applied to existing documents. Which procedure would be of greatest value and benefit to teachers and students?

No doubt the decision-making processes of a group has been researched and written about. Allow me to supplement such reports with a few personal anecdotal observations. The question, "Why does this appear at the Grade 3 level?" will probably elicit the response "It is in the WCP/Ontario Strand." The question, "Should it be there?" may not be entertained. If, by chance, it is, the following scenarios are plausible. Utterances directly from the affective domain, such as "I like it" or "Teachers like it," may sway a group into accepting an idea. A passionate oration by someone, which might make reference to having students "see the beauty of..." can influence decision making (especially if the speaker is an administrator). Many times, it is assumed that a member of a group, such as a primary teacher, is a spokesperson with whom all members of that group will agree, and, as a result, statements by that member are not questioned, especially if he or she is the only member present. In instances like these, some sort of consensus is reached based on aspects of social interaction that are far removed from an examination of an overall growth plan for a given topic. Should that ever be the case?

Some revision meetings begin in a relaxed or rather open-ended way without a specific agenda. It inevitably happens that, at the end of the scheduled day(s), group members leave at different intervals (early). As a result, "legislation via exhaustion" occurs and fatigue sets in. Decisions are made rapidly and they may not be based on what was agreed on earlier. I have often wondered whether some chairs actually had this outcome in mind as part of their "hidden curriculum" when they made decisions about an agenda.

Decisions about revising, updating or creating curriculum are complex—more complex than many of the ideas in this article suggest. Perhaps some of these ideas can become part of a discussion about that complex process and procedure. That is my hope.

Why Do Numerate Students/Adults Lack Conceptual Understanding of Division?

Werner Liedtke

Division will be used as the focus of the discussion, which could be applied to other topics as well. The following questions will be addressed in this article: Are any concerns about the numeracy of our students warranted? What data exist to suggest that our students lack conceptual understanding of division? What are some possible reasons for students lacking conceptual understanding? What might be done to have students acquire conceptual understanding of division? and What are some of the key components of conceptual understanding of division?

Numeracy Concerns

There are mathematics educators who reference "rising scores on certain tests" and "results of performances on contests, nationally and internationally" to suggest that, as far as mathematics learning is concerned, things are improving or even all is well. These types of conclusions could be countered with information released by universities that not only expresses concerns about the lack of basic literacy and numeracy skills of its graduates but also talks about intervention programs for these students (who represent the top 15 to 20 per cent of our population—so what might be said about the others?).

I have collected competency test scores from education students for more than 30 years, and have seen that these scores have not risen. In fact, the opposite is true. Some colleagues have tried to explain this trend by suggesting that top students who at one time would have enrolled in education are now attracted to other areas or professions. Even if this assumption is true, changes to the system these students went through should result in higher levels of numeracy. As far as my observations tell me, that is not the case. I still encounter as many people as I did many years ago who somewhat proudly proclaim that they not only do not like math but also are not good at math or unable to do math. Comments like these, reports in newspapers and on the news about unwise consumers all lead me to believe that, despite all of our efforts, and it hurts more than a little to say this, things have not changed since Paulos (1998) wrote the book about innumeracy. I think there are as many people now who, as Paulos suggests, read number as (numb) (er) as there were when he wrote the book.

If there is any hope, or if there are any positive signs, it would be the implementation of the British Columbia Association of Mathematics Teachers' suggestions of increased emphasis on conceptual understanding and fostering the development of number sense—the key foundation for numeracy. Without these areas of emphases, things are not likely to change and improve. However, the mere identification of these areas is by no means sufficient. As will be seen, I think more needs to be done.

Conceptual Understanding

If the opposite of rote procedural knowledge is well-defined procedural knowledge, possible indicators of the latter include knowing how and why something works and being able to illustrate procedures with base-10 blocks or diagrams. Possible indicators of conceptual knowledge include the ability to connect ideas to one's experience, that is, being able to create relevant word problems; being able to simulate a procedure (algorithm) with appropriate denominations of money and explaining it in one's own words; using more than one method to find an answer; making predictions about an answer or commenting on the reasonableness of results and knowing why and how another operation might be used to check an answer.

Many topics in elementary mathematics are presented to students in a rote-procedural fashion. (I almost used *taught* rather than *presented* in the last sentence. This would be inappropriate for someone who agrees with those who believe rote learning is an oxymoron.) I think many aspects of geometry and telling time rank high among these topics, but division is at or very near the top of the list.

The competency test items that include aspects of division always present the greatest difficulty for the students in my courses. Most of the students recall being taught an algorithm by way of a rule method that is still used and displayed in some classrooms around the province. This method of presentation, along with inappropriate practice, no matter how extensive, will not result in the acquisition of conceptual understanding. Charles and Lobato (1998, 17) define practice as "appropriate when it involves or is connected to the process of doing mathematics; that is, reasoning, communicating, connecting, and problem solving."

Over the years, students who enrolled in a course on diagnosis and intervention strategies have interviewed hundreds of students from the intermediate grades and up. Occasionally, a student is encountered whose responses are indicators of the presence of some well-defined procedural knowledge for division. However, rarely, if ever, did indicators of conceptual knowledge surface. It never fails to amaze us how little understanding of division secondary school students have and how little they remember.

Early this year, I addressed a group of teachers in Port McNeill, British Columbia. After one session, a lady who teaches secondary school students approached me. She did mention that her mathematics training was not received in North America, and she shared her dismay that students come to her classes confused about division. These students lack understanding and there was no doubt in her mind that it could all be blamed on the symbol that we use for long division. She may be right, but could it also have something to do with the language that is used while the symbol and procedure are presented?

The secondary students who lack understanding become adult consumers. Some enter the teaching profession. Perhaps that, in part, explains the results reported by Howe (1999) that not one of the American elementary teachers included in a comparison study with Chinese counterparts was able to come up with a meaningful word problem for a division equation with fractions.

What is true for division is true for other topics as well. My conclusion is that the majority of our students lack conceptual understanding. I have encountered people who will say, "Why is conceptual understanding important? I did not need it and I did all right!" (Comments like these make me think of statements uttered by a former premier and a minister of education, respectively: "Just give everyone a shovel and they will have an opportunity to become millionaires. It worked for me."; and "I came from a one-room school. Look at me. Why do we need more?" How can one argue with that kind of logic?).

If numerate persons are able to connect numerals and operations to life experiences and actions arising in real-life situations, then conceptual understanding is required and necessary for our students.

Possible Reasons for Lack of Conceptual Understanding

A profession more valuable to society than teaching does not exist. Teachers are the greatest resource society can have, but it is true that the majority of them are not majors of mathematics or in mathematics education. Since that is the case, the mathematics program in most classrooms is only as good as the main references available for the teacher and the students.

Several dilemmas exist because many classrooms in British Columbia, for example, do not have enough texts for each student. That is only part of the problem if the references that are available do not clearly specify for a teacher how to put emphasis on fostering the development of conceptual knowledge. As far as I am concerned, the references available for teachers lack the necessary specificity and detailed growth plans that are required for this endeavour. The reasons for that being the case are easy to explain. Let's assume that authors of these references take the learning outcomes they are writing for from the provincial guides (for example, Integrated Resource Package [IRP]) or from documents like the Western Canadian Protocol (WCP). The problem with this procedure is that the majority of the outcomes in these are much too general to be of value for preparing content with a focus on conceptual knowledge.

To illustrate the dilemma described in the last paragraph, consider an example from the IRP for Grades 2–3. The Prescribed Learning Outcome states, "It is expected that students will explore and demonstrate the process of division up to 50, using manipulatives, diagrams, and symbols." The WCP includes a very similar statement for Grade 3. There are two basic things teachers and authors take away from this statement, and they can be heard time and time again: manipulatives are important and the teaching sequence should go from concrete to abstract. The implicit assumption is that these two ideas suffice and will result in conceptual understanding. Do these generalizations contribute to or result in conceptual understanding of division? Based on all of the data I have collected and described earlier, they do not. Much more is needed if we are serious about emphasizing conceptual understanding. Teachers require specific information for guidance.

Over the years, quite a few teachers have told me that they design their own mathematics program for their students. Now, I do not want to take anything away from the experts who have the know-how to do this, but in the majority of cases, teachers do not have the time to develop growth plans for well-defined procedural as well as conceptual knowledge. My greatest fear is always that many students in these settings do not acquire conceptual understanding. The same feeling is true for another setting. From time to time reports in newspapers remind us that home schooling is on the increase in our province. How would most parents know how to focus on fostering conceptual knowledge and sense of number if they did not experience these types of settings and if the references they use are not of any assistance and without any professional training? I also would like to ask, "Is it possible for students to develop and acquire conceptual knowledge and sense of number if mathematics is presented in a language they are just beginning to learn?" My answer would be that this is unlikely. I have encountered quite a few adult students in my courses who did attribute their difficulties to having been in such settings.

One procedural weakness can be very detrimental to students, and it has to do with staffing. A few years ago, a mother presented me with a problem her daughter in secondary school was experiencing. After having done well in mathematics, not just enjoying it but understanding it as well, a teacher who had no mathematics background whatsoever was assigned to teach her class. The classroom setting became rote-procedural and rule-oriented. Things fell apart for this young lady and the desperate mother was searching for a possible solution. At the highest district level, she was told that that is how things work and nothing could be done about it. Does conceptual understanding have to be forfeited in this way, or is there a way out?

What Might Be Done to Have Students Acquire Conceptual Understanding?

First and foremost, Prescribed Specific Learning Outcomes are needed that clearly indicate to teachers what is expected or required for conceptual understanding. To illustrate this point, consider the sample Prescribed Learning Outcome from the IRP and the WCP quoted earlier. Teachers need to be informed of how objects are to be manipulated, that is, which type of division is to be used and why; the natural and mathematical language students are expected to use as manipulations are carried out (to avoid guzinta); and what the diagrams the students learn to draw are supposed to show and why. Would an effort to provide this type of information, take up a little more paper and be a little more work? If that is the case, so be it, but without this information the exploring and demonstrating that will be done by many students in different classrooms will not be the same and may even be done improperly. An emphasis on conceptual understanding requires that, whenever necessary and whenever possible, teachers are aware of the indicators of this type of understanding that they need to look for. Making these necessary changes can be done without spending a great amount of money. Publishers can then use these blueprints to provide the appropriate reference materials.

When learning outcomes and content are considered during meetings that involve IRP revisions, the questions, "Why this?" and "Why this at this grade level?" are sometimes posed. The most common response usually refers to the fact that it is in the (WCP) or some other important document. The assumption is made that, because it is in the WCP, it is appropriate and good. This type of rationale must be questioned. The somewhat circular procedure of making use of a cut-and-paste to create one type of document from another and then vice versa needs to be broken. Growth plans for different topics that consist of key specific learning outcomes for well-defined procedural and conceptual knowledge must be created, and that may have to be done without being based on something that exists. Perhaps somewhere along the way we may also need to inform those who are skeptical of the research evidence, which indicates that the necessary conceptual understanding can be developed without hurting skill development.

I have read in more than one reference that one of the greatest weaknesses of the American education system is that teachers seldom know what happens in previous grades or what will happen in subsequent grades. This type of outcome is reinforced in many settings that I have been a part of. More often than not people are placed in groups that deal with the grade level they are teaching. (I have heard some people announce that they are not interested in anything that is not related to the grades they teach.) It would be advantageous for those involved, as well as for continuity, if a group that is involved in revising or creating new curriculum materials would develop a complete growth plan for a topic rather than just for one grade or for a narrow range of grades. Once completed, computers should make it possible and easy for teachers to call up such complete growth plans for any topic they are about to teach. That would give them a complete picture of what conceptual understanding of a topic entails and what is learned along the way. It would be made clear to them why something is done at a given level and how it contributes to success in later grades.

Publishers and authors who prepare references for schools must clearly show teachers that what they have prepared includes growth plans for both well-defined procedural and conceptual knowledge. Also the practice activities they have devised must be appropriate; that is, that they meet the criteria suggested by Charles and Lobato (1998). These activities should tease students to think and to advance their thinking about what is being practised.

Authors of assessment instruments need to be made aware that both types of knowledge must be assessed. Once conceptual understanding becomes an important part of such instruments, there is no doubt that many teachers will strive to refocus the emphasis of their teaching of mathematics.

By the time students complete junior high school, they should have acquired conceptual understanding of division. What should these students be able to say and do? What skills, procedures and ideas should they have learned in the first six or seven grades?

The following are key components of conceptual understanding (Liedtke 1998). Following in parentheses are possible responses from students who lack this understanding. Readers are invited to present these types of tasks and then draw their own conclusions about understanding division.

- Show $6 \div 3$. Ask, "How would you read this?" (Students may say "goes into" or "into," and many will say that both "3 into 6" and "6 into 3" are acceptable. Students are not aware from their experience of the two types of action that can be matched with every numerical statement of this type. The two interpretations of division are not known.)
- Show $12 \div 3$. Ask, "Try to make up a word problem for this. (The students will focus on the answer and may work backward to make up an answer. They are likely to get confused about the divisor and the quotient. Neither of the two possible interpretations of division may be referred to.)
- Show a big handful or a jar filled with chips or beads. Ask, "What would you do to divide by three?" or show $(\ ) \div 3$ and ask what they could do to find the answer. (They may declare that it cannot be done unless they know how many chips or beads are in the hand or jar.)
- Show $6 \div 3$. Request, "Try to make a sketch to show the action. (The sketches may not illustrate the actions for either of the two interpretations of division. Confusion about the divisor and the quotient may surface.)
- Show one or two basic facts, that is, $56 \div 7$ and $72 \div 9$. Ask, "Pretend you have forgotten the

answers, what would or could you do to figure out what they are?" "What else could you do?" If the student knows the answers ask, "How could you check to find out that the answers are correct?" (A lack of strategies to reinvent forgotten answers or to get unstuck may become evident.) It never fails to amaze me that teachers have different definitions of basic facts, even those who work on tasks related to the mathematics curriculum. Mind you, very few references exist that offer these definitions. However, it seems logical to assume that those who teach the basic facts should know what they are.

- Show 3 ÷ 0 and 0 ÷ 5. Ask, "What are the answers and how do you know the answers are correct?" (Strategies that involve connecting may not be available to reinvent generalizations. These types of tasks may be classified as being the same. Division by zero may not be an issue.)
- Show 624 ÷ 4. Request a recording other than the short form that shows how to find the answer. Ask questions about the value of the partial dividends and products. (The partial dividends and products may be referred to in terms of ones, or tens and ones, rather than the actual value.)
- Show an item, one at a time, and ask for an explanation of estimation strategies to make predictions about the answer. Is the answer greater than one or less than one (or about one)? How do you know? What number is the answer close to? How do you know? For example, 37,642 ÷ 13, 6.85 ÷ 0.25, 0.35 ÷ 0.5, 1 ¾ ÷ ½, or S! ÷ ¾. (Students may not have estimation strategies at their disposal.)
- Show items like those from the last example. Ask, "Who would want to find the answers for these items? When? Why? Try to make up a meaningful word problem for each one. (Students may not know which interpretation of division would be best to use for making up meaningful word problems.)

The list of possible indicators of conceptual knowledge can be extended, but the goal is to identify some of the key components that need to be part of a growth plan for division if it is to be taught for understanding. At present, many of these ideas are not specifically identified and clearly stated in the key references that teachers use.

There exists one more important issue. After all of the many interview transcripts I have read, all of the interviews I have discussed with my students and with classroom- and special-education teachers and all of the interviews I have conducted, one thing has become very clear to me—number sense is the key foundation for conceptual understanding. Without it, the goal of developing conceptual understanding will not be reached, no matter who tries to undertake the task and how much they charge.

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Erratum

Natali Hritonenko

With respect to *delta-K*, Volume 40, Number 2, September 2003, page 49, problem 4, the correct answer is (a) and not (c), because the domain of the problem is: Squaring both sides we increase our domain. Therefore, 5 does not satisfy the initial problem.