + C = 0$ actually represents a circle. One answer depends on the following two observations: (1) by completion of squares, we see that the graph of any equation of this form is either a circle, a singleton set (that is, a set consisting of just one point) or the empty set and (2) the graph of this equation does pass through the three distinct points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ and $P_3(x_3, y_3)$, because the above system of linear equations is satisfied and, hence, must be a circle.

I turn next to the second "algebraic" method promised in the title of this section. Quite simply, this method presents the following equation for the circle passing through three given noncollinear points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ and $P_3(x_3, y_3)$ in the plane:

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0.$$

For our usual example, the above equation is

$$\begin{matrix} x^2 + y^2 & x & y & 1 \\ (-9)^2 + 11^2 & -9 & 11 & 1 \\ 2^2 + (-4)^2 & 2 & -4 & 1 \\ 6^2 + 8^2 & 6 & 8 & 1 \end{matrix} = 0.$$

By expanding the determinant along its first row, we can rewrite this equation as

$$192(x^2 + y^2) + 984x - 1{,}608y - 12{,}240 = 0$$

or, equivalently,

$$x^2 + y^2 + (41/8)x - (67/8)y - 255/4 = 0,$$

thus agreeing with the result already obtained twice above by other methods.

Why is the above determinental method valid in general? To answer this question, first notice that the proposed equation is satisfied by each of the points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ and $P_3(x_3, y_3)$ because the determinant vanishes for any square matrix having two equal rows. Moreover, by the above comments, if the proposed equation is truly quadratic, then its graph must be a circle (because we now know that it is neither a singleton set nor empty). Finally, the proposed equation is truly quadratic. Indeed, expanding along the first row of the determinant appearing in this equation, we see that the coefficient of $x^2 + y^2$ is

$$\begin{matrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{matrix}.$$

This determinant is nonzero (thus, the proposed equation is truly quadratic) for a fundamental geometric reason. In fact, the absolute value of this determinant can be shown to be twice the area of ΔABC (see Dobbs and Peterson [1993, 537]). Verification of this assertion makes for an accessible computational exercise early in a precalculus course (and a much easier, more conceptual exercise later for a student who knows about the crossproduct of vectors).

I next give another justification for the above determinental method. Still working in the Euclidean plane, let $Q(x, y)$ be another point on the circle passing through the three noncollinear points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ and $P_3(x_3, y_3)$. Consider the system of linear equations

$$(x_2 + y_2)S + xA + yB + 1 \bullet C = 0$$
$$(x_1^2 + y_1^2)S + x_1A + y_1B + 1 \bullet C = 0$$
$$(x_2^2 + y_2^2)S + x_2A + y_2B + 1 \bullet C = 0$$
$$x_3^2 + y_3^2 + x_3A + y_3B + 1 \bullet C = 0$$

for the unknowns S, A, B and C. Because the four points Q, P_1, P_2 and P_3 all lie on some circle $x^2 + y^2 + Ax + By + C = 0$, there is a nontrivial solution for these unknowns (in which $S = 1$). Consequently, by Cramer's rule, the coefficient matrix of the above linear system has its determinant equal to 0. The statement that this determinant equals 0 is precisely the proposed determinental method; and we can see that the proposed equation is truly quadratic (that is, after we expand along the top row, the coefficient of $x^2 + y^2$ is nonzero) as explained above.

In closing, I note that the second "algebraic" method can be modified to give an equation for the line passing through any two distinct points, $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$. (This observation was made by Nathan Mendelsohn in a lecture I attended in 1962, but I have not seen it elsewhere. Nor have I seen the analogous description of an equation for the sphere passing through four given noncoplanar points, but that would not be as useful, because it involves a 5×5 determinant.) More specifically,

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

is an equation for the line passing through the two given points P_1 and P_2. As above, the verification can proceed in either of two ways: (1) invoke Cramer's rule or (2) note that the expansion of the determinant gives a nontrivial linear equation satisfied by both the given points. Because the first postulate of Euclid's *Elements* states that exactly one line passes through any given pair of distinct points, we are done. In addition, by appealing to the very foundations of Euclidean geometry, this algebraic activity has served to illustrate the unity of mathematics.

References

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Dobbs, D. E., and J. Peterson. *Precalculus*. Dubuque, Iowa: William C. Brown, 1993.

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A is 25 per cent of *B*, and *B* is 30 per cent of *C*. What percentage of *C* is *A*? What percentage of *A* is *C*?