On Vectorial Proofs, Circumradii and Equilateral Triangles

David E. Dobbs

Introduction

In a recent note, Norman Schaumberger (2002) has given a new proof of the following fact. Let A, B and C be noncollinear points (in, say, the Euclidean plane or three-space) and let G be the centroid of the triangle $\Delta = \Delta ABC$; then, as P varies over all points, $|PB|^2 + |PB|^2 + |PC|^2$, the sum of the squares of the distances from P to the vertices of Δ is as small as possible if and only if $P = G$. The preceding assertion is an immediate consequence of what Schaumberger actually proves, namely, that for any point P, $|PA|^2 + |PB|^2 + |PC|^2 = 3|PG|^2 + |GA|^2 + |GB|^2 + |GC|^2$. The preceding equation is not new, although earlier proofs of it were not especially illuminating either: see, for instance, the proof via seemingly unmotivated calculations in analytic geometry published about 50 years ago in ([Altshiller-Court 1952], Theorem 109, 70). Nevertheless, Schaumberger's proof has considerable merit, for it does illuminate the situation, prompting Schaumberger (2002) to precede his proof with the declaration, "Here is why." Indeed, Schaumberger's proof is able to address the "why" of the situation because it uses vectorial methods to simultaneously discover and prove the underlying facts. In this way, Schaumberger's argument fits well into the "investigative, quantitative" pedagogic philosophy described in Dobbs (2001, 28), and thus could be used as enrichment material for precalculus students who are familiar with the basic properties of the dot product of vectors.

In fact, the argument of Schaumberger (2002) leads to additional ways of providing enrichment material for a precalculus course. Suppose, in the above notation, that P is taken to be the circumcentre K of Δ (that is, the centre of the circum[scribed]circle of Δ). Then the equation that was established by Schaumberger (and already known in [Altshiller-Court 1952]) leads at once to a formula for the circumradius R of Δ; that is, the radius of the circumcircle of Δ. Although this formula for R is already known [(Altshiller-Court 1952), Corollary 110, 71], we believe that there would be much pedagogic value in making available a proof of it that is based on Schaumberger's vectorial methods. Doing so is the first main purpose of the present note. While this note could be used in conjunction with other enrichment material involving circumcircles, such as (Dobbs, to appear), we have arranged it to be essentially self-contained. The presentation of the first four results would be appropriate for classes acquainted with dot product and the material on centroids typically covered in the prerequisite course on geometry.

Now, let k denote the incentre of Δ (that is, the centre of the in(scribed)circle of Δ), and let r be the inradius of Δ. According to a theorem of Euler (see [(Coxeter and Greitzer 1967), Theorem 2.12]), $|Kk|^2 = R^2 - 2Rr$. As noted in [Exercise 5, 31], it is an easy consequence that $R \geq 2r$. Accordingly, it seems natural to ask which triangles Δ satisfy $R = 2r$. By the above-cited result of Euler, it is equivalent to ask which triangles Δ satisfy $K = k$. The second main purpose of this article is to answer this and related questions: see Theorem 2.5. While it may be possible to prove Theorem 2.5 using some arcane, sophisticated methods, the proof given for it in this paper can be presented in the typical Precalculus class. In fact, Theorem 2.5 may be read/presented independently of the first four results in the paper. Among the topics reinforced by the proof of Theorem 2.5 are the following: the midpoint formula, slope, equations of lines, absolute value, the distance formula (between two points), the formula for the distance from a point to a line and criteria for congruence of triangles (as covered in the typical geometry course that is a prerequisite for precalculus).

We next describe the organization of this note. Lemma 2.1 gives the well-known formula for the position vector of the centroid G in terms of the position vectors of the vertices A, B and C. Proposition 2.2 (a) develops useful expressions for the vectors $\overrightarrow{GA}$, $\overrightarrow{GB}$ and $\overrightarrow{GC}$; then Proposition 2.2 (b) uses these expressions to prove a fact that was stated without proof in Schaumberger (2002), namely, that $\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC}$ is the zero vector. Proposition

2.3 (a) uses vectorial methods to prove a new formula for the sum $|GA|^2 + |GB|^2 + |GC|^2$, while, for the sake of completeness, Proposition 2.3 (b) includes the proof of Schaumberger. Then Corollary 2.4 gives the promised formula for the circumradius R. Finally, Theorem 2.5 shows that the equality of the circumcentre, centroid and incentre characterizes equilateral triangles.

Results

First, we recall the notion of a position vector. If a fixed point P is taken as the origin for a vectorial representation of the points of the Euclidean plane, then the *position vector of a point Q (relative to P)* is $\vec{PQ}$.

Lemma 2.1. *If the vertices of $\Delta = \Delta ABC$ have position vectors* $u = \vec{PA}$, $v = \vec{PB}$ *and* $w = \vec{PC}$ *and if G is the centroid of Δ, then the position vector of G is* $\vec{PG} = (1/3)(u + v + w)$.

Proof. Let D be the midpoint of the segment AB. Observe that the position vector of D is

$$\vec{PD} = \vec{PA} + \vec{AD} = u + ½\vec{AB} = u + ½(\vec{PB} - \vec{PA}) = u + ½(v - u) = ½(u + v).$$

As $\vec{PG} = \vec{PD} + \vec{DG}$, we proceed next to describe $\vec{DG}$. It is well known that G is located two-thirds of the way along the median from the vertex C to the midpoint D, and so

$$\vec{DG} = ⅓\vec{DC} = ⅓(\vec{PC} - \vec{PD}) = ⅓(w - ½(u + v)) = ⅓w - ⅙(u + v)$$

Therefore,

$$\vec{PG} = \vec{PD} + \vec{DG} = ½(u + v) + ⅓w - ⅙(u + v),$$

which simplifies to $(⅓)(u + v + w)$, to complete the proof.

Proposition 2.2 (b) isolates a well-known fact that forms the starting point in the proof of Schaumberger (2002).

Proposition 2.2. *Let G be the centroid of $\Delta = \Delta ABC$. Then:*

(a) If P is any point and the vertices of Δ have the position vectors $u = \vec{PA}$, $v = \vec{PB}$ *and* $w = \vec{PC}$, *then* $\vec{GA} = (⅓)(2u - v - w)$, $\vec{GB} = (⅓)(2v - u - w)$, *and* $\vec{GC} = (⅓)(2w - u - v)$.

(b) $\vec{GA} + \vec{GB} + \vec{GC} = 0$, *the zero vector.*

Proof. (a) Using Lemma 2.1, observe that

$$\vec{GA} = \vec{PA} - \vec{PG} = u - 1/3(u + v + w),$$

which simplifies to $(⅓)(2u - v - w)$, as asserted. The proofs for $\vec{GB}$ and $\vec{GC}$ are similar and, hence, left for the reader.

(b) Using the results of (a), we find, after algebraic simplification, that $\vec{GA} + \vec{GB} + \vec{GC}$ can be rewritten as $0u + 0v + 0w = 0 + 0 + 0 = 0$. The proof is complete.

The above proofs have made use of addition and subtraction of vectors, as well as scalar multiplication of vectors. The next proof also uses the properties of the dot product of vectors. Recall, in particular, that if v is a vector, then the square of its length is given by the dot product $v \times v$.

Proposition 2.3. *Let G be the centroid of $\Delta = \Delta ABC$. Then:*

(a) $|GA|^2 + |GB|^2 + |GC|^2 = (⅓)(|AB|^2 + |BC|^2 + |CA|^2)$.

(b) If P is any point, then $|PA|^2 + |PB|^2 + |PC|^2 = 3|PG|^2 + |GA|^2 + |GB|^2 + |GC|^2$.

Proof. (a) Let u, v and w be as in the statement of Proposition 2.2 (a). Then, by the above remark and Proposition 2.2 (a), we can rewrite $|GA|^2 + |GB|^2 + |GC|^2$ as

$$⅓(2u - v - w) \times ⅓(2u - v - w) + ⅓(2v - u - w) \times ⅓(2v - u - w) + ⅓(2w - u - v) \times ⅓(2w - u - v),$$

which simplifies algebraically to

$$⅔(u \times u + v \times v + w \times w - u \times v - u \times w - v \times w).$$

On the other hand, $|AB|^2 + |BC|^2 + |CA|^2$ can be rewritten as

$$(\vec{PB} - \vec{PA}) \times (\vec{PB} - \vec{PA}) + (\vec{PC} - \vec{PB}) \times (\vec{PC} - \vec{PB}) + (\vec{PA} - \vec{PC}) \times (\vec{PA} - \vec{PC}),$$

that is, as

$$(v - u) \times (v - u) + (w - v) \times (w - v) + (u - w) \times (u - w),$$

which simplifies algebraically to

$$2(u \times u + v \times v + w \times w - u \times v - u \times w - v \times w).$$

The assertion follows immediately.

(b) (Schaumberger 2002) Just as in the proof of (a), the proof of (b) involves a vectorial reformulation of the assertion. Once again, we let u, v and w denote the position vectors of A, B and C, respectively. Observe that $|PA|^2 + |PB|^2 + |PC|^2 = u \times u + v \times v + w \times w$. Rewriting u as $\vec{PG} + \vec{GA}$, with analogous renderings of v and w, we see, after algebraic simplification, that the above sum of dot products differs from the sum

$$3\vec{PG} \times \vec{PG} + \vec{GA} \times \vec{GA} + \vec{GB} \times \vec{GB} + \vec{GC} \times \vec{GC} = 3|PG|^2 + |GA|^2 + |GB|^2 + |GC|^2$$

by

$$2\vec{PG} \times (\vec{PA} + \vec{PB} + \vec{PC}).$$

According to Proposition 2.2 (b), this difference is $2\vec{PG} \times 0 = 0$. Since the difference is 0, we have that $|PA|^2 + |PB|^2 + |PC|^2 = 3|PG|^2 + |GA|^2 + |GB|^2 + |GC|^2$, to complete the proof.

We can now complete an essentially vectorial proof of a formula for the circumradius of a triangle that was given in [(Altshiller-Court 1952), Corollary 110, p. 71].

Corollary 2.4. *If G is the centroid and R is the circumradius of $\Delta = \Delta ABC$, then* $R = \sqrt{|KG|^2 + (1/9)(|AB|^2 + |BC|^2 = |CA|^2)}$.

Proof. In Proposition 2.3, take the "origin" P to be K, the circumcentre of Δ. Then $|PA| = |PB| = |PC| = R$, and so, combining parts (b) and (a) of Proposition 2.3, we have that

$$3R2 = |KA|^2 + |KB|^2 + |KC|^2 = 3|KG|^2 + 1/3(|AB|^2 + |BC|^2 + |CA|^2).$$

By straightforward algebra, we can solve for R^2 and then for R, thus yielding the asserted formula. The proof is complete.

The formula for R in Corollary 2.4 takes a particularly simple form in case $K = G$. Accordingly, it seems natural to ask which triangles have the property that their circumcentre coincides with their centroid. One could also ask which triangles have the property that their circumcentre coincides with their incentre. The answers to these questions are given in Theorem 2.5, which includes several characterizations of equilateral triangles.

First, it is convenient to recall the following consequences of congruence criteria: the circumcentre K (respectively, the incentre k) of Δ is the intersection of the perpendicular bisectors of the sides (respectively, the intersection of the bisectors of the interior angles) of Δ.

Theorem 2.5. *Let G be the centroid, K the circumcentre, k the incentre, R the circumradius, and r the inradius of $\Delta = \Delta ABC$. Then the following six statements are equivalent:*

(1) $K = G$;
(2) $K = k$;
(3) $k = G$;
(4) $K = k = G$;
(5) $r = R/2$;
(6) Δ is an equilateral triangle.

Proof. (6) ⇒ (4): Besides the material recalled above, we mention two additional facts that should be familiar. The first of these is the following consequence of the side–angle–side congruence criterion: the bisector of the vertical angle of an isosceles triangle is the perpendicular bisector of the base of the triangle. The second fact is actually a definition: the centroid of a triangle is the intersection of the medians of the triangle. Taking all the "recalled" information into account, we see at once that if Δ is equilateral and a line L is the bisector of an interior angle of Δ, then K, k and G all lie on L, whence (by intersecting two such angle bisectors) $K = k = G$. Therefore, (6) ⇒ (4).

(2) ⇒ (6): Assume (2); that is, $K = k$. To prove (6), it suffices to show that any two sides of Δ have the same length. We shall show that $|CA| = |CB|$. To show this, it is enough to prove that $\angle CAB \cong \angle CBA$. (Indeed, given this congruence of angles, if the altitude from C meets the side AB at the point P, then $\Delta CAP \cong \Delta CBP$ by the angle–angle–side congruence criterion, whence $|CA| = |CB|$.) Observe that the bisectors of angles $\angle CAB$ and $\angle CBA$ meet at $k = K$. As K is on the perpendicular bisector of the side AB, it now follows easily (via side–angle–side) that K is equidistant from A and B; that is, $|KA| = |KB|$. Then, since the base angles of an isosceles triangle are congruent (another consequence of the side–angle–side criterion), $\angle KAB \cong \angle KBA$. Therefore, $\angle CAB = 2\angle KAB \cong 2\angle KBA = \angle CBA$, thus proving (6).

(1) ⇒ (2): Assume (1): that is, $K = G$. Let the line CK meet the side AB at D; the line AK meet the side BC at E; and the line BK meet the side CA at F. Since K is the centroid of Δ and two points determine a line, it follows that D, E and F are the midpoints of the sides AB, BC and CA, respectively, and, moreover, that the lines CD, AE and BF are the perpendicular bisectors of the sides AB, BC and CA, respectively. Therefore, the distances $|KA|$, $|KB|$ and $|KC|$ are equal. As the base angles of an isosceles triangle are congruent, if follows that $\angle KAD \cong \angle KBD$, $\angle KBE \cong \angle KCE$ and $\angle KCF \cong \angle KAF$. Then $\Delta KAD \cong \Delta KBD$ (by either the angle–angle–side criterion or the hypotenuse–side criterion), whence $\angle AKD \cong \angle BKD$ as corresponding parts of congruent triangles. Similarly, $\angle BKE \cong \angle CKE$ and $\angle CKF \cong \angle AKF$. Using the fact that vertically opposite angles are congruent, we now see that all the above six angles with vertex at K are congruent. (For instance, considering the vertically opposite angles relative to the lines CD and FB leads to the conclusion that $\angle BKD \cong \angle CKF$.) Then the angle–angle–side criterion yields that $\Delta KFA \cong \Delta KDA$, whence $\angle FAK \cong \angle DAK$. In other words, K is on the bisector of the angle $\angle CAB$. Similarly, K is also on the bisectors of $\angle ABC$ and $\angle BCA$, and so $K = k$, completing the proof of (2).

(3) ⇒ (6): Assume (3); that is, $k = G$. Orient matters so that Δ is ΔABC, with vertices $A(0, 0)$, $B(c, 0)$ and $C(d, e)$, for some real numbers c, d and e such that $c > 0$ and $e > 0$. Our task is to prove (6), namely, that the sides of Δ all have the same length. By the distance formula, this means that our task is to prove that $c = \sqrt{(d-c)^2 + e^2} = \sqrt{d^2 + e^2}$.

Observe that the coordinates of $k = G$ are $((d + c)/3, e/3)$. One way to see this is to find the coordinates of the point of intersection of the medians $y = (e/(d + c))x$ and $y = (e/(2d - c))(2x - c)$; another way is to use Lemma 2.1. The details are straightforward and, hence, left to the reader.

We next find the (perpendicular) distances from k to the sides of Δ. Of course, the distance from k to the side AB is $e/3$. Next, observe that the line CA has equation $y = (e/d)x$, or equivalently, $ex - dy = 0$. Therefore, the distance from k to the side CA is

$$\frac{\left| e\left(\frac{d+c}{3}\right) - d\left(\frac{e}{3}\right) \right|}{\sqrt{e^2 + d^2}} = \frac{\frac{ec}{3}}{\sqrt{e^2 + d)^2}}.$$

Similarly, the line BC has equation $y = (e/(d - c))(x - c)$, or, equivalently, $ex + (c - d)y - ec = 0$. Therefore, the distance from k to the side BC is

$$\frac{\left| e\left(\frac{d+c}{3}\right) + (c - d)\left(\frac{e}{3}\right) - ec \right|}{\sqrt{e^2 + (c - d)^2}} = \frac{\frac{ec}{3}}{\sqrt{e^2 + (c - d)^2}}.$$

As the incentre k is equidistant from the sides of Δ, we now have that

$$\frac{e}{3} = \frac{\frac{ec}{3}}{\sqrt{e^2 + d^2}} = \frac{\frac{ec}{3}}{\sqrt{e^2 + (c - d)^2}}.$$

Cancelling the positive quantity $e/3$ and cross-multiplying, we find that $c = \sqrt{e^2 + d^2} = \sqrt{e^2 + (c - d)^2}$, thus completing the proof of (6).

Next, notice that (4) trivially implies each of (1), (2) and (3). In view of the implications that were proved above, this completes the proof that conditions (1), (2), (3), (4) and (6) are equivalent.

Finally, it suffices to prove that (2) $\Rightarrow$ (5). According to the theorem of Euler [(Coxeter and Greitzer 1967) Theorem 2.12] that was recalled in the introduction, $|Kk|^2 = R^2 - 2Rr$. Hence, (2) $\Rightarrow |Kk|^2 = 0 \Rightarrow R(R - 2r) = R^2 - 2Rr = 0 \Rightarrow R = 2r$ (since $R > 0$) $\Rightarrow$ (5). The proof is complete.

We close with two points. First, the proof given above for the equivalence of conditions (1), (2), (3), (4) and (6) in Theorem 2.5 is self-contained, making no reference to the theorem of Euler [(Coxeter and Greitzer 1967) Theorem 2.12] that was mentioned in the Introduction. Second, the proof given above that (2) $\Rightarrow$ (6) in Theorem 2.5 was shown to the author by his colleague, Pavlos Tzermias. This proof replaces the author's original analytic proof, which was considerably longer. The proof given here that (2) $\Rightarrow$ (6) was seen by Tzermias as a high school student in Greece approximately 20 years ago, thus providing additional anecdotal evidence of a significant difference between the typical North American and European high school curricula in geometry.

References

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Srinivasa Ramanujan (1887–1920)

One day this Indian mathematician drove with the British mathematician Godfrey Harold Hardy (1877–1947) in a taxi, which was marked with the number 1729. "A very boring number," commented Hardy. "But quite the opposite!" responded Ramanujan immediately. "It is a very interesting number in that it is the smallest number which can be written as the sum of two cubic numbers in two different ways." What are the two different ways?