Gauss and the Regular Heptadecagon

William Dunham

In 1796, the German journal *Allgemeine Literaturzeitung* carried the announcement of a remarkable mathematical discovery. The discoverer was an unknown adolescent named Carl Friedrich Gauss, and his achievement was the construction, with compass and straightedge only, of a regular heptadecagon; that is, a regular polygon of 17 sides. At the conclusion of this announcement, Professor E. A. W. Zimmermann added these words of endorsement:

> It deserves mentioning that Mr. Gauss is now in his 18th year, and devoted himself here in Brunswick with equal success to philosophy and classical literature as well as higher mathematics.

Why was Professor Zimmermann so enthusiastic, and what made this so significant a moment in mathematics history?

Constructability problems date back to the Greek geometers. It was they who established the compass and straightedge as the allowable tools of geometry and who thereby determined the sorts of constructions that were and were not possible.

It is intriguing to speculate as to why the compass and straightedge were adopted to the exclusion of other tools. Two reasons suggest themselves. First, the compass and straightedge existed as easy-to-use mechanical devices that could be employed by the geometer to draw figures on papyrus or in the sand. Second, they produced, respectively, circles and straight lines—the perfect two- and one-dimensional figures that so appealed to the Greeks' sense of beauty. By combining ease of use with a beautiful output, these geometrical tools won the hearts of the Greek mathematicians.

Although the compass and straightedge tradition predates Euclid, it is in his *Elements* that one finds its most perfect manifestation. Euclid enshrined them in his first three postulates and thereafter restricted himself to constructions involving them and them only. Such constructions included simple operations like bisecting lines and erecting perpendiculars, but by Book IV of the *Elements*, Euclid had demonstrated the more sophisticated constructions of regular pentagons and pentadecagons (15-gons). One wonders how many math teachers today could pull out a compass and straightedge and construct a regular 15-sided polygon. Euclid could.

Interestingly, nowhere in the *Elements* did Euclid explicitly state that a regular n-gon is constructible if and only if a regular $2n$-gon is, even though he clearly recognized this principle. Thus, among polygons with fewer than 50 sides, Euclid could construct regular 3, 4, 5, 6, 8, 10, 12, 15, 16, 20, 24, 30, 32, 40 and 48-gons. Of course he could continue *ad infinitum* by repeated doubling of the number of sides.

There was a general sense that this list included all the constructible regular polygons. Centuries passed—and mathematicians like Archimedes, Al-Khowârizmî, Newton, Leibniz and Euler came and went—but no one augmented the collection. No one, that is, until the young Carl Friedrich Gauss, 2100 years after Euclid, conquered the 17-gon.

Before addressing his discovery, we might mention other constructability challenges bequeathed to posterity by the Greeks. The three best known were the trisection of the angle; the duplication of the cube (that is, given the side of a cube, construct the side of a cube with double the volume); and the quadrature of the circle (given a circle, construct a square of equal area). These remained unresolved in classical times and intrigued a host of mathematicians well into the modern era.

All three were still open questions when Gauss announced his heptadecagon breakthrough in 1796. No one had suspected that such a construction was possible, and if less effort had been expended trying to construct 17-gons than trying to trisect angles, it was because the former seemed much more improbable than the latter. Gauss's amazing construction must have given renewed hope to the angle trisectors and circle squarers, who could reasonably argue that if a regular 17-gon were constructible, then a dose of Gaussian ingenuity might lead to a successful trisection or circle squaring as well.

Of course, things did not turn out that way. In 1837, the French mathematician Pierre Laurent Wantzel proved that both angle trisection and cube duplication were beyond the power of compass and straightedge. As a matter of fact, he proved both of these on the same page of a groundbreaking paper in Liouville's journal!

Squaring the circle held out a bit longer, until an 1882 theorem by the German Ferdinand Lindemann established that π was a transcendental number and thus not constructible. From this it followed that circles could not be squared with compass and straightedge.

We now return to the 17-gon. Needless to say, Gauss's reasoning cannot be squeezed into a few paragraphs; if it were that simple, someone would have stumbled upon it during the previous 21 centuries. Yet the difficulty lies more in its intricate interconnection of ideas rather than in the profundity of any idea in particular. What follows is the basic plot, a sort of "Cliff Notes" version of his proof:

The argument rested upon the postulates of Euclid that allowed us to construct a straight line between two points (Postulate 1), to extend a straight line segment as far as we wish (Postulate 2), and to construct a circle with a given point as centre and given length as radius (Postulate 3). Thereafter, Gauss's reasoning assumed an algebraic character—following the insights of Descartes—by noting that, if a magnitude can be expressed using only integers and a finite number of operations of addition, subtraction, multiplication, division and the extraction of square roots, then that magnitude can be constructed with compass and straightedge. For instance, beginning with a given unit length, one could construct a segment of length

$$\frac{2+\sqrt{3+\sqrt{5}+\sqrt{7}}}{4+\sqrt{1}+\sqrt{11}}$$

(although I wouldn't necessarily want to).

Next, Gauss invoked trigonometry by proving the following constructability condition:

> A regular n-gon is constructible if and only if $\cos(2p/n)$ is constructible.

In the context of regular heptadecagons, this naturally led him to consider $\cos(2\pi/17)$.

It is at this point that things became truly amazing, for young Gauss veered off into the world of complex numbers. This may seem bizarre—given that all geometric constructions occur in the real world—but the key to the problem lay in DeMoivre's theorem and the roots of unity. Recall that DeMoivre had proved that

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

Letting $\theta = 2\pi/17$, we begin to sense a link between DeMoivre's theorem and the constructability condition above.

With inspired algebraic insights and deft manipulations, Gauss managed to show the hitherto unexpected fact that

$$\cos(2\pi/17) = -\frac{1}{16} + \frac{1}{16}\sqrt{17} + \frac{1}{16}\sqrt{34-2\sqrt{17}} + \frac{1}{8}\sqrt{17+3\sqrt{17}-\sqrt{34-2\sqrt{17}}-2\sqrt{34+2\sqrt{17}}}.$$

Because the expression on the right is built of integers that are added, subtracted, multiplied, divided and square-rooted, it is constructible. Thus $\cos(2\pi/17)$ is constructible, and it follows from the constructability criterion that the regular 17-gon is as well. QED.

Of course, besides the 17-gon, Gauss could construct 34-gons, 68-gons and others by repeated doubling. In addition, from the regular 17-gon he could construct a regular $3 \times 17 = 51$-gon (and all of its doubles) and a regular $5 \times 17 = 87$-gon (and all of its doubles). His general theory showed that,

> If $2^{2^n} + 1$ is a prime, then a $2^{2^n} + 1$-gon is constructible with compass and straightedge.

For $n = 0$, 1 and 2, this yields regular 3-gons, 5-gons and 17-gons. For $n = 3$, we get $2^8 + 1 = 257$, a prime, and so a regular 257-gon is constructible (and hence a regular 514-gon and so on). For $n = 4$, $2^{16} + 1 = 65{,}537$, another prime, so along comes another batch of constructible polygons.

Unfortunately for polygon constructors,

$$2^{2^5} + 1 = 2^{32} + 1 = 4{,}294{,}967{,}297$$

is not prime, for it has a factor of 641 as the incomparable Euler had observed 50 years earlier. No further constructible polygons have been found since Gauss laid down his pen two centuries ago.

Legend says that an aging Gauss wished to have the regular 17-gon inscribed on his tombstone. The validity of the legend notwithstanding, it certainly would have been a fitting memorial to so great a scholar. Unfortunately, no such monument was executed. For what it's worth, however, the German government has seen fit to put Gauss's countenance upon its ten mark note.

Of course, the practical importance of this construction is nil. The regular heptadecagon won't help anyone balance a checkbook, build a quieter refrigerator or find a quick way through the Lincoln Tunnel at rush hour. Sad to say, it is essentially useless. But . . . it is as beautiful and unexpected a piece of mathematical reasoning as there is. If mathematical theorems can ever be termed breathtaking, this one qualifies.

The construction has a final claim to fame. In later years, when Gauss had achieved international repute as the Prince of Mathematicians, he observed that it was this discovery that propelled him into mathematics. His observation raises the unsettling possibility that, had it not been for the regular heptadecagon, Carl Friedrich Gauss might have become a cobbler or wine taster. Clearly, the mathematical world owes this polygon a huge debt of gratitude.

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Brahmagupta (c. 628)

He is the most prominent of the Hindu mathematicians of the 7th century. In his book *Cutta ca*, he writes:

If one reduces the number of days by 1, divides the difference by 6 and adds 3, the result is always one-fifth of the original number of days. What is the original number of days?