Fractions Are Much More Than Pies

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Fraction fun begins as early as Kindergarten where students have been known to say, "Cut the cookie in half and give me the bigger half." It continues as students encounter fraction notation, fraction equivalency and operations with fractions. Why do fractions seem to be a difficult notion for students?

Our notation system can be a source of difficulty. When students encounter, for example, 2/3, they must understand that writing down two whole numbers does not represent a whole number. This can be magical from their point of view (Moss and Case 1999). A glimpse of early fraction notation (Cajori 1993) may help the reader appreciate students' struggles with fraction notation (see Table 1).

Table 1
Ancient Symbols for Fractions

Ancient Culture	Culture's Symbol	Our Symbol
Egypt		1/5
Greece	θʹ	1/9
Mesopotamia		1,132/100 (11.32)
Rome		7/12
India		2/3
China		48,125/1,000 (48.125)

Ancient cultures invented a variety of notation systems for fractions. The Mesopotamian and Chinese cultures did not invent special symbols for them, but they extended their place value system to name fractional amounts. Most ancient cultures invented a notation system for fractions that involved some form of fraction indicator. For example, the early Greeks placed an apostrophe-like mark above a counting number symbol. Some of the fraction notation systems were limited in the kinds of fractions they named. The early Egyptians, with a few exceptions, had symbols only for unit fractions. The Romans had symbols for a restricted set of fractions whose denominators were 12, 24, 48 and 72.

While a notation system can partly explain why students have difficulty with fractions, the complexity of fraction meanings is another reason. The literature indicates that students are unlikely to address problem situations involving fractions well or understand fraction arithmetic unless they clearly understand the full range of meanings of fractions (Tzur 1999). Five meanings of fractions seem important for teaching middle-year students.

Five Meanings of Fractions

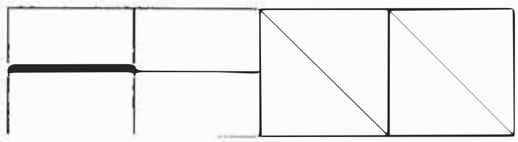
The five meanings recommended here serve as conceptual models or tools for thinking about and working with fractions. They also serve as a framework for designing teaching activities that engage students in sense making as they construct knowledge about fractions. A discussion of these meanings follows. The order of presentation does not imply an order of teaching.

The Part of a Whole Meaning

When a five-year-old child says, "I ate half of the cookie," he or she is expressing a part-whole relationship. The child uses "half" not in the sense of a number but in the sense of an actual or imagined action that involves cutting a whole physical object in the middle. The imagined or actual action of cutting a whole object into n equivalent/equal parts underlies the part of whole meaning of fraction. We represent each part symbolically by the fraction notation $1/n$.

Equivalent/equal may have different implications in different contexts. When we cut a rectangular granola bar into eight equivalent pieces and call each piece 1/8, we mean equivalent in the sense of area. The pieces normally look the same but they need not be. Figure 1 illustrates one way of cutting a granola bar into eight pieces equal in area but where not all of the pieces are the same shape. When we fold a rope into eight equal sections by making a succession of half folds, we mean equal in the sense of length.

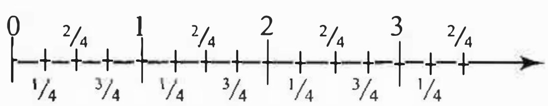
Figure 1
Cutting a Granola Bar into Eight Equivalent Pieces



The Name for a Point Meaning

This meaning best illustrates fractions as numbers distinct from counting numbers. The name for a point meaning grows out of the part of a whole meaning—the latter is useful for identifying and naming the points that lie between the points on a line that name whole numbers. These in-between points are obtained by applying the part of a whole meaning to a succession of unit segments of the number line (see Figure 2). This results in unit segments that are subdivided in the same way into equal lengths (into quarters in Figure 2). The segments differ in that each one begins with a different whole number name. In this way, a different number name is attached to each identified point on the number line (for example, 1/4, 1 1/4, 2 ¼ and so on). A number line that contains fraction-named points can serve as a ruler for measuring length using fractional amounts of units of length.

Figure 2
A Number Line Naming Quarters



The Part of a Group/Set Meaning

The part of a group/set meaning of a fraction involves a subgroup/subset of a collection of discrete things. For example, if there are 23 books of varying size and content on a shelf and 14 of them are novels, we can represent this situation by the fraction 14/23. In this case, by 14/23, we mean that 14 out of the 23 books are novels. This meaning is significantly different from the "part of the whole" and "name for a point" meanings. The part of a group/set meaning involves mentally placing discrete things into categories (for example, red, prime)—a different enterprise than cutting up a whole into equivalent parts or naming a point on a number line.

The quantification of probability involves the part of a group/set meaning of a fraction. For example, suppose a bag contains 20 marbles of various sizes and types. Seven of the marbles are crystals (big ones, medium ones and small ones). You reach into the bag and pull out one marble. The probability of that marble being a crystal is 7/20.

Fractions that are derived from the part of a group/set meaning have an interesting property. They are not additive in the linear measurement sense. Consider the following situation.

A football team played 8 away games and won 4 of them. It played 10 home games and won 9 of them. We can use the part of a group/set meaning to translate this situation into fraction form. The team won 4/8 of the away games and 9/10 of the home games. We can ask the question, "How many games did it win altogether, expressed as a fraction?" The answer is 13/18, and by this we mean that the team won 13 out of the 18 games it played. In obtaining the fraction 13/18, we added the away wins and home wins and added the away games and home games. If we represent what we did using fraction notation, we are compelled to write $4/8 + 9/10 = (4 + 9)/(8 + 10) = 13/18$. In other words, we added two fractions by adding numerators to numerators and denominators to denominators. This unorthodox algorithm produces the answer that makes sense (13/18 in this case). The standard fraction addition algorithm would not produce the answer that makes sense.

This situation is not restricted to football games. It occurs whenever the part of a group/set meaning is the conceptual basis for using fractions. I leave it to the reader to work out the details for a probability example. Imagine two bags containing dimes. Bag #1 has 5 dimes, 3 of which were minted in 1990. Bag #2 has 10 dimes, 4 of which were minted in 1990. What is the probability of pulling out a dime minted in 1990 from bag #1? From bag #2? Now imagine combining both bags into one bag (thus adding the contents). What is the probability of pulling out a dime minted in 1990 now? Which fraction addition algorithm produces that result? The standard one or the unorthodox one suggested above?

The intent of the above is not to encourage the teaching of an unorthodox fraction addition algorithm. Rather, it is to call attention to a fundamental quality of mathematics and its applications. Within formal mathematics, the rules are clear. But applications of mathematics (involving meanings) are always at our risk. When a mathematical notation is given more than one meaning, this can lead to confusing and even conflicting uses of the notation.

The Ratio Meaning

Ratio is a statement about a numerical relationship between two quantities that may or may not involve different constructs. Suppose the ratio between flour and butter in a recipe is 5 cups flour to 3 tablespoons butter. In this situation, the relationship is between the same construct (volume) but it does involve different units. Suppose that in an ecosystem, the optimal ratio between deer and forage is 1 deer to 2 tons of forage per square mile. This ratio involves different constructs, a count of discrete entities and a measurement of weight and area. Ratio may be explicit as in the examples above or it may be implicit. If I have two-thirds as many notebooks as Harry, this implies a ratio of 2 to 3 (I have 2 notebooks for each 3 that Harry has).

We can name ratios using a variety of notations (for example, 5 : 3, 5/3). Fraction notation seems to have an advantage for working with ratios. For example, we can equate two ratios when we are solving problems about similar triangles (for example, $2/3 = x/12$). The use of fraction notation for ratio leads to a comfortable algorithm for solving such problems. The danger is that students may think of 2/3 (for example) as having to do with pies (the part of a whole meaning of fraction) or some other meaning. That kind of thinking easily leads to confusion about solving ratio problems that use fraction notation to express ratio.

The Indicated Division Meaning

Another name for the indicated division meaning is the quotient meaning. It concerns the arithmetic of division where the symbol "/" is used to call for the division of two numbers. One application of this meaning is converting fraction notation to decimal notation, in which we divide the numerator of a fraction by its denominator (for example, 1/2 is $1 \div 2$ or .5).

There does not seem to be an effective way to build the indicated division meaning on concrete experience, yet this meaning is, increasingly, one of the most commonly used meanings (for example, in formulas such as density = mass/volume). It is therefore too often introduced by rote, if introduced at all.

The Meanings and Working with Students

How can we help students gain the power that ensues from understanding the above five meanings of fractions and that allows them to move flexibly among the various meanings? How do we help them see fraction notation as something that makes personal sense? A partial response to these questions is implicit in the following description of my work with three Grade 5 students on two Saturday mornings (part of a long-term study).

By the end of Grade 4, the students had had some fraction instruction in school about the part of a whole meaning. They were proficient at naming pieces of pies/pizzas using fractions but they did not see 1/3 (for example) as a number distinct from a counting number. They considered a fraction to be two whole numbers that describe what happens when you cut up things like pies. They also had some difficulty understanding wholes and parts of wholes. For example, when asked which is larger, 3 or 1/3, one of the students responded, "They are the same. If you used a whole pie there would be three thirds plus 3 on the bottom." As well, the students had little sense of why fractions were invented. In short, their knowledge of fractions was limited and confused.

I decided to redevelop fractions by first considering a reason for their invention. I used a measurement context for this, one that also made it possible to develop the name for a point meaning of a fraction. We imagined we were people from long ago that used a stick for measuring length. The students were asked to use the stick to measure the length of a table as accurately as possible. When asked how we could come up with a number for a part of a stick, the students suggested making equal marks on it and numbering them 1, 2, 3 and so on. The measurement for the part of a stick would be the number closest to the actual length of the part. They were still thinking in terms of whole numbers even though they were subdividing the stick into smaller units of length.

When asked if there was another kind of number that could be used for naming a part of a stick, the students initially were unable to make use of their school-acquired fraction knowledge. The area model (pies/pizzas) that had been used for teaching fractions to them had not empowered them to carry fractions to other situations that might involve a different meaning. After we discussed sticks as objects that could be cut into pieces equal in length where each piece could be given a fraction name, the students realized that sticks were just like pies. They decided to split the stick into 8 pieces by making a succession of half marks on it and mentally attaching a fraction name to each mark (1/8, 2/8 and so on). We generalized measuring with a stick to constructing a number line (a ruler) that was then used to measure whole and fractional lengths.

We revisited the part of a whole meaning of a fraction, using it to attach fraction names in a variety of contexts (for example, a loaf of bread). We discussed how we had made use of the part of a whole idea to make the marks on the ruler and how the fraction name for each mark was a different name for a point on the ruler. Labels for the two meanings were part of the discussion.

The students now had a good sense of fraction as a part of whole and as a name for a point on a number line and an emerging understanding of the relationship between the two meanings. They realized that fractions were useful when measuring and when describing parts of things. They did not yet realize that people did arithmetic with fractions to solve problems. This would be the next step in the development of their fraction numeracy, a step that would utilize the name for a point meaning of fraction (measurement) as the vehicle.

References

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Adam Riese (1492–1559)

He was Germany's best-known Rechenmeister of that century. He was the most influential of the German writers in the movement to replace the old computation by means of counters by the more modern written computation. He posed the following problem:

Three bachelors wanted to buy a house for 204 Gulden. The first bachelor contributed three times as much as the second bachelor, who contributed four times as much as the third bachelor. How many Gulden did each of the bachelors pay?