Fun with Mathematics—Challenging the Reader

Andy Liu

Each issue of delta-K *will contain problem sets, which will also be posted on the MCATA website (www.mathteachers.ab.ca).*

The spring issue will contain a set of problems for January, March and May, and the fall issue will contain a set of problems for September and November. Teachers and students are invited to participate by submitting the full solution to each problem by the deadline stated on the website.

Note that the solutions to the problem sets will be published in delta-K *only, with fall issue containing the solutions for the January, March and May problems and the spring issue containing the solutions for the September and November problems.*

Submit your full solutions to Andy Liu, Department of Mathematics, University of Alberta, Edmonton, Alberta, T6G 2G1; fax (780) 492-6826, e-mail aliumath@telus.net.

The solutions to the January, March and May problems are shown below, followed by the September and November problem sets.

Solutions to January 2003 Problems

Problem 1

The numbers 1, 2, . . ., 16 are placed in the cells of a 4 × 4 table as shown below:

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

One may add 1 to all numbers of any row or subtract 1 from all numbers of any column. How can one obtain, using these operations, the table shown below?

1	5	9	13
2	6	10	14
3	7	11	15
4	8	12	16

Solution

Denote the number of addition operations applied to the rows by a_1, a_2, a_3, a_4 and the number of subtraction operations applied to the columns by b_1, b_2, b_3, b_4. Comparing the initial and required tables, we see the necessity of the following relations: $a_1 = b_1$, $a_2 = b_2$, $a_3 = b_3$, $a_4 = b_4$, $a_1 - b_2 = 3$, $a_1 - b_3 = 6$, $a_1 - b_4 = 9$. Thus, letting a_4 be an arbitrary non-negative integer, we solve the problem; the order of performing the operations does not matter. One of the solutions is $a_1 = 9, a_2 = 6, a_3 = 3, a_4 = 0, b_1 = 9, b_2 = 6, b_3 = 3, b_4 = 0$.

Problem 2

There are four kinds of bills: $1, $10, $100 and $1,000. Can one have exactly half a million bills worth exactly $1 million?

Solution

Assume there exists a set of notes as described in the problem. Let a, b, c and d be the numbers of notes in the set whose values are $1, $10, $100 and $1,000, respectively. Then we have two equations:

$$a + b + c + d = 500{,}000$$
$$a + 10b + 100c + 1{,}000d = 1{,}000{,}000$$

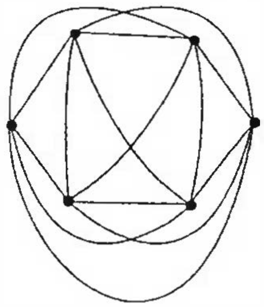
Subtracting the first equation from the second, we get $9b + 99c + 999d = 500{,}000$, which is impossible because 500,000 is not divisible by 9 whereas the left side is clearly a multiple of 9.

Problem 3

The king intends to build six fortresses in his realm and to connect each pair with a road. Draw a diagram of the fortresses and roads so that there are exactly three intersections and exactly two roads crossing at each intersection.

Solution

The appropriate diagram is shown



Problem 4

If each boy purchases a pencil and each girl purchases a pen, they will spend a total of 1¢ more than if each boy purchases a pen and each girl purchases a pencil. There are more boys than girls. What is the difference between the number of boys and the number of girls?

Solution

Denote the number of boys and the number of girls by B and G, respectively, and the prices of a pencil and a pen by x and y, respectively. Then we have the equation $Bx + Gy = By + Gx + 1$, that is, $(B - G)(x - y) = 1$. But the product of two integers can be equal to 1 only if both are equal to either 1 or -1. Since we know that the difference of $B - G$ is positive, we conclude that it is equal to 1.

Problem 5

A six-digit number from 000000 to 999999 is called lucky if the sum of its first three digits is equal to the sum of its last three digits. How many consecutive numbers must we have to be sure of including a lucky number if the first number is chosen at random?

Solution

The answer is 1,001. Note that if the first ticket we buy happens to be 000001, then the first lucky ticket we can get is 001001; that is, there are cases when to purchase fewer than 1,001 tickets is not sufficient.

Now we have to show that 1,001 is a sufficient number of tickets for our aim. Write the six-digit number of the first bought ticket as AB, where A represents the number formed by the first three digits and B the number formed by the last three. If $A \geq B$, we can buy $A - B \leq 1{,}000$ tickets and obtain lucky ticket AA. If $A < B$, the purchase of $1{,}001 - B$ tickets leads us to ticket $A'B'$, with $A' = A + 1$ and $B' = 0$. Then we buy an additional $A + 1$ tickets and obtain the lucky ticket. So we have reached our aim this time having bought $1{,}002 - (B - A)$ tickets, and since $B - A \geq 1$, we conclude that 1,001 is indeed a sufficient number of tickets to buy.

Problem 6

Two players play the following game on a 9 × 9 chessboard. They write in succession one of two signs in any empty cell of the board: the player making the first move writes a plus sign (+) and the other player writes a minus sign (–). When all the squares of the board are filled, the scores of the players are tabulated. The number of rows and columns containing more plus signs than minus signs is the score of the first player, and the number of all other rows and columns is the score of the second player. What is the highest number of points the first player can gain in a perfectly played game?

Solution

It is easy to devise a strategy providing 10 points for the first player. She has to make her first move in the central square of the board and then write a plus sign each time in the square symmetric (with respect to the centre of the board) to the square the second player has filled in at the previous move. This strategy guarantees that the central row and column bring two points to the first player. Further, all other rows can be split into pairs of symmetric rows, and we see that each pair shares the points among both players equally. The same is true for the columns, and thus the first player gets exactly 10 points.

Now we have to prove that the second player is able to play so that he earns at least eight points (the total number of the rows and columns on the board is 18). The main idea is that the second player also can achieve the symmetric filling of the board, which, as we have seen, leaves him with the desired eight points. If the first player follows the preceding strategy, the actions of the second player are of no importance, but if the first player makes a nonsymmetric move, her opponent should begin to support symmetry. If, at the very beginning, the first player makes her first move in a square, other than the central square, the second player can still support the necessary symmetry, and since the last move is made by the first player, she will be compelled to complete the symmetric filling of the board. Thus, we have proved that the answer is 10.

Solutions to March 2003 Problems

Problem 1

Initially, there is a 0 in each cell of a 3 × 3 table. One may choose any 2 × 2 subtable and add 1 to all numbers in it. Can one obtain, using this operation a number of times, the table shown below?

4	9	5
10	18	12
6	13	7

Solution

Each 2 × 2 subtable contains the central box and exactly one of the comer boxes. Thus, the number in the central box must be equal to the sum of the numbers in the corners. But this relation does not hold true for the pictured table, and thus the solution follows.

Problem 2

A teacher plays a game with 30 students. Each writes the numbers 1, 2, . . ., 30 in any order. Then the teacher compares the sequences. A student earns a point each time the same number appears in the same place in the sequences of that student and of the teacher. It turns out that each student earns a different number of points. Prove that at least one student's sequence is the same as the teacher's.

Solution

Each player could gain from 0 to 30 aces. However, he cannot earn exactly 29 aces. Consequently, using the pigeonhole principle, we conclude that one of the players gained exactly 30 aces, and so his sequence coincided with that of the leader.

Problem 3

Is it possible to write the numbers 1, 2, . . ., 100 in a row so that the difference between any two adjacent numbers is not less than 50?

Solution

Yes. One can write the following sequence: 51, 1, 52, 2, 53, 3, . . . , 49, 100, 50.

Problem 4

Do there exist two nonzero integers such that one is divisible by their sum and the other is divisible by their difference?

Solution

No, such numbers A and B do not exist. If A and B are nonzero integers, then either $A + B$ or $A - B$ has an absolute value greater than the absolute values of A and B. This can be checked from the fact that either sign (A) = sign (B), or sign (A) = – sign (B). It is now sufficient to recall that, if a nonzero integer X is divisible by Y, then $|X| \geq |Y|$.

Problem 5

A game starts with a pile of 1,001 stones. In each move, choose any pile containing at least two stones and remove one of them, and then split any pile containing at least two stones into two nonempty piles, which need not be of equal size. Is it possible for all remaining piles to have exactly three stones after a sequence of moves?

Solution

The answer is no. The basic idea of the solution is the very popular idea (in mathematics and science) of invariance. Define the quantity S to be the sum of the number of piles, and the number of stones. Under the conditions described in the problem, S is invariant. The initial value of S is 1,002, and if it is possible to obtain n piles each containing exactly three stones, then S should be equal to n (the number of piles) + $3n$ (the number of stones) = $4n$, which gives a contradiction, since 1,002 is not divisible by 4.

Problem 6

A square castle is divided into 64 rooms in an 8 × 8 configuration. Each room has a door on each wall and a white floor. Each day, a painter walks through the castle, repainting the floors of all the rooms he passes, so that white is changed to black and vice versa. Can he do this so that, after several days, the floors in the castle will be coloured like a chessboard?

Solution

Yes, the painter can do it. Assume he walks from any room to another room A and then returns the same way; then the color of A changes while all other rooms in the castle retain their colors. Surely, using these operations, the painter can arrange the chess pattern on the castle floor.

Solutions to May 2003 Problems

Problem 1

A jury makes up problems for an Olympiad, with a paper for each of Grades 7–12. The jury decides that each paper should consist of seven problems, with exactly four of them not appearing on any other paper of the Olympiad. What is the greatest number of distinct problems that could be included in the Olympiad?

Solution

The answer is 33 problems. There are exactly 24 problems such that each of them is included in exactly one list, and since each of k other problems belongs to more than one list, the total number of problems for all grades is no less than $24 + 2k$. But this amount is equal to $6 \times 7 = 42$, which implies that $k \leq 9$ On the other hand, the jury is clearly able to make up the lists containing 33 distinct problems—it is sufficient to include three common problems in the lists of the fifth and sixth grades, the seventh and eighth grades, and the ninth and tenth grades, respectively.

Problem 2

A six-digit number (from 000000 to 999999) is called lucky if the sum of its first three digits is equal to the sum of its last three digits. Prove that the number of lucky numbers is equal to the number of six-digit numbers with a digit sum of 27.

Solution

Clearly, if we change the first three digits a, b, c in a lucky number to $9 - a$, $9 - b$, $9 - c$, we obtain a number whose digits' sum is equal to 27, and vice versa. (For instance, 273390 is a lucky ticket and $(9-2)+(9-7)+(9-3)+3+9+0=27$). This one-to-one correspondence between tickets of the considered types implies that the number of tickets in one group is equal to the number of tickets in the other.

Problem 3

Given 32 stones of distinct weights, prove that 35 weightings on a balance are sufficient to determine which are the heaviest and the second heaviest.

Solution

First, dividing 32 stones into pairs and using 16 weighings, we extract a set of 16 stones that contains the heaviest stone. Further, performing an analogous operation for the extracted set, we reduce the number of candidates to eight stones and so on. So, using $16 + 8 + 4 + 2 + 1$ weighings in five stages, we can determine the heaviest stone. Now we should notice that the second heaviest stone is necessarily one of those five that were in the same pair with the heaviest one. Thus, to complete the solution, we must find the heaviest of five stones with four weighings. We leave this very simple exercise for the reader.

Problem 4

Find two six-digit numbers such that the number obtained by writing them one after another is divisible by their product.

Solution

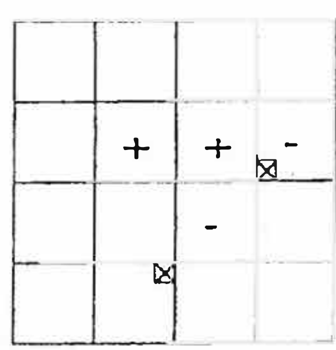
The answer is 166667 and 333334. Check that $166667333334 = 3 \times 166667 \times 333334$. By the way, it is the only answer.

Problem 5

Two players play a game of wild tic-tac-toe on a 10×10 board. They take turns putting either an X or an O in any empty cell on the board. Both players can use X or O, and not necessarily consistently. A player wins the game by making three identical symbols appear in consecutive cells horizontally, vertically or diagonally. Can either player have a winning strategy? If so, is it the player who moves first or the one who moves second?

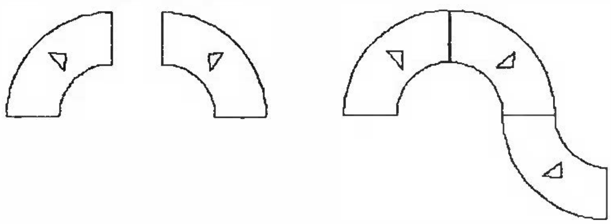
Solution

Let us call the player who makes the first move in the game "First" and his opponent "Second." We show that Second wins in an errorless game. She has to use the following strategy. If her current move may be a winning one (that is, she can complete a chain of three identical signs) she undoubtedly should do it. Otherwise, she must "reverse" the last move of her opponent; that is, place an opposite sign in the square symmetric to that occupied by the previous move with respect to the centre of the board. The described strategy ensures that First can never win this game (check it), and we have only to show that eventually Second will gain a victory. Consider the central 4×4 fragment of the board just after the second move of First in the central 2×2 square—it looks like the one in the figure below (modulo the reversion of all the signs). If the square with the shaded bottom corner were occupied by a plus, then First would be able to win, but we know that this is impossible. On the other hand, if this square is empty, Second wins by writing a plus in it. So the unique interesting case is when the square with the shaded bottom corner contains a minus. In this case, if the square with the shaded top corner is empty or contains a plus, then Second can terminate the game by writing minus or plus into the relevant square. All that is left is to notice that the square with the shaded top corner cannot contain a minus because otherwise the game would already be finished. These arguments complete the solution.



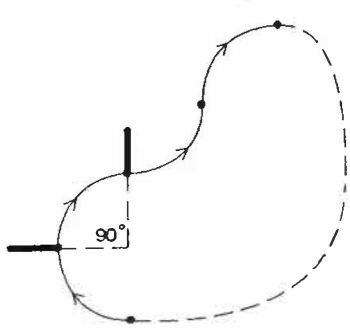
Problem 6

Each section of tracks in a model railway is a quadrant of a circle directed either clockwise or counterclockwise, as shown below in the diagram on the left. One may only assemble the track in such a way that the directions of the sections are consistent along the whole track, as illustrated below in the diagram on the right. If such a closed track can be assembled using given sections, prove that this is no longer the case if one clockwise section is replaced by a counterclockwise section.



Solution

Note that if we have assembled a legitimate closed track using k sections of type 1 and m of type 2, then $k - m$ must be divisible by 4. This can be clarified using the following reasoning. Choose the beginning of some section and travel along the track in the direction indicated by the section arrows until we return to the chosen point. Imagine that while we move along the track, we drag a small arrow that is perpendicular to the track (more strictly, perpendicular to the tangent line of the track) and that points to the outside of the track (see figure). The small arrow turns by an angle of 90 degrees in a clockwise direction when we move along a section of type 1 and in a counterclockwise direction when we move along a section of type 2. Therefore, it makes k turns in a clockwise direction and m turns in a counterclockwise direction. Assume (without loss of generality) that $k \geq m$. Since the initial and final positions of the arrow coincide, we conclude that $k - m$ turns in a clockwise direction must mean a complete turn by an angle whose measure is a multiple of 360 degrees. But this means that $k - m$ is divisible by 4. We are done now, because we see at once that $(k - 1) - (m + 1) = (k - m) - 2$ cannot be a multiple of 4.



September 2003 Problems

1. Paula bought a notebook with 96 sheets and numbered its pages in sequence from 1 to 192. Paula pulled out 25 sheets at random and added together all 50 numbers written on them. Prove that this sum cannot be equal to 1,990.
2. Exactly one of 101 coins is counterfeit. The 100 genuine coins have the same weight, but a different weight from that of the counterfeit coin. It is not known whether the counterfeit coin is heavier or lighter than a genuine coin. How can this question be resolved by two weighings on a balance scale? It is not necessary to identify the counterfeit coin.
3. Is it possible to dissect a 39×55 rectangle into 5×11 rectangles?
4. A two-player game starts with the number 1,234. Tom moves first, and he and Jerry make alternate moves thereafter. In each move, the player subtracts from the current number one of its nonzero digits. The player who obtains zero wins. Who should win in a perfectly played game?
5. Three students together solved 100 problems from the textbook. Each of them solved exactly 60 problems individually. We call a problem difficult if it is solved by only one of them, and easy if it was solved by all of them. Prove that the number of difficult problems exceeds the number of easy problems by 20.
6. For every boy in a certain village, all his female acquaintances know one another. Among the acquaintances of any girl, the number of boys is greater than the number of girls. Prove that, in this village, the number of girls is not greater than the number of boys.

November 2003 Problems

1. Each of 40 students at a technical institute has several nails, screws and bolts. Exactly 15 of them have unequal numbers of nails and bolts, and exactly 10 of them have equal numbers of screws and nails. Prove that at least 15 students have unequal numbers of screws and bolts.
2. In the stock exchange of Funny City, one can exchange any two shares for three others and vice versa. Can John exchange 100 shares of Fun Oil for 100 shares of Auto Fun, giving exactly 1,991 shares in the process?
3. Cars A, B, C and D start simultaneously from the same point on a circular racetrack. The first two cars move in a clockwise direction while the other

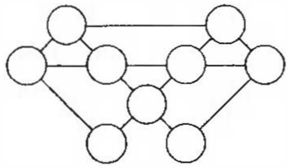
two move in a counterclockwise direction. Each moves at a constant speed, and A is faster than B. If A meets C for the first time at the same moment as B meets D for the first time, prove that A catches up with B for the first time at the same moment as C catches up with D for the first time.

4. Beginning on August 1, 1991, Baron Munchhausen tells his cook, "Today I will bring home more ducks than two days ago but fewer than one week ago." What is the greatest number of days that the baron can repeat this and not be caught in a lie?
5. A red stick, a white stick and a blue stick have the same length. Julie breaks the red stick into three parts; then Ben does the same with the white stick; and then Julie breaks the blue stick into three parts. Can Julie break the sticks so that, no matter what Ben does, it would be possible to assemble from the nine parts three triangles such that each has sides of different colours?
6. Nine teams took part in a tournament in which every two teams played against each other exactly once. Does there necessarily exist two teams such that every other team has lost to at least one of them in the tournament?

Albert Einstein (1879–1955)

This German-American theoretical physicist and philosopher, despite his fame, continued to pose mathematical problems in *Frankfurter Zeitung*.

In the diagram below, the nine spheres represent vertices of four small and three larger equilateral triangles. Place the numbers from 1 to 9 in the spheres in such a way so that the sum in each of the seven triangles is always the same.



Corner triangles are equilateral