Edmonton Junior High Mathematics Contest 2001

Andy Liu

The Edmonton Junior High Mathematics Contest is designed to challenge the top 5 per cent of Grade 9 math students in Edmonton. The annual contest is run by a group of mathematics teachers and is written by Andy Liu. The main sponsors are the Association of Professional Engineers, Geologists and Geophysicists of Alberta (APEGGA), IBM, MCATA, Edmonton Public Schools and Edmonton Catholic Schools. After the contest, the top 50 students are recognized at a dinner banquet also attended by parents and teachers.

Part 1: Multiple Choice

1. The last digits of 49^{2001} are

   (a) 01 (b) 49 (c) 69 (d) 81

   Answer: (b)

2. The sum $7^7 + 7^7 + 7^7 + 7^7 + 7^7 + 7^7 + 7^7$ is equal to

   (a) 8^7 (b) 7^8 (c) 49^7 (d) 7^{49}

   Answer: (b)

3. For any numbers x and y, define $x \oplus y = x + y + xy - 1$ and $x \otimes y = a^2 + b^2 - ab$.
   The value of $3 \oplus (2 \otimes 4)$ is

   (a) 36 (b) 42 (c) 48 (d) 50

   Answer: (d)

4. The price is first increased by r% and then reduced by r%. If the final price is divided by the original price, the quotient is

   (a) 1 (b) $1 - \frac{r}{10000}$ (c) $1 + \frac{r2}{10000}$ (d) $1 - \frac{r2}{10000}$

   Answer: (d)

5. There is enough cabbage to last the goat x days and the rabbit y days. The number of days the cabbage will last both the goat and the rabbit is

   (a) $\frac{1}{x+y}$ (b) $\frac{1}{x} + \frac{1}{y}$ (c) $\frac{xy}{x+y}$ (d) $\frac{1}{xy}$

   Answer: (c)

6. When Ace was as old as Bea is now, Bea was 10 years old. When Bea is as old as Ace is now, Ace will be 25 years old. Ace is older than Bea by

   (a) 5 years (b) 10 years (c) 15 years (d) none of these

   Answer: (a)

7. The length of each side of a triangle is a positive integer and the sum of the these three integers is odd. If the difference between two of them is 5, the smallest possible value of the third is

   (a) 4 (b) 6 (c) 7 (d) 8

   Answer: (b)

8. The sum of the angles of a polygon is less than 2001°. The largest possible number of sides of this polygon is

   (a) 11 (b) 12 (c) 13 (d) 14

   Answer: (c)

9. In triangle ABC, $AB = AC$. E is the point on AC such that BE is perpendicular to AC. F is the midpoint of AB. If $BE = EF$, then the measure of $\angle C$ is

   (a) 65° (b) 70° (c) 75° (d) 80°

   Answer: (c)

10. If the number x satisfies $\frac{2}{x} - |x| = 1$, what is the value of $\frac{2}{x} + |x|$.
    Hint: $|2| = 2$, and $|-2| = 2$.

    (a) –3 (b) –1 (c) 1 (d) 3

    Answer: (d)

Part 2: Numeric Response

1. The value of $\dfrac{-2 \div (-2)^{-2} + (-2)^3}{\left(-\frac{1}{2}\right)^{-1} + \frac{1}{2}}$ in fractional form.

   Answer: 10 $^2/_3$ or $^{32}/_3$

2. In a student union election, 1,500 votes are cast. Of the first 1,000, Ace receives 350, Bea 370 and Cec 280. Of the remaining 500 votes, at least how many must Ace receive in order for him to have more votes than either Bea or Cec?
Answer: 261

3. The ratio of Peter's and Steven's running speed is 5:3. They start from the same point on a circular track at the same time. After some time, they meet again at the starting point, and Peter has run 4 more laps than Steven. How many laps has Steven run?
Answer: 6

4. What is the smallest positive integer n such that $n + 13$ is a multiple of 5 and $n - 13$ is a positive multiple of 6?
Answer: 37

5. The positive integers w, x, y and z are such that $\frac{w}{x} = \frac{x}{y} = \frac{y}{z} = \frac{5}{8}$. What is the smallest possible value of $w + x + y + z$?
Answer: 1,157

6. D is a point on the side BC of triangle ABC. If $AC = 5$, $AD = 6$, $BD = 10$ and $CD = 5$, what is the area of triangle ABC?
Answer: 36

7. What is the unit digit of $1^3 - 2^3 + 3^3 - 4^3 + \ldots + 1{,}999^3 - 2{,}000^3 + 2{,}001^3$?
Answer: 1

8. How many four-digit multiples of 11 are there in which each of the digits 1, 2, 3 and 4 appears?
Answer: 8

9. Three thousand three hundred and seventy-five 1-cm cubes are used to form a larger cube. The outer surfaces of the newly assembled cube are painted. After drying the paint, the cubes are knocked apart. Find the total surface area of the unpainted surfaces on the 3,375 cubes.
Answer: 18,900

10. $ABCD$ is a rectangle of area 24. E, F and G are points on AB, BC and CD, respectively such that $BE = 3AE$, $CF = 2BF$ and $DG = CG$. What is the area of triangle EFG?
Answer: 8

Christian Goldbach (1690–1764)

This mathematician was born in Prussia, lived in various Western European countries and settled in Russia. In the Goldbach conjecture, it is claimed that every even number (except 2) is equal to the sum of two prime numbers. This conjecture has been verified up to 2×10^{10}. Check this conjecture yourself for numbers up to 50.

(In 1742 Goldbach wrote about this conjecture to the Swiss mathematician Leonard Euler. This conjecture remains unproved until today.)

Edmonton Junior High Mathematics Invitational 2001

Andy Liu

The Edmonton Junior High Mathematics Invitational is a follow-up exam to the Edmonton Junior High Mathematics Contest. The top 50 Grade 9 students on the Edmonton Junior High Mathematics Contest are invited back to participate on the Edmonton Junior High Mathematics Invitational.

The annual invitational exam is run by a group of mathematics teachers and is written be Andy Liu. The main sponsors are the Association of Professional Engineers, Geologists and Geophysicists of Alberta (APEGGA), IBM, MCATA, Edmonton Public Schools and Edmonton Catholic Schools.

2001 Solution

In what follows, we give a detailed discussion on the approach to each problem. Thus the write-up is much longer than a formal solution. We hope that reading through this additional material will help you improve your problem-solving skill.

1. Find 100 different positive integers such that the product of any five of them is divisible by the sum of these five numbers.

Solution

Some problems look hard because the numbers involved are large. As a preliminary analysis, cut them down to size. For instance, we may replace 100 with a smaller number. The smallest number which makes sense is 5. If we simply take 1, 2, 3, 4 and 5, $1 \times 2 \times 3 \times 4 \times 5 = 120$ is indeed divisible by $1 + 2 + 3 + 4 + 5 = 15$. If we want 6 numbers, will 1, 2, 3, 4, 5 and 6 do? Let us see.

(a) If we leave out 1, $2 \times 3 \times 4 \times 5 \times 6 = 720$ is divisible by $2 + 3 + 4 + 5 + 6 = 20$.

(b) If we leave out 2, then $1 \times 3 \times 4 \times 5 \times 6 = 360$ is not divisible by $1 + 3 + 4 + 5 + 6 = 19$.

(c) If we leave out 3, then $1 \times 2 \times 4 \times 5 \times 6 = 240$ is not divisible by $1 + 2 + 4 + 5 + 6 = 18$.

(d) If we leave out 4, then $1 \times 2 \times 3 \times 5 \times 6 = 180$ is not divisible by $1 + 2 + 3 + 5 + 6 = 17$.

(e) If we leave out 5, $1 \times 2 \times 3 \times 4 \times 6 = 144$ is divisible by $1 + 2 + 3 + 4 + 6 = 16$.

(f) If we leave out 6, $1 \times 2 \times 3 \times 4 \times 5 = 120$ is divisible by $1 + 2 + 3 + 4 + 5 = 15$ as we have already observed.

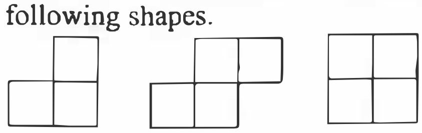
Let us try to fix the problem in case (c). Here we need an extra factor of 3. Had we started with 3, 6, 9, 12, 15 and 18 instead, then $3 \times 6 \times 12 \times 15 \times 18 = 240 \times 3^5$ will be divisible by $3 + 6 + 12 + 15 + 18 = 18 \times 3$. However, neither case (b) nor case (d) is fixed this way. We would start instead with 19, 38, 57, 76, 95 and 114 in the former, and 17, 34, 51, 68, 85 and 102 in the latter.

Nevertheless, this analysis suggests that we should start with M, $2M$, $3M$, $4M$, $5M$ and $6M$ for a suitably chosen M. Certainly, $M = 3 \times 17 \times 19 = 969$ will work here. Hence 969, 1,938, 2,907, 3,876, 4,845 and 5,814 solve the problem with 100 replaced by 6.

We are now ready to tackle the original problem. Our numbers are M, $2M$, $3M$, ..., $100M$. All we have to do is to choose a suitable M. There are too many cases for us to repeat the earlier analysis. Fortunately, it is not necessary to do so. Let aM, bM, cM, dM and eM be any five of the numbers. Then their product is $P = abcdeM^5$ while their sum is $S = (a+b+c+d+e)M$. Since $15 = 1+2+3+4+5 \leq a+b+c+d+e \leq 96+97+98+99+100 = 490$, we can choose $M = 490 \times 489 \times 488 \times \cdots \times 17 \times 16 \times 15$. Then M is a multiple of $a+b+c+d+e$, so that P is indeed divisible by S.

The value of M is larger than is necessary, but we do not have to pay for it. It saves us a lot of work. Going back to the earlier example with 6 numbers, we could have avoided the case analysis by taking $M = 20 \times 19 \times 18 \times 17 \times 16 \times 15$. However, that much work was valuable because it points us in the right direction.

2. You have a large number of pieces of each of the following shapes.

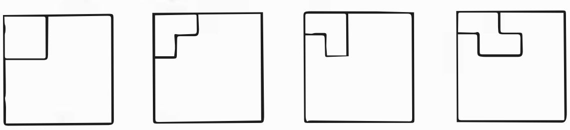


Each square of a piece covers a square of the chessboard. No overlapping within or protrusions beyond are allowed. You do not have to use all three shapes. Is it possible to cover every square of 5 × 5 chessboard using any combination of these pieces? Either give such a covering or give a short proof why no such coverings exist.

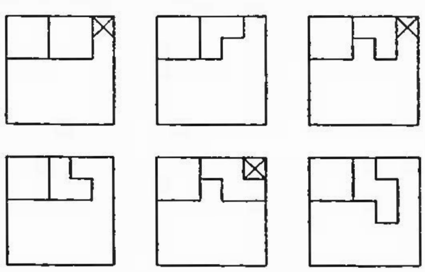
Solution

The 5 × 5 board is not that big, and it does not take long to convince ourselves that the desired covering does not exist. However, convincing other people that we know what we are talking about takes more doing. It is not enough to just show a few failed attempts. Perhaps we have overlooked a solution.

One approach, favoured by computing scientists, is known as backtracking. It is an exhaustive analysis of all cases by brute force. We number the square from 1 to 25 row by row from left to right and top to bottom. At each point, we examine all possible ways of covering the uncovered square with the lowest number. Thus at the start, we attempt to cover square #1. There are six cases, but the two not shown are equivalent to the third and fourth ones below.



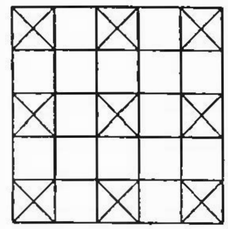
From the first case, six subcases are generated according to how square #3 is to be covered. Three of these lead immediately to impossible situations, but further analysis are needed for the remaining subcases. Thus this is not an attractive approach for mortal souls without the backing of immense computational power.



So what can we do? Let us try an algebraic approach. Suppose we use x, y and z copies of the three shapes, respectively. Then $3x + 4(y + z) = 25$. We can have $x = 7$ and $y + z = 1$, or $x = 3$ and $y + z = 4$. So the total number of pieces used in any successful covering is 8 or 7. This gives us more information, but the road ahead is still unclear.

Perhaps we should consider a smaller board, say the 3 × 3. Now $3x + 4(y + z) = 9$ and $x = 3$, $y = 0$ and $z = 0$ is the only solution. It is easy enough to use backtracking here to show that 3 pieces of the first shape cannot cover the board, but we need an argument which can be extended to the 5 × 5 board.

Backtracking has focused our attention to the corner squares of the board. The 3 × 3 board has four, and the first shape can cover at most one of them. Hence we need at least 4 pieces, and there is simply not enough room for them.



The nine squares marked × in the 5 × 5 board above place the roles of the comer squares in the 3 × 3 board. Each of the three shapes can cover at most one of them. Hence at least 9 pieces are needed. This is a contradiction since we have already determined that at most 8 pieces are to be used. Hence the desired covering cannot exist.

3. Let a, b, c and d be any numbers. Prove that $(1 + ab)^2 + (1 + cd)^2 + (ab)^2 + (cd)^2 \geq 1$.

Solution

First, we can simplify the problem by replacing ab with x and cd with y. We want to prove that $(1 + x)^2 + (1 + y)^2 + x^2 + y^2 \geq 1$ for any numbers x and y. Note that x or y may be negative, as otherwise the expression is certainly at least 2.

Since the expression contains a lot of squares, it may be a good idea to examine squares more closely. The square of 0 is 0. The square of a positive number is positive. The square of a negative number is also positive. Hence a square is never negative.

Expanding the squares, we obtain $1 + 2x + x^2 + 1 + 2y + y^2 + x^2 + y^2$. If we set aside a 1 and can combine $1 + 2x + 2x^2 + 2y + 2y^2$ into squares, we will have the desired result. By separating x from y and splitting the 1 between them, we have $\frac{1}{2} + 2x + 2x^2 = \frac{1}{2}(1 + 4x + 4x^2) = \frac{1}{2}(1 + 2x)^2$. An analogous expression exists for y. It follows that

$$(1 + x)^2 + (1 + y)^2 + x^2 + y^2 =$$
$$1 + \tfrac{1}{2}(1 + 2x)^2 + \tfrac{1}{2}(1 + 2y)^2 \geq 1.$$

Another way is to observe that $(1 + x + y)^2 = 1 + 2x + x^2 + 2y + y^2 + 2xy$, so that

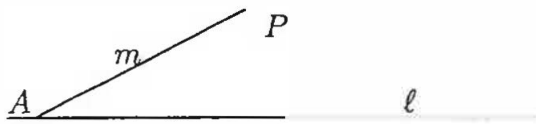
$$(1 + x)^2 + (1 + y)^2 + x^2 + y^2 =$$
$$1 + (1 + x + y)^2 + x^2 + y^2 - 2xy$$
$$= 1 + (1 + x + y)^2 + (x - y)^2$$
$$\geq 1.$$

Note that equality holds if and only if each of $1 + 2x$, $1 + 2y$, $1 + x + y$ and $x - y$ is 0. In other words, the minimum value of 1 of the given expression is attained if and only if $x = y = -1$(over)2.

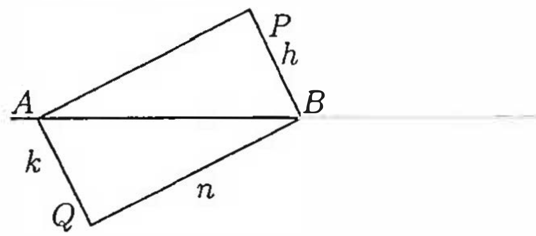
4. You have an instrument that allows you to draw the line segment connecting two given points, and to draw through a point on the line perpendicular to the given line. Given a point P not on a given line l, describe a construction using your instrument only the line through P parallel to l. You do not have to justify your construction.

Solution

At the start, we have a line l and a point P not on it. Thus there is no immediate way of applying either of the allowable operations. So we should choose a second point A somewhere, and it makes sense for this point to lie on l. Now we can join A to P, or draw a line through A perpendicular to l. The latter does not lead us anywhere, so we connect A and P by a line m.



What can we do now? Still rejecting the drawing of a perpendicular to l through A, we can instead draw perpendiculars h and k to m at P and A respectively. Now h will intersect l at a point B. Drawing a perpendicular n to h at B, it will intersect k at a point Q.



Note that $PAQB$ is a rectangle with A and B on l. In particular, P and Q are equidistant from l. Repeating the above process with Q playing the role of P, we can construct a rectangle $QCRD$ with C and D on l. Then Q and R are equidistant from l, so that PR is the desired line parallel to l.