Teaching the Mean Meaningfully

John C. Uccellini

Ask a group of middle school students what the average (mean) of two, eight, four, six, three and seven is, and they will probably give the answer five. Ask these same students what the number five represents in relationship to the six numbers given and the response usually heard is an explanation of the algorithm. "Add them up and divide by the number of them that you have." The response is no different if the problem is given in a real-life context. For example, the foregoing six numbers could represent the number of pencils that six students have in their desks. In either situation, the almost-universal response of students when questioned about the meaning of five from the "add and divide" algorithm demonstrates that students have not gained a conceptual understanding of this basic statistic. This same phenomenon exists throughout mathematics and is demonstrated whenever students try to explain subtraction by describing the vertical algorithm or the Pythagorean theorem by stating that $c^2 = a^2 + b^2$. Through the use of simple manipulative activities and graphing, however, middle school students can be taught the mean meaningfully.

Equal Distribution

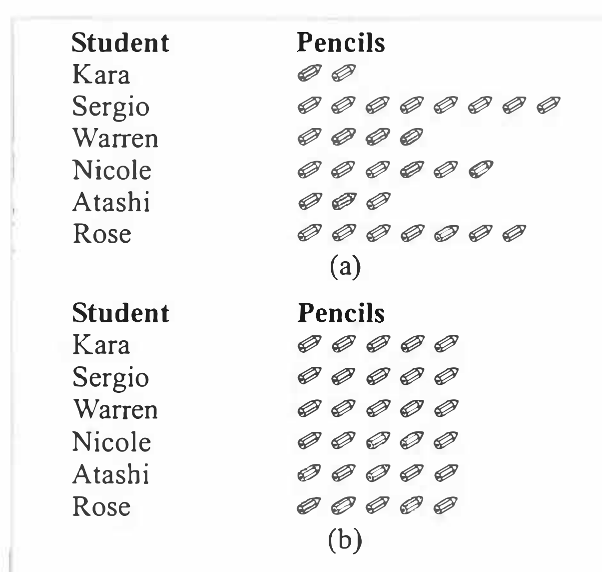
The mean can be interpreted in two different ways for a given set of data. The first interpretation relates the mean to an equal distribution of data items, whereas the second interpretation relates the mean to a balancing point of the data. Of the two interpretations, the concept of equal distribution is well known to both teachers and students but is rarely connected to the mean. In the context of the foregoing problem, the number five represents the equal distribution of the thirty pencils among the six students. In other words, the number five represents the number of pencils each student would have if the pencils were distributed equally to all six students. This sense of mean is easy for students to comprehend, since equal distribution of candy, crayons, baseball cards and so on is a common occurrence in their lives.

Before teaching the add-and-divide algorithm to students, their rich experiential understanding of equal distribution should be maximized. As in the pencil problem, students should be given thirty pencils, or some other appropriate manipulative, and asked to line them up on their desks in groups corresponding to the number of pencils the six students have in the problem (see Figure 1a). When they have arranged them correctly, the students can then move the pencils or markers from the larger groups to the smaller groups until an equal distribution has been made (see Figure 1b). At this point, the solution, five, should be identified as the mean. Begin by first introducing problems in which the mean is a natural number. Once students are comfortable with the concept of equal distribution, introduce problems in which the mean is not a natural number. Consider the following:

> Six children counted the number of chocolate bars that they won at the school fair. They had won two, three, two, six, three and five bars, respectively. What is the mean (average) number of chocolate bars that they won?

Figure 1

Pencils distributed among students. The original distribution of pencils is shown in (a) and an equal distribution of pencils is shown in (b).



By using two linking cubes to represent each chocolate bar, ask the students to solve for the mean by creating an equal distribution of the objects (see Figure 2). The solution, three and one-half chocolate bars per child, is easily found once the three remaining bars are broken in half and distributed among the six children.

By exploring sample problems involving objects, children gain experience in dividing the objects into halves, thirds, fourths and so on, which (1) builds on their prior experiences to further their understanding of the mean and (2) reinforces their use and understanding of fractions. It is important also to include examples in which the mean is not a natural number and the context of the problem concerns objects that cannot physically be broken apart. Consider the following problem:

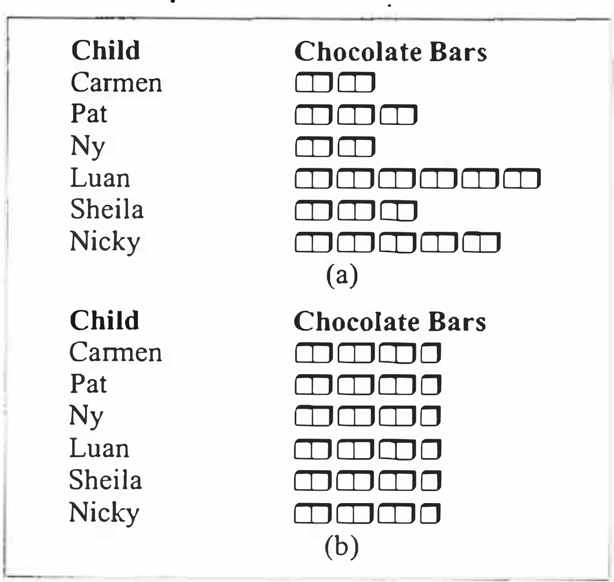
> Six children were asked how many brothers and sisters they had, and the following data were collected from the children: two, one, zero, three, six and one. What is the mean (average) number of brothers and sisters of these children?

When students attempt to solve this problem by manipulating objects representing the 13 siblings to form an equal distribution, they are faced with two dilemmas (see Figure 3): (1) how to divide the extra sibling among the six students, and (2) how to interpret the solution of the problem, which is two and one-sixth.

The students should be encouraged to discuss the difference between those examples in which the physical division of the objects is possible and feasible and those in which it is not. This discussion should also engage the students in attempting to interpret how two and one-sixth is an appropriate description of the mean number of siblings, even though it is not physically possible to achieve. Students should realize that the mean is a measure or a number that summarizes a set of data. Even in the pencil problem, wherein an equal distribution is possible, no student actually has five pencils; unless they mutually agree to share their pencils, each of them will not have the mean number of pencils.

Figure 2

Chocolate bars distributed among the children. The original distribution of bars is shown in (a), whereas (b) illustrates the equal distribution of the bars.



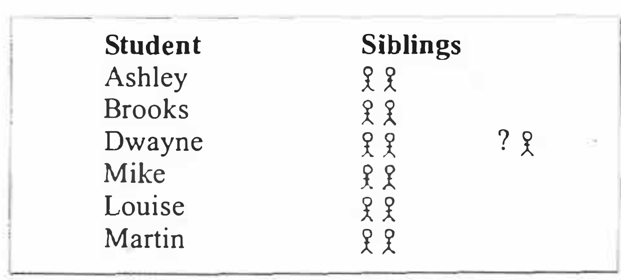
Once the class demonstrates a general understanding of the equal-distribution sense of the mean, whole-class data should be collected and the mean found. At this point, the need for the algorithm will become apparent to the students as they try to distribute equally large numbers of data items or to work with data that involve large numbers. For example, students could be asked to find the mean distance that the students live from school or the mean height of all the students in the class. If members of the class have not already come up with the add-and-divide algorithm, it should be introduced. The algorithm is faster and more practical to use, and, more important, the groundwork has been laid for the solution found by employing the algorithm to have real meaning and not just be a number.

The Balancing Point

The second interpretation of the mean, that of being the "balancing point" of a set of data, is not well known and is rarely taught to students. Consider the problem at the beginning of this article. The data can be placed on a number line with "X" representing each of the six numbers, as shown in Figure 4. Knowing that the mean for these data items is five, the respective distances between the other data items and five can be found and displayed below the number line as shown in Figure 4. First take the data items

Figure 3

The dilemma of equally distributing the 13th sibling



less than five and find their respective distances from five. Find the sum of these distances. Repeat this procedure for those data items greater than five. This equality in the sum of the distances of the data items above and below the mean leads to the interpretation of the mean as the balancing point of the data. When students understand this balancing-point interpretation of the mean, they begin to understand how the mean can be viewed as the centre of the data. The mean can next be discussed as being one of the measures of central tendency that statisticians use to describe sets of data along with the median and the mode.

Figure 4
Number line showing 5 as the balancing point of the data

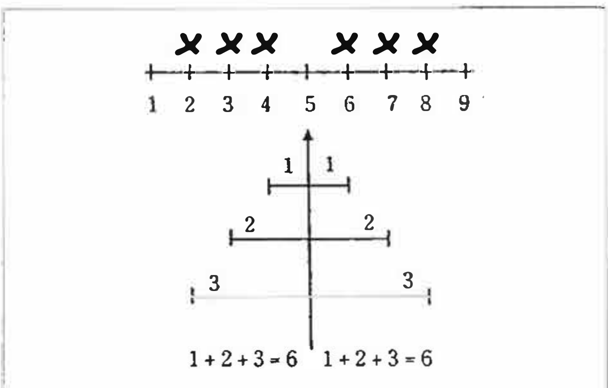
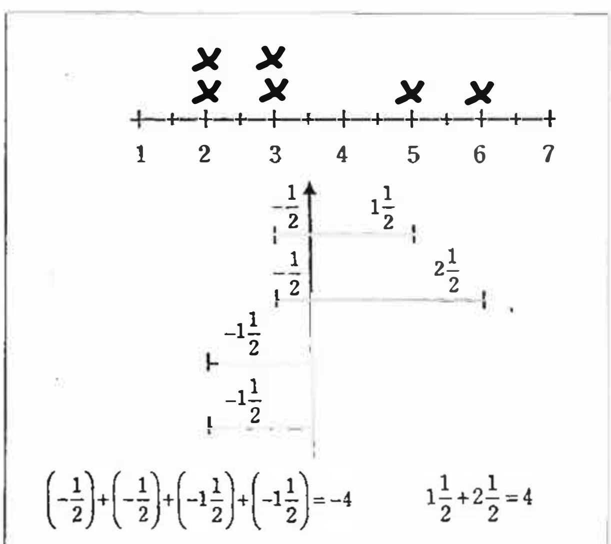


Figure 5
Number line showing that the sums of the differences from the mean are additive inverses, or opposites



Having students construct number lines to illustrate the balancing-point interpretation of the mean can also reinforce their understanding of positive and negative numbers and additive inverses, or opposites. Figure 5 shows the data from the chocolate-bar problem. Instead of calculating the respective distances of each data item from the mean, students calculate the difference between the value of each data item and that of the mean. In using the formula $D - M$, where D represents the value of each data item and M represents the value of the mean, data items larger that the mean yield positive differences and those less than the mean yield negative differences. The sum of the differences below the mean is the additive inverse, or opposite, of the sum of the differences above the mean. This situation always holds true and reinforces the concept of the mean as being a balancing point, or centre, of the data.

Teaching for understanding is difficult. In the instance of the mean, in which a simple algorithm is widely known, teachers frequently assume students' understanding of the mean in relation to a data set when students demonstrate a mastery of the algorithm. This article presents two conceptual interpretations of the mean and simple manipulative and graphing activities that can help students form a deeper understanding of this important statistic. In a world that overwhelms us with quantitative information, it is important that students be taught the mean meaningfully.

References

Friel, S. N., and R. B. Corwin. "Implementing the Standards: The Statistics Standards in K–8 Mathematics." *Arithmetic Teacher* 38 (October 1990): 35–39.

Friel, S. N, J. R. Mokros and S. J. Russell. *Used Numbers: Middles, Means, and In-Betweens.* Palo Alto, Calif.: Dale Seymour, 1992.

Grass, B. A. "Statistics Made Simple." *Arithmetic Teacher* 12 (March 1965): 196–98.

Landwehr, J. M, and A. E. Watkins. *Exploring Data.* Palo Alto, Calif.: Dale Seymour, 1987.

National Council of Teachers of Mathematics. *Curriculum and Evaluation Standards for School Mathematics.* Reston, Va.: Author, 1989.

Paull, S. "Not Just an Average Unit." *Arithmetic Teacher* 38 (December 1990):54–58

Zawojewski, J. S. "Research into Practice: Teaching Statistics: Mean, Median and Mode." *Arithmetic Teacher* 35 (March 1988): 25–26.

Reprinted with permission from Mathematics Teaching in the Middle School, *Volume 2, Number 2, November–December 1996, pages 112–15, an NCTM publication. Minor changes have been made to spelling and punctuation to fit ATA style.*