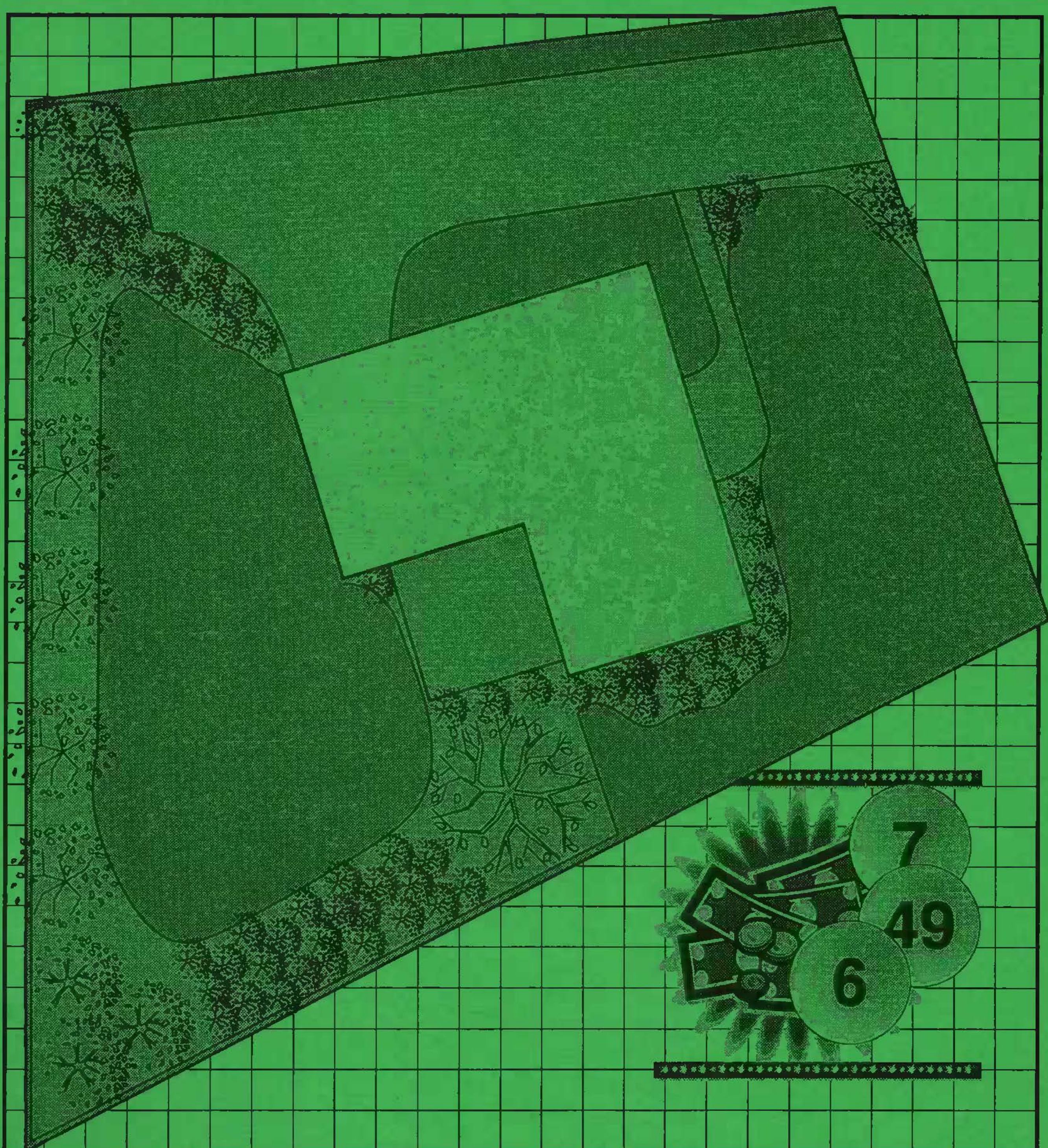
delta-k

JOURNAL OF THE
MATHEMATICS COUNCIL
OF THE ALBERTA
TEACHERS' ASSOCIATION

Volume 41, Number 1 February 2004



GUIDELINES FOR MANUSCRIPTS

delta-K is a professional journal for mathematics teachers in Alberta. It is published to

- promote the professional development of mathematics educators, and
- stimulate thinking, explore new ideas and offer various viewpoints.

Submissions are requested that have a classroom as well as a scholarly focus. They may include

- personal explorations of significant classroom experiences;
- descriptions of innovative classroom and school practices;
- reviews or evaluations of instructional and curricular methods, programs or materials;
- discussions of trends, issues or policies;
- a specific focus on technology in the classroom; and
- a focus on the curriculum, professional and assessment standards of the NCTM.

Manuscript Guidelines

1. All manuscripts should be typewritten, double-spaced and properly referenced.
2. Preference will be given to manuscripts submitted on 3.5-inch disks using WordPerfect 5.1 or 6.0 or a generic ASCII file. Microsoft Word and AmiPro are also acceptable formats.
3. Pictures or illustrations should be clearly labelled and placed where you want them to appear in the article. A caption and photo credit should accompany each photograph.
4. If any student sample work is included, please provide a release letter from the student's parent allowing publication in the journal.
5. Limit your manuscripts to no more than eight pages double-spaced.
6. A 250–350-word abstract should accompany your manuscript for inclusion on the Mathematics Council's website.
7. Letters to the editor or reviews of curriculum materials are welcome.
8. *delta-K* is not refereed. Contributions are reviewed by the editor(s), who reserve the right to edit for clarity and space. **The editor shall have the final decision to publish any article.** Send manuscripts to A. Craig Loewen, Editor, 414 25 Street S, Lethbridge, Alberta T1J 3P3; fax (403) 329-2412, e-mail loewen@uleth.ca.

Submission Deadlines

delta-K is published twice a year. Submissions must be received by August 31 for the fall issue and December 15 for the spring issue.

MCATA Mission Statement

Providing leadership to encourage the continuing enhancement of teaching, learning and understanding mathematics.

delta-K

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COMMENTS ON CONTRIBUTORS

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Cynthia Ballheim is a vice-principal at Father Lacombe High School in Calgary, Alberta.

Harry Bohan is a professor of mathematics education at Sam Houston State University in Huntsville, Texas.

David E. Dobbs is a professor of mathematics at the University of Tennessee in Knoxville, Tennessee.

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David R. Duncan is a professor of mathematics at the University of Northern Iowa in Cedar Falls, Iowa.

William Dunham is the Koehler professor of mathematics at Muhlenberg College in Allentown, Pennsylvania, and the author of *Journey Through Genius: The Great Theorems of Mathematics* and of *Euler: The Master of Us All.*

Natali Hritonenko is a professor of mathematics at the University of Texas in Dallas, Texas.

Beverly Irby is a professor in mathematics education at Sam Houston State University in Huntsville, Texas.

Art Jorgensen is a retired junior high school principal from Edson, Alberta, and a former longtime MCATA executive member.

Werner Liedtke is a professor emeritus, University of Victoria in Victoria, British Columbia.

Bonnie H. Litwiller is a professor of mathematics at the University of Northern Iowa in Cedar Falls, Iowa.

Andy Liu is a professor of mathematics in the Department of Mathematical and Statistical Sciences at the University of Alberta in Edmonton, Alberta.

A. Craig Loewen is a professor at the University of Lethbridge in Lethbridge, Alberta.

Emeric T. Noone teaches at Longwood College in Farmville, Virginia. He is primarily interested in undergraduate probability and statistics.

Klaus Puhlmann is a retired superintendent of schools for Grande Yellowhead Regional Division No. 35 and MCATA journal editor.

Sandra M. Pulver is a professor of mathematics at Pace University in New York.

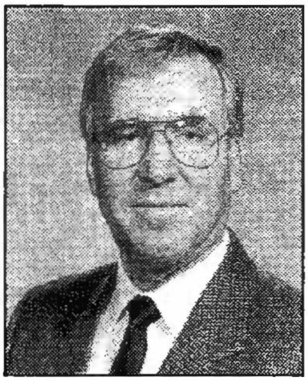
Deanna Shostak is high school mathematics teacher at Alberta College in Edmonton, Alberta, and is on secondment with Alberta Learning as the Applied Mathematics 30 exam manager. She is also MCATA's Alberta Learning representative.

John C. Uccellini teaches at Indiana Area Senior High School in Indiana, Pennsylvania. He is actively involved with training elementary and middle school teachers in quantitative literacy.

Dolly Vogel is a graduate student at Sam Houston State University and a teacher of sixth-grade mathematics at Houser Intermediate School, Conroe Independent School District in Conroe, Texas.

Yuri Yatsenko is a professor in computer information systems at Houston Baptist University in Texas. His interests include teaching and research in mathematics, economics and computer-related areas.

Farewell and Welcome to the New Editor



As I write this editorial, I am fully aware that it is my last one. It will be January/February 2004 by the time you receive this issue. My successor will assume the editorship with the spring 2004 issue. Please send all future *delta-K* submissions to A. Craig Loewen, Editor, *delta-K*, 414 25 Street South, Lethbridge, AB T1J 3P3; e-mail loewen@uleth.ca; phone (403) 327-8765.

I cannot help but have mixed feelings about retiring from the editorship because I have truly enjoyed editing *delta-K* for the past eight years. My enjoyment was largely derived from several sources. Networking with many of the contributors was one important source and always professionally rewarding. Their submissions were interesting to read and contributed to my own professional growth immensely.

Working with the executive of the Mathematics Council of the Alberta Teachers' Association (MCATA) was the most satisfying experience. Comprising representatives from the teaching profession, the University of Alberta Faculty of Education and mathematics department, Alberta Learning and the Alberta Teachers' Association (ATA), all members were dedicated and gave generously of their time to achieve the mission, which is providing leadership to encourage the continuing enhancement of teaching, learning and understanding mathematics. The feedback and suggestions received from the executive are largely responsible for making *delta-K* such an invaluable resource for mathematics educators in Alberta as well as in other Canadian provinces and many states in the United States.

Working with the Document Production staff of the ATA has been a most pleasant experience for me. Each one of the staff members is dedicated and committed to producing an end product that is of the highest quality. The creation of *delta-K* from start to finish is very complex, involving many people—authors, editors and Document Production staff. There is no doubt that the Document Production staff deserves our collective commendation and much of the credit in producing a professionally looking journal.

To maintain a high quality mathematics journal or enhance the quality of *delta-K*, thereby making it an important vehicle for communicating teaching and learning experiences in mathematics, it must contain relevant content for our readers. This can only be achieved with your help. We all know that good things are happening in the mathematics classrooms of our schools, colleges and universities. It is important that they do not go unnoticed and that they are shared with our readers. It is in that spirit that I urge you—teachers, students, graduate students, and college and university teachers—to submit manuscripts for inclusion in *delta-K*.

delta-K must become a journal written by teachers for teachers. Every section of *delta-K* is open for submissions. Those of you who do not have time to write a feature article may choose to share a teaching idea or comment on something that has been written in a previous issue. Student projects are produced in every high school mathematics classroom in Alberta, yet few student projects are submitted for publication. Please read the editor's comments under Student Corner for direction.

delta-K allows everyone to share his or her experiences in teaching/learning mathematics. Please consider making a contribution to *delta-K*. Only through your active participation can the editor plan, design and produce a mathematics journal that is relevant, meaningful and a useful resource for mathematics educators.

Thank you for the opportunity to work with so many dedicated people. It has been a real pleasure working with you and being the editor of *delta-K*.

Klaus Puhlmann

From the President's Pen

Back in the Saddle Again

After two years out of the classroom (a sabbatical year followed by an administrative stint), I'm "back in the saddle again." As I acclimate myself to my new school, new home and new country, I have begun to look at all aspects of teaching through different lenses.

Assessment is a key aspect of teaching that has captured my fancy in a big way. In particular, questioning has become my focus. Linda G. Barton (1997) has summarized questioning leading to critical thinking using Bloom's Taxonomy as her framework. According to Barton, there are six levels of questioning. They are knowledge, comprehension, application, analysis, synthesis and evaluation. These are arranged hierarchically, from bottom to top.

Knowledge questions are the base and test the recall of facts. Comprehension questions allow students to demonstrate their understanding of facts and ideas by organizing, comparing or interpreting. Application questions cause students to solve problems involving new situations by applying knowledge in different ways. Analysis questions enable students to break information into parts and to make inferences from these parts. Synthesis questions prompt students to compile information in different ways and to propose alternative solutions. Evaluation questions guide students to present and defend opinions by making judgements about information based on certain criteria.

This hierarchy of questioning is hardly new, but my choice of verbs to describe these levels was deliberate—allow, cause, enable, prompt and guide. Questions are gateways to knowledge. If we truly want to give our students every opportunity to tell us what they know, we must craft our questions carefully and use them as segues to new knowledge. Whether in the classroom or on tests, we all need to focus on the kinds of conversation that will occur if we ask the right kinds of questions. I challenge each of you to craft one different kind of question during each of your classes and for each of your written assessments. I know that you will be very pleased with the mathematical conversation that will result. Questions are the building blocks of communities, and our classrooms are mathematical communities. Have you asked a good question lately?

Reference

Barton, L. G. *Quick Flip Questions for Critical Thinking.* Dana Point, Calif.: Edupress, 1997.

Cynthia Ballheim

The Right Angle

Deanna Shostak

Curriculum—IOP

Thirty-four teachers have been selected to take part in field-validating the following revised IOP programs of study, beginning in September 2003:

- IOP program vision, philosophy and rationale statements
- IOP English Language Arts, Grades 8–12
- IOP Mathematics, IOP Science, IOP Social Studies, Grades 8–11
- IOP Occupational component, Grades 8 and 9

Teachers who take part in field-validating IOP programs of study will also provide feedback on the corresponding component(s) of the IOP Studio (online guide to implementation) and the Career Development component.

Teachers who take part in field-validating IOP programs of study will also participate in developing and field-testing items for the following:

- IOP English Language Arts, Grade 9
- IOP Mathematics, Grade 9
- IOP Science, Grade 9

Teachers across the province are invited to use and provide feedback on the new IOP programs of study and online guide to implementation. Provincial workshops are being planned to inform teachers and administrators about the revised program. We will be informing the field regarding locations and dates of the IOP workshops through various methods, such as *Connections for Teachers* online magazine, regional learning consortia and IOP updates to superintendents.

Learner Assessment Branch

The following documents are available on Alberta Learning's website (www.learning.gov.ab.ca):

- Information bulletins for both Pure and Applied Mathematics 30
- Projects and teacher notes for both Pure and Applied Mathematics 30 (project solutions are found on the extranet)
- Exam manager reports for January and June 2003

Teachers should also note that all diploma examinations will be held secure until released to the public by the minister. However, for the January and June 2004 examinations, teachers will be allowed access to a Teacher Perusal Copy for review purposes one hour after the examination has started. Representative portions of the January and June examinations will be released, along with an Examination Manager's Report for each subject early in the fall 2004.

Bhaskara I (c. 1114–c. 1185)

The prominent Hindu mathematician, who wrote chiefly on astronomy, arithmetic, mensuration and algebra, posed this problem:

Find natural numbers which, when divided by 2, 3, 4, 5 and 6, have a remainder of 1 and in addition are divisible by 7.

MCATA Executive in Action

Klaus Puhlmann

Your MCATA executive met on September 26–27, 2003, in Edmonton. The executive consists of 21 members with representatives from the teaching profession, Alberta Learning, the University of Alberta Faculty of Education and Department of Mathematics and the Alberta Teachers' Association. Every year, new members join the executive, replacing those members who leave.

The membership must know who the executive members are so that they can contact them directly on issues related to the teaching and learning of mathematics. The executive works diligently on behalf of members, keeping a strong focus on MCATA's mission: "Providing leadership to encourage the continuing enhancement of teaching, learning and understanding mathematics."

As always, important issues filled the agenda. At this meeting, the executive discussed and finalized arrangements for the November 19–22, 2003, NCTM/MCATA Regional Conference in Edmonton. The executive also discussed and approved a budget for the 2003/04 school year. The key topic for the day was mathematical literacy, an initiative promoted by MCATA for the past two years.

Renata Taylor-Majeau, examination manager for Grade 3 mathematics provincial achievement test, spoke about the minister's revisioning of mathematics and how it will change the focus of Grade 3 mathematics. It is encouraging that the revisioning will include elements from the MCATA initiative on mathematical literacy. Please watch for further developments in this area and let your executive members know how you feel.

MCATA executive welcomes the new members, who are briefly introduced below. The pictures that follow depict your MCATA executive members during the September meeting.

New Executive Members

Sharon Gach has been teaching senior high mathematics for the past 20 years at Vegreville Composite High School in Vegreville and Bev Facey Community High School in Sherwood Park. Last year, she was the senior high AISI math coordinator for Elk Island Public Schools. She has just completed her master of education degree in education administration and leadership at the University of Alberta.

A. Craig Loewen teaches courses in mathematics methods and curriculum to both the math majors and minors at the University of Lethbridge in the Faculty of Education. For the past three years, Craig has been serving as the assistant dean of Student Programs in the Faculty of Education. He is on study leave until April 2004, when he will return as the assistant dean for a second term. Craig is particularly interested in writing and developing teaching resources for problem-solving instruction and for teaching mathematics in a meaningful way. During his leave, he hopes to develop a new course, finish a few articles that have been started and begin a book on teaching problem solving in the middle grades.

Geri Lorway is a consultant and staff developer with a passion for mathematics. Her current portfolio includes staff development projects with Northern Lights School Division No. 69, St. Paul Education Regional Division No. 1, Lakeland RCSS District No. 150, Buffalo Trail Regional Division No. 28, and High Prairie School Division No. 48. She is also a staff-development consultant with Learning Network (East Central Consortium). Her professional affiliations include the National Council of Teachers of Mathematics (NCTM), the Association for Supervision and Curriculum Development, the National Staff Development Council, College of Alberta School Superintendents, and the Canadian Math Educators Study Group. You may have seen her at a MCATA or NCTM regional or national conference, as she loves to fill last-minute vacancies when speakers are forced to cancel.

Problem-centred learning and visual and spatial reasoning are her current research interests. She believes that understanding how the brain processes information is the key to increasing student learning, and that our focus must be working together from Kindergarten to Grade 12.

Martina Metz has taught elementary math and science at the Calgary Science School for the past four years. Prior to that, she spent seven years teaching elementary and junior high at a small rural school in Hays, Alberta. Martina enjoys reading, philosophizing and spending time in outdoor pursuits.

Charlotte White is in her 12th year of teaching at St. Mary's High School in Calgary. A University of Calgary graduate, she uses her oil-industry experience in geophysics and economics to build mathematical connections in the classroom.

Charlotte has worked on several committees developing and revising the standards for the new curriculum. She is a strong proponent of cooperative learning, the use of technology to enhance understanding and the use of alternative assessment in the classroom.

(Note: The following members are not shown: Cynthia Ballheim, president, Calgary; Robert Wong, webmaster, Edmonton; Indy Lagu, mathematics representative, Calgary.)

A. Craig Loewen, delta-K *editor*

l–r: Daryl M. J. Chichak, membership director, Edmonton; Len Bonifacio, vice president, publications, Edmonton; Sandra Unrau, past president, 2004 conference director, Calgary; Janis Kristjansson, director, professional development, Calgary; Charlotte White, NCTM representative, Calgary.

l–r: Carol Henderson, PEC liaison, Calgary; Deanna Shostak, Alberta Learning representative, Edmonton; David Jeary, ATA staff advisor, Calgary; Geri Lorway, director, awards and grants, Bonnyville; Anne MacQuarrie, newsletter editor, Calgary; Dale Burnett, Faculty of Education representative, Lethbridge.

l–r: Charlotte White, NCTM representative, Calgary; Martina Metz, director at large, Calgary; Renata Taylor-Majeau, examination manager for Grade 3 mathematics provincial achievement test and guest presenter, Edmonton; Carol Henderson, PEC liaison, Calgary; Deanna Shostak, Alberta Learning representative, Edmonton.

l–r: Geri Lorway, director, awards and grants, Bonnyville; Daryl M. J. Chichak, membership director, Edmonton; Elaine Manzer, vice president, professional development, Peace River; Donna Chanasyk, secretary, Edmonton; Sharon Gach, director at large, Sherwood Park; Len Bonifacio, vice president, publications, Edmonton; Sandra Unrau, past president, 2004 conference director, Calgary.

l–r: Shauna Boyce, director, publications, Spruce Grove; Doug Weisbeck, treasurer, St. Albert; Carol Henderson, PEC liaison, Calgary; Deanna Shostak, Alberta Learning representative, Edmonton; David Jeary, ATA staff advisor, Calgary; Anne MacQuarrie, newsletter editor, Calgary.

l–r: Sandra Unrau, past president, 2004 conference director, Calgary; Charlotte White, NCTM representative, Calgary; Martina Metz, director at large, Calgary; Shauna Boyce, director, publications, Spruce Grove; Anne MacQuarrie, newsletter editor, Calgary.

NCTM/MCATA 2003 Regional Conference

Message from the Conference Chair

Cynthia Ballheim

The 2003 annual conference, "Taking Mathematics to the Nth Degree," was held November 20–22 at the Shaw Conference Centre and Westin Hotel in Edmonton. This conference was held jointly with the National Council of Teachers of Mathematics (NCTM) and allowed registrants to mix with colleagues, share ideas, attend sessions with international and local speakers, and visit exhibits.

Despite lots of snow, the conference opened with a wine and cheese and entertainment with the Atomic Improvisation. NCTM president Johnny Lott gave the opening general session, "Who Is the Ultimate Mathematics Teacher?" and NCTM president-elect Cathy Seely closed the conference with "Personal Leadership to Transform Your Classroom and Your Community." Sixteen major speakers were featured throughout the conference, including Thomas Kieren, David Schwartz, James Schultz, Trevor Calkins, Keith Devlin, Helaman and Claire Fergusson, Kanwal Neel, Dale Burnett, John Van de Walle and Brent Davis.

Sharon Barry from Grande Prairie Composite High School was honoured during the opening session with MCATA's Mathematics Educator of the Year Award. MCATA members who renewed or purchased a new membership received a purple presentation folder, Penatia Multi Function pen, Calgary conference pen, two MCATA student certificates, a business card and a MCATA sticker.

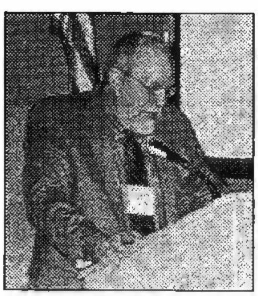
Mark your calendars for the 2004 MCATA annual Conference in Calgary on October 29–30. See you there.

A Pictorial Potpourri

Klaus Puhlmann

NCTM's Canadian Regional Conference was held in Edmonton from November 19 to 22, 2003. The theme of the conference was "Taking Mathematics to the Nth Degree." In excess of 200 sessions, workshops and mini-courses were offered, with presenters from all across Canada and the United States.

The next few pages are an attempt to capture some of the presenters, participants and events.



Johnny W. Lott, NCTM president, one of many major speakers, is exploring the question, "Who is the ultimate mathematics teacher?"

NCTM members taking turns at the registration desk. (l–r) Betty Morris and Shauna Boyce.

James E. Schultz, Ohio University, Athens, Ohio, has his focus on "Technology: The Good, the Bad and the Ugly," providing many examples of proper use, misuse and inappropriate use leading to misconceptions.

Johnny W. Lott, NCTM president, making a presentation to Sandra Unrau (l), past president and 2003 program chair, and Cynthia Ballheim (r), MCATA president and 2003 conference chair, in appreciation of their work in organizing this year's conference.

Sharon Barry (l), Grande Prairie Composite High School, is the recipient of the 2003 Mathematics Educator of the Year Award, with Cynthia Ballheim (r) making the presentation.

MCATA president Cynthia Ballheim made the following comments prior to presenting the award to Sharon Barry.

One of my favourite parts of our annual conference is the presentation of the Mathematics Educator of the Year Award. This year is no exception.

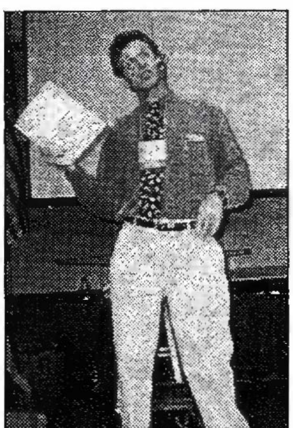
Our winner is an exemplary teacher who loves teaching mathematics. She spends hours preparing her class so that each and every student will succeed in mathematics. Her lessons are creative and she motivates her students to do their best. She spends countless hours before and after school and during her lunch hour to prepare her students for all manner of assessments. These students are not just her students. They flock to her from all classes because of her mathematical knowledge and caring ways. Her mathematical skills and knowledge, combined with her dedication to students have made her a valued member of committees at the local and provincial level. Her contributions to her own department at the school level have encouraged many of her colleagues to improve their teaching techniques.

The staff at Grande Prairie Composite High School considers her to be an inspiration to staff and students alike.

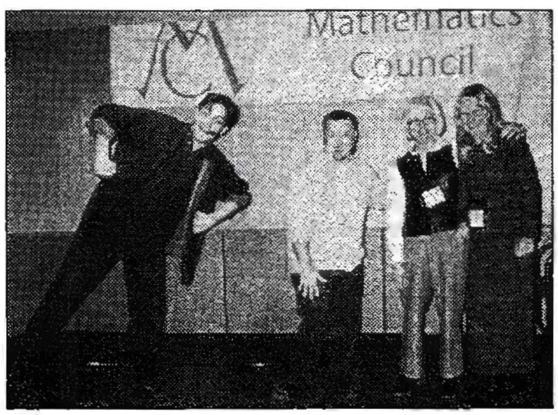
It is with great pleasure that I present to you, Sharon Barry, our 2003 Mathematics Educator of the Year.

Lise Bureau, H. E. Bourgoin School, Bonnyville, Alberta, shares a teacher's success and concerns as she moved toward a model of assessment that encourages students in their own evaluation. Her topic was "Students' Focus, Students' Assessment: Nuturing Lifelong Learners."

Thomas E. Kieren, University of Alberta, Edmonton, Alberta, delivering a major session on "Teaching in the Middle School: Using Interactive Intelligence and Collective Understanding."



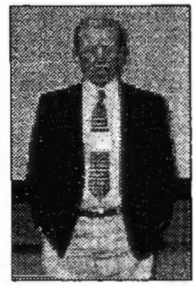
David M. Schwartz, author from Oakland, California, shows how exciting the results can be when children begin mathematical explorations inspired by literature. His topic was, "Students, Teachers, Authors: A Golden Triangle for Mathematical Learning."



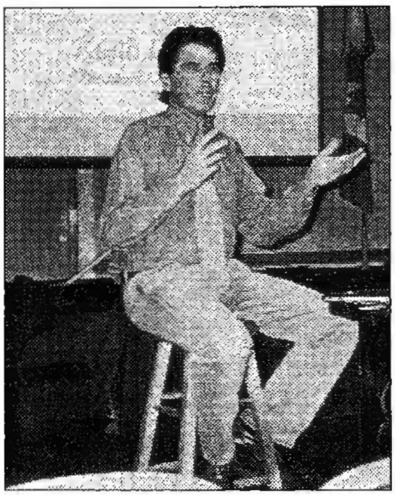
Conference attendees are entertained by Mark Meer and Donovan Workun of Atomic Improv Company, with Cynthia Ballheim, MCATA president, and Sandra Unrau, past president, getting involved in the act.

Phoebe Astra Arcills and Ryan F. Mann, both from Parkland School Division No. 70, Stony Plain, Alberta, conducted a workshop on "What's Up with Applied Mathematics?" which contains a project-driven strand that challenges the typical classroom.

Frances L. Schatz, Ontario Association of Mathematics Educators, Kitchner, Ontario, shares insights, experiences, instructional strategies and observations from her very successful career as a tutor.

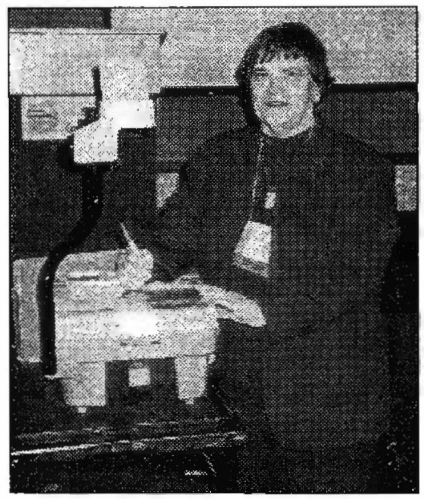


Ron Larson, Penn State University at Erie, Erie, Pennsylvania, presents "The Three Most Important Things in Teaching Algebra" to his participants.



Keith Devlin, Standford University, Standford, California, making a major presentation on "The Evolutionary Origins of Mathematical Thinking Ability."

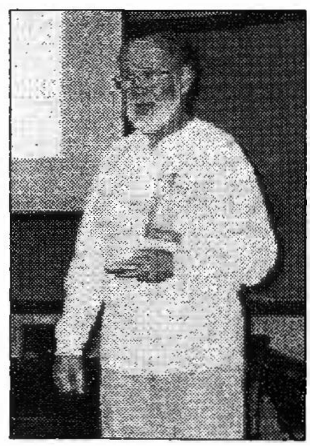
Klaus Puhlmann, retiring editor of delta-K, *received an engraved glass plaque in appreciation for his work as editor from Cynthia Ballheim, president, at the MCATA annual general meeting. Geri Lorway (r), MCATA executive member looking on.*

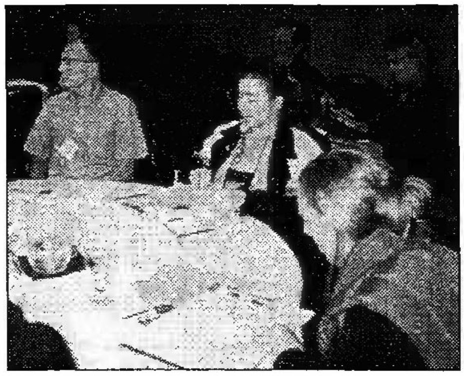


Bonnie Litwiller, University of Northern Iowa, Cedar Falls, Iowa, is "Navigating Through Principles and Standards: Activities from the Grades 3–5 Navigations Books."

Werner W. Liedtke, University of Victoria, Victoria, B.C., presents a popular session dealing with "Number Sense: The Key to Success and Numeracy."

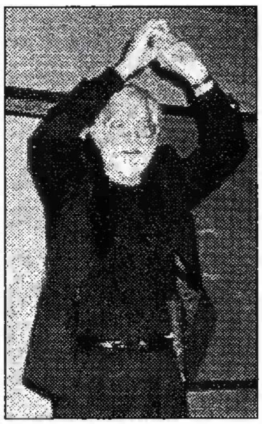
Hubert J. Ludwig, Ball State University (Emeritus), Muncie, Indiana, enlightened his audience with ancient and modern points of view, historical comments and a variety of mathematical procedures involving pie and fractals. His topic was "Pie: From Measuring Fields to Fractals."



MCATA annual general meeting open to all MCATA members to attend. In addition to receiving information, this is an opportunity for input by the membership.

Kanwal S. Neel, British Columbia Association of Mathematics Teachers, Richmond, B.C., is engaging teachers in "Getting Ready to Teach Mathematics to the Nth Degree."

Helaman P. Ferguson and Claire Ferguson, Helaman Ferguson Sculpture, Laurel, Maryland, in their major session on "Mathematics in Stone and Bronze" are using slides and video to trace Helaman's creations from initial conception through mathematical design and computer graphics to their final form.

Daniel H. Jarvis, University of Western Ontario, London, Ontario, conducted a workshop on "Di Divina Proportione: The Art and Science of the Golden Ratio."

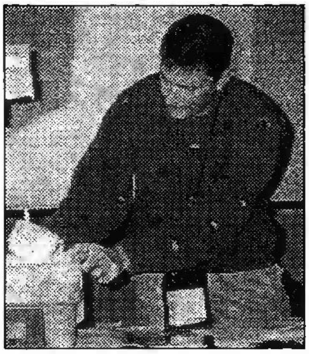
Nola E. Aitken, University of Lethbridge, Lethbridge, Alberta, presents teacher education programs and mathematics achievement results of Grades 4–9 mathematics questions of Western Australia and the University of Lethbridge preservice teachers and examines the implications of the findings.

Marian S. Small, University of New Brunswick, Fredericton, New Brunswick, engages the participants in activities presented in "Navigating Through Principles and Standards: Activities from the Grades Pre-K–2 Navigation Books."

Dale Burnett, University of Lethbridge, Lethbridge, Alberta, describes the importance of technologically based notation for mathematical thinking and problem solving in his major session on "Tools and Notation: A Symbolic Relationship for the Future."

Rudy V. Neufeld, Neufeld Learning Systems, London, Ontario, is presenting strategies for using interactive software to help students understand fractions, integers, per cent, exponents, equations, algebra, graphing, probability, geometry and measurement.

Dave Wagner, University of Alberta, Edmonton, Alberta, deals with the topic, "Teaching Mathematics for Peace." Mathematics is often used as a tool for destruction and injustice. He asks, Can it equip young people for peace?

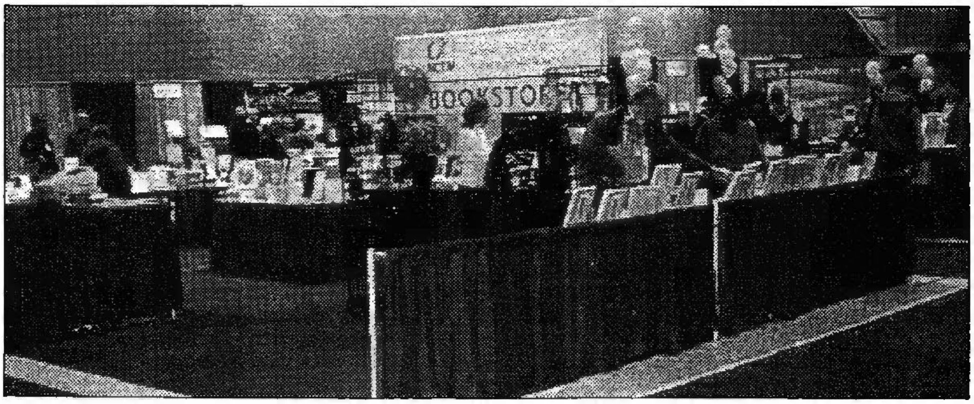


Todd Steinhauer, T. D. Baker Junior High School, Edmonton, Alberta, conducted a workshop dealing with "Critical-Thinking Games and Activities (Shape-Space and Statistics-Probability)." A group of teachers actively engaged in the activities.

Anna M. Zukowski, Graminia Community School, Spruce Grove, Alberta, presents "Make a Difference: Differentiate," in which she asserts that differentiating instruction provides success for all learners.

John A. Van de Walle, Virginia Commonwealth University, Richmond, Virginia, takes a look at what a student-centred or problem-based lesson looks like in his presentation entitled, "Let Kids Be the Sense Makers, Not You! Planning for Problem-Based Lessons."

Jo Towers, University of Calgary, Calgary, Alberta, speaks about her research on "Understanding Area: Ways of Knowing," with special attention to different ways of knowing.



The NCTM bookstore and the commercial exhibits are always a favourite with the teachers and they are an important part of regional and annual conferences.

In this section, we will share your points of view on teaching and learning mathematics and your responses to anything contained in this journal. We appreciate your interest and value the views of those who write.

Some Processes for Changing Curriculum Beg Revisiting

Werner Liedtke

Over the years, I have had many opportunities to be a member of groups that were involved in updating and creating elementary mathematics reference and assessment materials for teachers and students. The settings were all similar: a large room, a large table and a group of people representing various grade levels and institutions. The goals for these settings may have differed in that they involved reviewing and reacting to newly created materials, revising existing materials or creating new materials. However, there existed common aspects that were part of the procedures associated with these tasks. The sole purpose of the ideas recorded in this article is to focus on aspects of the procedures. The intent is not to be critical in any way of the participants. The assumption is made that the question, "How do revised and new documents or reference materials come into existence?" might be of interest not only to those who use them but also to those who will work with committees of this type in the future.

A group can find itself invited to a meeting where the terms of reference are very restrictive. As a result, any forthcoming revisions and changes are limited. Members may end up asking whether calling the meeting was necessary or worth it, or whether the same results could have been reached in a more efficient manner. Would it not be advantageous to inform participants of the terms of reference and a detailed agenda prior to calling a meeting? Based on my experience, this was rarely done.

A group may be in a position of having to base all of its revision decisions on existing documents or even on one existing document. Questions making reference to content like, "Why this?" or "Where does this come from?" are answered with, "It is in the Ontario (California) Strands." or "It is in the Western Canadian Protocol (WCP)." Some assume that, because it appears in this document, it is appropriate and good and should not be questioned. Decisions for revision or inclusion are made accordingly.

Because the WCP is viewed by some as *the* document for revision, content development or even research projects, a few comments and questions are in order. A lot of good things can be said about this document, but that is not the point of this article. On the other hand, this document does include very general statements that are difficult to interpret or are open for interpretation. Because some terms and expressions that are part of the text are undefined, they require examples. Without these, only the author or authors of these statements may know the intended meanings. Some illustrative examples do not make sense. Some are not appropriate for the listed learning outcomes. Some very important ideas lack illustrative examples. Some learning outcomes are inappropriate or do not make sense. Serious omissions exist.

Specific examples for these shortcomings could be presented, but neither is that the intent of this article. (These shortcomings will likely be addressed during upcoming revision meetings.) As far as this article is concerned, the most important question is, "Where did the entries (learning outcomes, illustrative examples) in the WCP come from?" Content from different provincial guides was used to make up the entries of the WCP, and I suggest that, for meaningful curriculum decisions related to updating, revising and especially for generating new material, this "circle may need to be unbroken." For example, an emphasis on conceptual understanding or on the development of number sense would require a brand new growth plan that then becomes a part of these new or revised documents, rather than creating something through a cut-and-paste strategy applied to existing documents. Which procedure would be of greatest value and benefit to teachers and students?

No doubt the decision-making processes of a group has been researched and written about. Allow me to supplement such reports with a few personal anecdotal observations. The question, "Why does this appear at the Grade 3 level?" will probably elicit the response "It is in the WCP/Ontario Strand." The question, "Should it be there?" may not be entertained. If, by chance, it is, the following scenarios are plausible. Utterances directly from the affective domain, such as "I like it" or "Teachers like it," may sway a group into accepting an idea. A passionate oration by someone, which might make reference to having students "see the beauty of..." can influence decision making (especially if the speaker is an administrator). Many times, it is assumed that a member of a group, such as a primary teacher, is a spokesperson with whom all members of that group will agree, and, as a result, statements by that member are not questioned, especially if he or she is the only member present. In instances like these, some sort of consensus is reached based on aspects of social interaction that are far removed from an examination of an overall growth plan for a given topic. Should that ever be the case?

Some revision meetings begin in a relaxed or rather open-ended way without a specific agenda. It inevitably happens that, at the end of the scheduled day(s), group members leave at different intervals (early). As a result, "legislation via exhaustion" occurs and fatigue sets in. Decisions are made rapidly and they may not be based on what was agreed on earlier. I have often wondered whether some chairs actually had this outcome in mind as part of their "hidden curriculum" when they made decisions about an agenda.

Decisions about revising, updating or creating curriculum are complex—more complex than many of the ideas in this article suggest. Perhaps some of these ideas can become part of a discussion about that complex process and procedure. That is my hope.

Why Do Numerate Students/Adults Lack Conceptual Understanding of Division?

Werner Liedtke

Division will be used as the focus of the discussion, which could be applied to other topics as well. The following questions will be addressed in this article: Are any concerns about the numeracy of our students warranted? What data exist to suggest that our students lack conceptual understanding of division? What are some possible reasons for students lacking conceptual understanding? What might be done to have students acquire conceptual understanding of division? and What are some of the key components of conceptual understanding of division?

Numeracy Concerns

There are mathematics educators who reference "rising scores on certain tests" and "results of performances on contests, nationally and internationally" to suggest that, as far as mathematics learning is concerned, things are improving or even all is well. These types of conclusions could be countered with information released by universities that not only expresses concerns about the lack of basic literacy and numeracy skills of its graduates but also talks about intervention programs for these students (who represent the top 15 to 20 per cent of our population—so what might be said about the others?).

I have collected competency test scores from education students for more than 30 years, and have seen that these scores have not risen. In fact, the opposite is true. Some colleagues have tried to explain this trend by suggesting that top students who at one time would have enrolled in education are now attracted to other areas or professions. Even if this assumption is true, changes to the system these students went through should result in higher levels of numeracy. As far as my observations tell me, that is not the case. I still encounter as many people as I did many years ago who somewhat proudly proclaim that they not only do not like math but also are not good at math or unable to do math. Comments like these, reports in newspapers and on the news about unwise consumers all lead me to believe that, despite all of our efforts, and it hurts more than a little to say this, things have not changed since Paulos (1998) wrote the book about innumeracy. I think there are as many people now who, as Paulos suggests, read number as (numb) (er) as there were when he wrote the book.

If there is any hope, or if there are any positive signs, it would be the implementation of the British Columbia Association of Mathematics Teachers' suggestions of increased emphasis on conceptual understanding and fostering the development of number sense—the key foundation for numeracy. Without these areas of emphases, things are not likely to change and improve. However, the mere identification of these areas is by no means sufficient. As will be seen, I think more needs to be done.

Conceptual Understanding

If the opposite of rote procedural knowledge is well-defined procedural knowledge, possible indicators of the latter include knowing how and why something works and being able to illustrate procedures with base-10 blocks or diagrams. Possible indicators of conceptual knowledge include the ability to connect ideas to one's experience, that is, being able to create relevant word problems; being able to simulate a procedure (algorithm) with appropriate denominations of money and explaining it in one's own words; using more than one method to find an answer; making predictions about an answer or commenting on the reasonableness of results and knowing why and how another operation might be used to check an answer.

Many topics in elementary mathematics are presented to students in a rote-procedural fashion. (I almost used *taught* rather than *presented* in the last sentence. This would be inappropriate for someone who agrees with those who believe rote learning is an oxymoron.) I think many aspects of geometry and telling time rank high among these topics, but division is at or very near the top of the list.

The competency test items that include aspects of division always present the greatest difficulty for the students in my courses. Most of the students recall being taught an algorithm by way of a rule method that is still used and displayed in some classrooms around the province. This method of presentation, along with inappropriate practice, no matter how extensive, will not result in the acquisition of conceptual understanding. Charles and Lobato (1998, 17) define practice as "appropriate when it involves or is connected to the process of doing mathematics; that is, reasoning, communicating, connecting, and problem solving."

Over the years, students who enrolled in a course on diagnosis and intervention strategies have interviewed hundreds of students from the intermediate grades and up. Occasionally, a student is encountered whose responses are indicators of the presence of some well-defined procedural knowledge for division. However, rarely, if ever, did indicators of conceptual knowledge surface. It never fails to amaze us how little understanding of division secondary school students have and how little they remember.

Early this year, I addressed a group of teachers in Port McNeill, British Columbia. After one session, a lady who teaches secondary school students approached me. She did mention that her mathematics training was not received in North America, and she shared her dismay that students come to her classes confused about division. These students lack understanding and there was no doubt in her mind that it could all be blamed on the symbol that we use for long division. She may be right, but could it also have something to do with the language that is used while the symbol and procedure are presented?

The secondary students who lack understanding become adult consumers. Some enter the teaching profession. Perhaps that, in part, explains the results reported by Howe (1999) that not one of the American elementary teachers included in a comparison study with Chinese counterparts was able to come up with a meaningful word problem for a division equation with fractions.

What is true for division is true for other topics as well. My conclusion is that the majority of our students lack conceptual understanding. I have encountered people who will say, "Why is conceptual understanding important? I did not need it and I did all right!" (Comments like these make me think of statements uttered by a former premier and a minister of education, respectively: "Just give everyone a shovel and they will have an opportunity to become millionaires. It worked for me."; and "I came from a one-room school. Look at me. Why do we need more?" How can one argue with that kind of logic?).

If numerate persons are able to connect numerals and operations to life experiences and actions arising in real-life situations, then conceptual understanding is required and necessary for our students.

Possible Reasons for Lack of Conceptual Understanding

A profession more valuable to society than teaching does not exist. Teachers are the greatest resource society can have, but it is true that the majority of them are not majors of mathematics or in mathematics education. Since that is the case, the mathematics program in most classrooms is only as good as the main references available for the teacher and the students.

Several dilemmas exist because many classrooms in British Columbia, for example, do not have enough texts for each student. That is only part of the problem if the references that are available do not clearly specify for a teacher how to put emphasis on fostering the development of conceptual knowledge. As far as I am concerned, the references available for teachers lack the necessary specificity and detailed growth plans that are required for this endeavour. The reasons for that being the case are easy to explain. Let's assume that authors of these references take the learning outcomes they are writing for from the provincial guides (for example, Integrated Resource Package [IRP]) or from documents like the Western Canadian Protocol (WCP). The problem with this procedure is that the majority of the outcomes in these are much too general to be of value for preparing content with a focus on conceptual knowledge.

To illustrate the dilemma described in the last paragraph, consider an example from the IRP for Grades 2–3. The Prescribed Learning Outcome states, "It is expected that students will explore and demonstrate the process of division up to 50, using manipulatives, diagrams, and symbols." The WCP includes a very similar statement for Grade 3. There are two basic things teachers and authors take away from this statement, and they can be heard time and time again: manipulatives are important and the teaching sequence should go from concrete to abstract. The implicit assumption is that these two ideas suffice and will result in conceptual understanding. Do these generalizations contribute to or result in conceptual understanding of division? Based on all of the data I have collected and described earlier, they do not. Much more is needed if we are serious about emphasizing conceptual understanding. Teachers require specific information for guidance.

Over the years, quite a few teachers have told me that they design their own mathematics program for their students. Now, I do not want to take anything away from the experts who have the know-how to do this, but in the majority of cases, teachers do not have the time to develop growth plans for well-defined procedural as well as conceptual knowledge. My greatest fear is always that many students in these settings do not acquire conceptual understanding. The same feeling is true for another setting. From time to time reports in newspapers remind us that home schooling is on the increase in our province. How would most parents know how to focus on fostering conceptual knowledge and sense of number if they did not experience these types of settings and if the references they use are not of any assistance and without any professional training? I also would like to ask, "Is it possible for students to develop and acquire conceptual knowledge and sense of number if mathematics is presented in a language they are just beginning to learn?" My answer would be that this is unlikely. I have encountered quite a few adult students in my courses who did attribute their difficulties to having been in such settings.

One procedural weakness can be very detrimental to students, and it has to do with staffing. A few years ago, a mother presented me with a problem her daughter in secondary school was experiencing. After having done well in mathematics, not just enjoying it but understanding it as well, a teacher who had no mathematics background whatsoever was assigned to teach her class. The classroom setting became rote-procedural and rule-oriented. Things fell apart for this young lady and the desperate mother was searching for a possible solution. At the highest district level, she was told that that is how things work and nothing could be done about it. Does conceptual understanding have to be forfeited in this way, or is there a way out?

What Might Be Done to Have Students Acquire Conceptual Understanding?

First and foremost, Prescribed Specific Learning Outcomes are needed that clearly indicate to teachers what is expected or required for conceptual understanding. To illustrate this point, consider the sample Prescribed Learning Outcome from the IRP and the WCP quoted earlier. Teachers need to be informed of how objects are to be manipulated, that is, which type of division is to be used and why; the natural and mathematical language students are expected to use as manipulations are carried out (to avoid guzinta); and what the diagrams the students learn to draw are supposed to show and why. Would an effort to provide this type of information, take up a little more paper and be a little more work? If that is the case, so be it, but without this information the exploring and demonstrating that will be done by many students in different classrooms will not be the same and may even be done improperly. An emphasis on conceptual understanding requires that, whenever necessary and whenever possible, teachers are aware of the indicators of this type of understanding that they need to look for. Making these necessary changes can be done without spending a great amount of money. Publishers can then use these blueprints to provide the appropriate reference materials.

When learning outcomes and content are considered during meetings that involve IRP revisions, the questions, "Why this?" and "Why this at this grade level?" are sometimes posed. The most common response usually refers to the fact that it is in the (WCP) or some other important document. The assumption is made that, because it is in the WCP, it is appropriate and good. This type of rationale must be questioned. The somewhat circular procedure of making use of a cut-and-paste to create one type of document from another and then vice versa needs to be broken. Growth plans for different topics that consist of key specific learning outcomes for well-defined procedural and conceptual knowledge must be created, and that may have to be done without being based on something that exists. Perhaps somewhere along the way we may also need to inform those who are skeptical of the research evidence, which indicates that the necessary conceptual understanding can be developed without hurting skill development.

I have read in more than one reference that one of the greatest weaknesses of the American education system is that teachers seldom know what happens in previous grades or what will happen in subsequent grades. This type of outcome is reinforced in many settings that I have been a part of. More often than not people are placed in groups that deal with the grade level they are teaching. (I have heard some people announce that they are not interested in anything that is not related to the grades they teach.) It would be advantageous for those involved, as well as for continuity, if a group that is involved in revising or creating new curriculum materials would develop a complete growth plan for a topic rather than just for one grade or for a narrow range of grades. Once completed, computers should make it possible and easy for teachers to call up such complete growth plans for any topic they are about to teach. That would give them a complete picture of what conceptual understanding of a topic entails and what is learned along the way. It would be made clear to them why something is done at a given level and how it contributes to success in later grades.

Publishers and authors who prepare references for schools must clearly show teachers that what they have prepared includes growth plans for both well-defined procedural and conceptual knowledge. Also the practice activities they have devised must be appropriate; that is, that they meet the criteria suggested by Charles and Lobato (1998). These activities should tease students to think and to advance their thinking about what is being practised.

Authors of assessment instruments need to be made aware that both types of knowledge must be assessed. Once conceptual understanding becomes an important part of such instruments, there is no doubt that many teachers will strive to refocus the emphasis of their teaching of mathematics.

By the time students complete junior high school, they should have acquired conceptual understanding of division. What should these students be able to say and do? What skills, procedures and ideas should they have learned in the first six or seven grades?

The following are key components of conceptual understanding (Liedtke 1998). Following in parentheses are possible responses from students who lack this understanding. Readers are invited to present these types of tasks and then draw their own conclusions about understanding division.

- Show $6 \div 3$. Ask, "How would you read this?" (Students may say "goes into" or "into," and many will say that both "3 into 6" and "6 into 3" are acceptable. Students are not aware from their experience of the two types of action that can be matched with every numerical statement of this type. The two interpretations of division are not known.)
- Show $12 \div 3$. Ask, "Try to make up a word problem for this. (The students will focus on the answer and may work backward to make up an answer. They are likely to get confused about the divisor and the quotient. Neither of the two possible interpretations of division may be referred to.)
- Show a big handful or a jar filled with chips or beads. Ask, "What would you do to divide by three?" or show $( ) \div 3$ and ask what they could do to find the answer. (They may declare that it cannot be done unless they know how many chips or beads are in the hand or jar.)
- Show $6 \div 3$. Request, "Try to make a sketch to show the action. (The sketches may not illustrate the actions for either of the two interpretations of division. Confusion about the divisor and the quotient may surface.)
- Show one or two basic facts, that is, $56 \div 7$ and $72 \div 9$. Ask, "Pretend you have forgotten the

answers, what would or could you do to figure out what they are?" "What else could you do?" If the student knows the answers ask, "How could you check to find out that the answers are correct?" (A lack of strategies to reinvent forgotten answers or to get unstuck may become evident.) It never fails to amaze me that teachers have different definitions of basic facts, even those who work on tasks related to the mathematics curriculum. Mind you, very few references exist that offer these definitions. However, it seems logical to assume that those who teach the basic facts should know what they are.

- Show 3 ÷ 0 and 0 ÷ 5. Ask, "What are the answers and how do you know the answers are correct?" (Strategies that involve connecting may not be available to reinvent generalizations. These types of tasks may be classified as being the same. Division by zero may not be an issue.)
- Show 624 ÷ 4. Request a recording other than the short form that shows how to find the answer. Ask questions about the value of the partial dividends and products. (The partial dividends and products may be referred to in terms of ones, or tens and ones, rather than the actual value.)
- Show an item, one at a time, and ask for an explanation of estimation strategies to make predictions about the answer. Is the answer greater than one or less than one (or about one)? How do you know? What number is the answer close to? How do you know? For example, 37,642 ÷ 13, 6.85 ÷ 0.25, 0.35 ÷ 0.5, 1 ¾ ÷ ½, or S! ÷ ¾. (Students may not have estimation strategies at their disposal.)
- Show items like those from the last example. Ask, "Who would want to find the answers for these items? When? Why? Try to make up a meaningful word problem for each one. (Students may not know which interpretation of division would be best to use for making up meaningful word problems.)

The list of possible indicators of conceptual knowledge can be extended, but the goal is to identify some of the key components that need to be part of a growth plan for division if it is to be taught for understanding. At present, many of these ideas are not specifically identified and clearly stated in the key references that teachers use.

There exists one more important issue. After all of the many interview transcripts I have read, all of the interviews I have discussed with my students and with classroom- and special-education teachers and all of the interviews I have conducted, one thing has become very clear to me—number sense is the key foundation for conceptual understanding. Without it, the goal of developing conceptual understanding will not be reached, no matter who tries to undertake the task and how much they charge.

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Liedtke, W. *Interview and Intervention Strategies for Mathematics.* Sherwood Park, Alberta: ECSI, 1998.

Paulos, J. *Innumeracy: Mathematical Illiteracy and Its Consequences.* New York: Vintage Books, 1998.

Erratum

Natali Hritonenko

With respect to *delta-K*, Volume 40, Number 2, September 2003, page 49, problem 4, the correct answer is (a) and not (c), because the domain of the problem is: Squaring both sides we increase our domain. Therefore, 5 does not satisfy the initial problem.

STUDENT CORNER

Communication is an important process standard in school mathematics; hence, the mathematics curriculum emphasizes the continued development of language and symbolism to communicate mathematical ideas. Communication includes regular opportunities to discuss mathematical ideas and to explain strategies and solutions using words, mathematical symbols, diagrams and graphs. Although all students need extensive experience in expressing mathematical ideas orally and in writing, some students may have the desire or should be encouraged by teachers to publish their work in journals.

delta-K *invites students to share their work with others beyond their classroom. Submissions could include papers on a mathematical topic, a mathematics project, an elegant solution to a mathematical problem, an interesting problem, an interesting discovery, a mathematical proof, a mathematical challenge, an alternative solution to a familiar problem, poetry about mathematics, a poster or anything of mathematical interest.*

Teachers are encouraged to review students' work prior to submission. Please attach a dated statement that permission is granted to the Mathematics Council of the Alberta Teachers' Association to publish the work in delta-K. *The student (or the parents if the student is under 18 years of age) must sign this statement, indicate the student's grade level and provide an address and telephone number.*

MCATA invites teachers of Pure Mathematics 30 or Applied Mathematics 30 to submit their best student projects. An independent panel will judge the projects. The students who submit the best two projects will each receive $50, and their work will be published in delta-K. *The students will also be acknowledged in the* Mathematics Council Newsletter. *Submissions must include both the project questions and answers. Students must submit both a hard copy of the project and a disk containing an electronic version. Entries must be accompanied with an application form that includes the student's name, home address and phone number; teacher's name; and school name and address. Students must sign a release form allowing the project to be published in* delta-K *and making it the property of MCATA. If the student is under 18 years of age, the parents must sign the release form. The project and the disk will be returned to the student, along with complimentary copies of* delta-K. *The submission deadlines are February 15 for first-semester projects and July 15 for second-semester projects. Submission/release forms are available on the MCATA website (www.mathteachers.ab.ca) under Grants/Awards. Send submissions to Geri Lorway, Director of Awards and Grants, Mathematics Council of the Alberta Teachers' Association, 4006 45 Avenue, Bonnyville T9N 1J4; phone (780) 826-2231, e-mail glorway@telusplanet.net.*

No submissions were received for this issue. We look forward to receiving your submissions for the next issue.

NCTM Standards in Action The Content Standard: Data Analysis and Probability

Klaus Puhlmann

The question, "What mathematical content and processes should students know, understand and be able to use in the K–12 curriculum?" is an ongoing preoccupation with curriculum developers. The National Council of Teachers of Mathematic's *Principles and Standards for School Mathematics* (NCTM 2000) strongly recommends 10 content standards for the K–12 school mathematics curriculum, one of which is data analysis and probability. Thus, data analysis and probability as a content strand is included at every level, albeit with varying degrees of depth and breadth. It is well articulated across the grades, thus ensuring the accumulation of important ideas and building successively deeper and more refined understanding.

The data analysis and probability standard identifies the broad areas of emphasis that enable all K–12 students to

- formulate questions that can be addressed with data and collect, organize and display relevant data to answer them;
- select and use appropriate statistical methods to analyze data;
- develop and evaluate inferences and predictions that are based on data; and
- understand and apply basic concepts of probability.

There is no doubt that our lives are inundated with large quantities of data. Data are often the basis for decision making in business, industry, politics, research and even in our personal lives. Consumer surveys guide marketing strategies; purchasing products, which is often based on data related to satisfaction levels about a particular product; and decisions that are often based on probabilistic reasoning. Therefore, it is important that students know about data analysis and probability to reason statistically and become informed citizens and intelligent consumers.

There are, of course, other valid reasons for including data analysis and probability as a content strand across the grades. The data analysis and probability strand allows teachers and students to make several important connections among ideas and procedures, from number, algebra, measurement and geometry. It also offers natural ways for students to connect mathematics with other subject areas and events occurring in their own lives. Engaging students in statistical reasoning about data not only serves them well as they enter the world of work and living independently but also teaches them that solutions to some problems depend on assumptions and have some degree of uncertainty. This kind of reasoning used in statistics and probability is not always intuitive; hence, students would not necessarily develop this important skill from other parts of the curriculum.

The expectations of students in this content strand are naturally age-appropriate. At the primary level, the expectations are that students

- pose questions and gather data about themselves and surroundings;
- sort and classify objects according to their attributes and organize data about the objects;
- represent data using concrete objects, pictures and graphs;
- describe parts of the data and the set of data as a whole to determine what the data show; and
- discuss events related to students' experience as likely or unlikely.

At the upper elementary level, students are

- designing investigations and considering how collection methods affect the nature of the data set;

- collecting data using observations, surveys and experiments;
- representing data using tables and graphs (for example, line plots, bar graphs and line graphs);
- describing the shape and important features of a set of data, comparing data and determining how the data are distributed;
- using measures of central tendency;
- comparing different representations of the same data;
- proposing and justifying conclusions and predictions, and designing studies;
- describing events as likely or unlikely, including such words as certain, equally likely and impossible;
- predicting the probability of outcomes; and
- gaining and understanding the likelihood of an event, represented by a number from 0 to 1.

At the middle school level, students are

- formulating questions, designing studies and collecting data about a characteristic shared by two populations or within one population;
- selecting, creating and using appropriate graphical representations of data (for example, histograms, box plots and scatter plots);
- finding, using and interpreting measures of centre and spread;
- discussing and understanding the correspondence between data sets and their graphical representations;
- using observations about differences between two or more samples to make conjectures about the population from which the samples were taken;
- making conjectures about possible relationships between two characteristics of a sample;
- using conjectures to formulate new questions;
- understanding and using appropriate terminology to describe complementary and mutually exclusive events;
- using proportionality and a basic understanding of probability to make and test conjectures; and
- comparing probabilities for simple compound events, using such methods as organized lists, tree diagrams and area models.

At the high school level, students are

- understanding the differences among various kinds of studies and the types of inferences that can be drawn from them;
- knowing the characteristics of well-designed studies, including randomization in surveys and experiments;
- understanding the meaning of measurement data and categorized data, of univariate and bivariate data and of the term *variable*;
- understanding histograms, parallel box plots and scatter plots, and their appropriate use;
- computing basic statistics and understanding the distinction between a statistic and a parameter;
- able to display the distribution for univariate measurement data;
- able to display a scatter plot for bivariate measurement data and determine regression;
- computing coefficients, regression equations and correlation coefficients using technology;
- displaying and discussing bivariate data with one categorized variable;
- recognizing how linear transformations of univariate data affect shape, centre and spread;
- identifying trends in bivariate data and finding functions that model the data;
- using simulations to explore the variability of sample statistics from a known population and constructing sampling distributions;
- understanding how sample statistics reflect the values of population parameters and using sampling distribution as the basis for informal inference;
- evaluating published reports that are based on data with a focus on the design of the study, the appropriateness of the data analysis and the validity of conclusions;
- understanding how basic statistical techniques are used in the workplace;
- understanding the concepts of sample space and probability distribution;
- using simulations to construct empirical probability distributions;
- compiling and interpreting the expected value of random variables;
- understanding the concept of conditional probability and independent events; and
- understanding how to compute the probability of a compound event.

This content standard is well articulated across the grades, moving from the simple notions and concepts to a relatively high level of sophistication at the high-school level. The study of data analysis and probability is based on the premise that students need to have hands-on experiences working with data derived from our everyday lives.

The content standard (data analysis and probability) recognizes the importance of having all students develop an awareness of the concepts and processes of data analysis and probability. It is essential that students learn that data analysis and probability is more than reading and interpreting graphs, but that it is also an effective tool for solving problems. The study of data analysis and probability highlights the importance of questioning, conjecturing and searching for relationships when formulating and solving real-world problems.

Students at all grade levels begin to understand that data come in various forms and in different quantities, requiring different treatment and organization to extrapolate meaningful information. Because of the prevalence of statistical data, summaries, graphs and probabilistic problems in our everyday lives and at work, it is important that data analysis and probability be given a more prominent place in the K–12 mathematics curriculum.

The three articles that follow relate to the data analysis and probability content standard. The first article presents an example from everyday life—applying basic probability concepts in a practical setting. The second article discusses how teachers can teach the concept of a mean in a meaningful way. The third article presents an elementary mathematics research model that allows the students to begin early to collect, organize and describe data. It is founded on problem-finding and problem-solving behaviours and is designed to support the development of high-level thinking as students' thoughts diverge and converge throughout the research process.

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Abul-Wefa (940–998)

The Persian mathematician credited for his improvements in trigonometry posed the following problem:

Two of three congruent squares are to be cut into eight pieces so that, together with the third square, they can be arranged into one larger square.

The Probability of Winning a Lotto Jackpot Twice

Emeric T. Noone

Studying state lotto games allows students to apply basic probability concepts in a practical setting. Students enjoy the discussion and become enthusiastic about investigating the probabilities of winning the jackpot. The probability of winning the jackpot in one play of a state lotto game is quite easy to compute. A more interesting problem is computing the probability of winning it twice. Some state lotteries have produced two-time winners of large jackpots. For example, an Ohio man won $4.3 million in August 1990, then won $12 million in March 1991. This article addresses the probability of winning twice.

A typical state lotto game is played by selecting one six-number combination from among the integers 1, 2, 3, . . . , 44. If the six-number combination selected matches the winning six-number combination that is randomly chosen, the player wins the jackpot. The number of six-number combinations possible in a play of the game is computed by using the formula for a combination of N objects taken r at a time. For the typical game that has been described, the formula yielded the following results, which can easily be computed with a hand-held calculator:

$$C(N, r) = \frac{N!}{r!(N-r)!}$$

$$C(44, 6) = \frac{44!}{6!38!}$$

$$C(44, 6) = 7{,}059{,}052$$

A total of 7,059,052 different six-number combinations are possible in one play of the game. Consequently, the probability of winning in one play of the game is

$$\frac{1}{7{,}059{,}052}.$$

Often when students are asked to compute the probability of winning the lotto game twice, they simply compute the following product, since the probability of success in two independent trials is the product of the probabilities of success in each trial.

$$\left(\frac{1}{7{,}059{,}052}\right)\left(\frac{1}{7{,}059{,}052}\right) \approx 2.0068 \times 10^{-14}$$

The likelihood of such events as winning a lotto game twice is frequently expressed as odds against the event's occurring. The odds against the event of winning the lotto twice are the ratio of the probability of not winning, $1 - 2.0068 \times 10^{-14}$, to the probability of winning, 2.0068×10^{-14}. If students use that reasoning to compute the probability of winning twice, then the odds against winning the jackpot twice are approximately 49,830,576,040,000:1. What is wrong with that reasoning? Nothing is wrong if the game is played exactly two times. The answer 2.0068×10^{-14} would be a reasonable estimate of the probability of winning twice, and 49,830,576,040,000:1 would be a reasonable approximation of the odds against winning twice.

However, what would be the effect on the probability of winning twice if the game was played repeatedly over a longer period of time and multiple six-number combinations were played each time? Two-time winners of large jackpots had most likely played the game for an extended time and had played more than one six-number combination per play.

For illustrative purposes, suppose that a player plays the game twice a week for four weeks and plays five different six-number combinations each time. What is the probability that the player will win the jackpot twice? In that situation, eight independent plays of the game occur, with the probability of winning on any given play of

$$\frac{5}{7{,}059{,}052}.$$

The number of successes, x, in n trials is a random variable X, which follows the binomial distribution denoted by $X \sim Bi(n, p)$, where p is the probability of success on a given trial. The probability of x successes in n trials is given by $P(X = x) = C(n, x)p^x \bullet (1-p)^{n-x}$. Hence, the probability of winning twice on eight independent plays of the lotto game when playing five six-number combinations each play is given by

$$P(X = 2) = C(8, 2)\left(\frac{5}{7{,}059{,}052}\right)^2\left(\frac{7{,}059{,}047}{7{,}059{,}052}\right)^6$$

$$\approx 1.40476 \times 10^{-11}.$$

Given the probability $P(X = 2) \approx 1.40476 \times 10^{-11}$, the odds against winning twice under these conditions are approximately 71,186,537,200:1. When the number of plays of the game is increased and the number of combinations played on each play of the game is increased, the probability of winning twice is enhanced. Although still bordering on the miraculous, the odds against winning twice are not quite as astronomical.

It is interesting to examine the effects on the probability of winning twice when the number of plays is increased. Table 1 shows the binomial probabilities of winning the jackpot twice when the number of independent plays, *n*, is increased significantly. The probabilities were computed using the formula $P(X = x) = C(n, x)p^x(1 - p)^{n-x}$ and a hand-held calculator.

The random variable $X \sim Bi(n, p)$ where *n* is large and *p* is small tends to behave like a random variable *Y*, which follows the Poisson distribution denoted by $Y \sim Po(m)$, where $m = np$. The probability that the random variable *Y* assumes the values $x = 0, 1, 2, \ldots$ is given by $P(Y = x) = e^{-m} m^x/x!$ The Poisson distribution $Y \sim Po(m)$ makes the computations easier and less tedious. For example, if $p = 5/7{,}052{,}059$ and $n = 100$, then

$$m = 100 \left(\frac{5}{7{,}052{,}059}\right)$$
$$= .00007.$$

Therefore, the probability of winning twice in 100 plays is

$$P(Y = 2) = e^{-.00007} \left(\frac{.00007^2}{2!}\right)$$

so $P(Y = 2) = 2.4498 \times 10^{-9}$.

Table 1 shows the Poisson probabilities for *n* independent plays of the game when playing five different six-number combinations each play. The probabilities are close to those of the binomial probabilities.

A total of 961.54 years would be needed to complete 100,000 plays of a lotto game when playing twice a week. If a player purchases five combinations on each play and each combination costs \$1, then the total cost would be \$500,000.

Using $Y \sim Po(m)$, one can investigate the probability of winning twice as the number of independent plays, *n*, approaches a much larger number and five different six-number combinations are played each time. If $n = 1{,}000{,}000$, then $P(Y = 2) = .12370$. The odds against winning would be 7.1:1. These odds are not too bad. However, if one played this version of the lotto game twice a week, 9,615.38 years would be needed to reach 1,000,000 plays. And at a \$1 per combination, the cost would be \$5,000,000.

Conclusion

State lotto games offer a wealth of problems and exercises for classroom activities. Students can create their own lotto games and study the probabilities associated with the game. For example, suppose that each student selects one three-number combination from among the integers 1, 2, 3, . . ., 21. The winning three-number combination is randomly chosen by drawing three slips of paper from among twenty-one slips numbered 1 to 21. Any student matching the winning combination wins the jackpot—perhaps a bag of M & M's. Before actually playing the game, the class can compute the probability of a given student's winning if the game is played repeatedly. Another interesting problem for students to explore is the likelihood that some student in the class wins on one play of the game. Also, they can compute the probability of winning twice if the game is played repeatedly. The students come away from this

Table 1
Comparison of Binomial and Poisson Probabilities

n	Binomial Probabilities of Winning Twice	Odds Against Winning Twice	Poisson Probabilities of Winning Twice
1,000	2.5043×10^{-7}	3,993,131:1	2.5124×10^{-7}
10,000	2.4906×10^{-5}	40,152:1	2.4956×10^{-5}
100,000	2.3369×10^{-3}	427:1	2.3349×10^{-3}

exercise with a much greater understanding of the remote chance that anyone has of winning a lotto game.

Yet another intriguing facet to study is the mathematical expectations of the random variables $X \sim Bi(n, p)$ and $Y \sim Po(m)$. The mathematical expectations $E(X) = np$ and $E(Y) = m$ are infinitesimally small and reveal further the futility of gambling.

Students can also simulate lotto games on the computer to determine empirically the probability of winning and compare the outcome with the theoretical probability of winning. As one can see, the possibilities for problems and exercises are numerous.

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Leonardo da Pisa (c.1175–c. 1250)

This Italian mathematician, who became famous under the name of Fibonacci, posed the following problem in his book *Liber Abaci*:

Determine five weights to be used to weigh objects, ranging from 1 kg to 30 kg, with the weight of the objects being whole numbers. A balance scale is being used. What must the weight of the five different weights be?

Teaching the Mean Meaningfully

John C. Uccellini

Ask a group of middle school students what the average (mean) of two, eight, four, six, three and seven is, and they will probably give the answer five. Ask these same students what the number five represents in relationship to the six numbers given and the response usually heard is an explanation of the algorithm. "Add them up and divide by the number of them that you have." The response is no different if the problem is given in a real-life context. For example, the foregoing six numbers could represent the number of pencils that six students have in their desks. In either situation, the almost-universal response of students when questioned about the meaning of five from the "add and divide" algorithm demonstrates that students have not gained a conceptual understanding of this basic statistic. This same phenomenon exists throughout mathematics and is demonstrated whenever students try to explain subtraction by describing the vertical algorithm or the Pythagorean theorem by stating that $c^2 = a^2 + b^2$. Through the use of simple manipulative activities and graphing, however, middle school students can be taught the mean meaningfully.

Equal Distribution

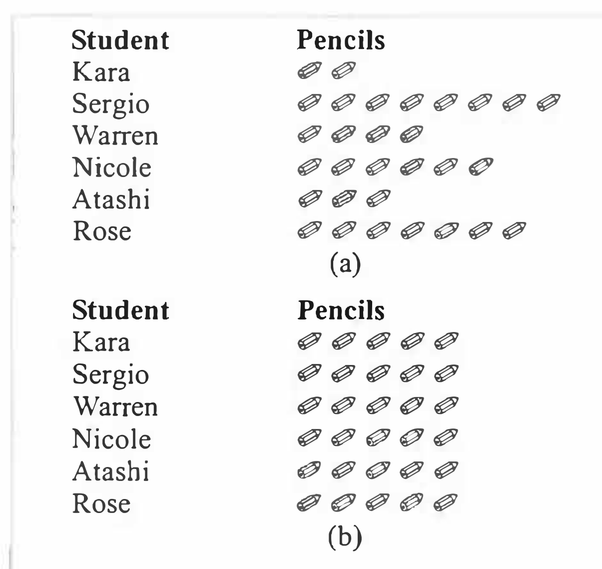
The mean can be interpreted in two different ways for a given set of data. The first interpretation relates the mean to an equal distribution of data items, whereas the second interpretation relates the mean to a balancing point of the data. Of the two interpretations, the concept of equal distribution is well known to both teachers and students but is rarely connected to the mean. In the context of the foregoing problem, the number five represents the equal distribution of the thirty pencils among the six students. In other words, the number five represents the number of pencils each student would have if the pencils were distributed equally to all six students. This sense of mean is easy for students to comprehend, since equal distribution of candy, crayons, baseball cards and so on is a common occurrence in their lives.

Before teaching the add-and-divide algorithm to students, their rich experiential understanding of equal distribution should be maximized. As in the pencil problem, students should be given thirty pencils, or some other appropriate manipulative, and asked to line them up on their desks in groups corresponding to the number of pencils the six students have in the problem (see Figure 1a). When they have arranged them correctly, the students can then move the pencils or markers from the larger groups to the smaller groups until an equal distribution has been made (see Figure 1b). At this point, the solution, five, should be identified as the mean. Begin by first introducing problems in which the mean is a natural number. Once students are comfortable with the concept of equal distribution, introduce problems in which the mean is not a natural number. Consider the following:

> Six children counted the number of chocolate bars that they won at the school fair. They had won two, three, two, six, three and five bars, respectively. What is the mean (average) number of chocolate bars that they won?

Figure 1

Pencils distributed among students. The original distribution of pencils is shown in (a) and an equal distribution of pencils is shown in (b).



By using two linking cubes to represent each chocolate bar, ask the students to solve for the mean by creating an equal distribution of the objects (see Figure 2). The solution, three and one-half chocolate bars per child, is easily found once the three remaining bars are broken in half and distributed among the six children.

By exploring sample problems involving objects, children gain experience in dividing the objects into halves, thirds, fourths and so on, which (1) builds on their prior experiences to further their understanding of the mean and (2) reinforces their use and understanding of fractions. It is important also to include examples in which the mean is not a natural number and the context of the problem concerns objects that cannot physically be broken apart. Consider the following problem:

> Six children were asked how many brothers and sisters they had, and the following data were collected from the children: two, one, zero, three, six and one. What is the mean (average) number of brothers and sisters of these children?

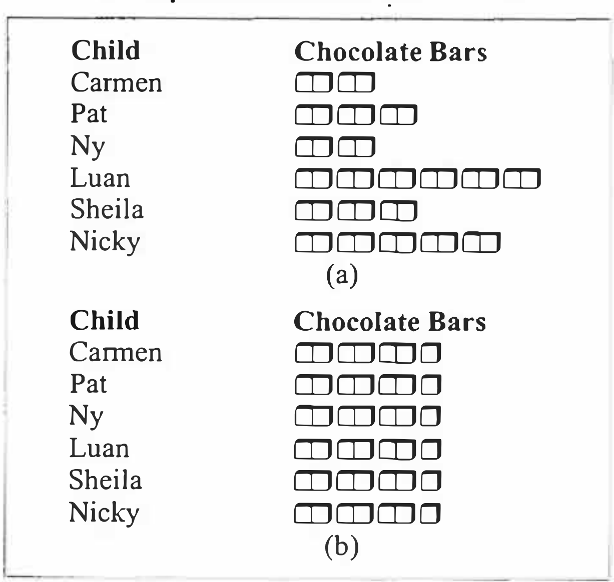
When students attempt to solve this problem by manipulating objects representing the 13 siblings to form an equal distribution, they are faced with two dilemmas (see Figure 3): (1) how to divide the extra sibling among the six students, and (2) how to interpret the solution of the problem, which is two and one-sixth.

The students should be encouraged to discuss the difference between those examples in which the physical division of the objects is possible and feasible and those in which it is not. This discussion should also engage the students in attempting to interpret how two and one-sixth is an appropriate description of the mean number of siblings, even though it is not physically possible to achieve. Students should realize that the mean is a measure or a number that summarizes a set of data. Even in the pencil problem, wherein an equal distribution is possible, no student actually has five pencils; unless they mutually agree to share their pencils, each of them will not have the mean number of pencils.

Once the class demonstrates a general understanding of the equal-distribution sense of the mean, whole-class data should be collected and the mean found. At this point, the need for the algorithm will become apparent to the students as they try to distribute equally large numbers of data items or to work with data that involve large numbers. For example, students could be asked to find the mean distance that the students live from school or the mean height of all the students in the class. If members of the class have not already come up with the add-and-divide algorithm, it should be introduced. The algorithm is faster and more practical to use, and, more important, the groundwork has been laid for the solution found by employing the algorithm to have real meaning and not just be a number.

Figure 2

Chocolate bars distributed among the children. The original distribution of bars is shown in (a), whereas (b) illustrates the equal distribution of the bars.

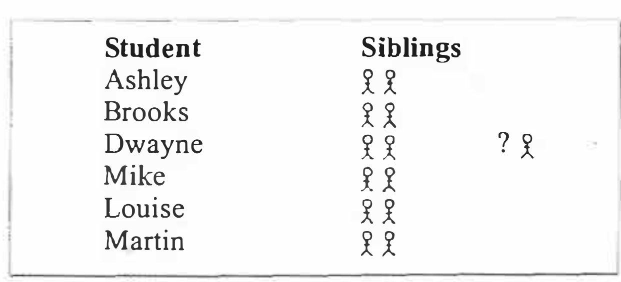


The Balancing Point

The second interpretation of the mean, that of being the "balancing point" of a set of data, is not well known and is rarely taught to students. Consider the problem at the beginning of this article. The data can be placed on a number line with "X" representing each of the six numbers, as shown in Figure 4. Knowing that the mean for these data items is five, the respective distances between the other data items and five can be found and displayed below the number line as shown in Figure 4. First take the data items

Figure 3

The dilemma of equally distributing the 13th sibling



less than five and find their respective distances from five. Find the sum of these distances. Repeat this procedure for those data items greater than five. This equality in the sum of the distances of the data items above and below the mean leads to the interpretation of the mean as the balancing point of the data. When students understand this balancing-point interpretation of the mean, they begin to understand how the mean can be viewed as the centre of the data. The mean can next be discussed as being one of the measures of central tendency that statisticians use to describe sets of data along with the median and the mode.

Figure 4
Number line showing 5 as the balancing point of the data

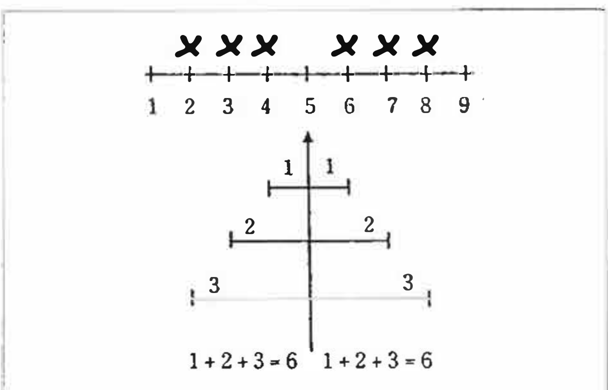
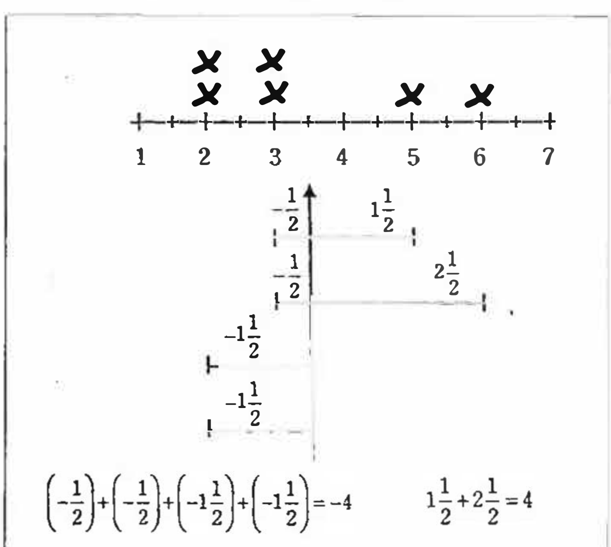


Figure 5
Number line showing that the sums of the differences from the mean are additive inverses, or opposites



Having students construct number lines to illustrate the balancing-point interpretation of the mean can also reinforce their understanding of positive and negative numbers and additive inverses, or opposites. Figure 5 shows the data from the chocolate-bar problem. Instead of calculating the respective distances of each data item from the mean, students calculate the difference between the value of each data item and that of the mean. In using the formula $D - M$, where D represents the value of each data item and M represents the value of the mean, data items larger that the mean yield positive differences and those less than the mean yield negative differences. The sum of the differences below the mean is the additive inverse, or opposite, of the sum of the differences above the mean. This situation always holds true and reinforces the concept of the mean as being a balancing point, or centre, of the data.

Teaching for understanding is difficult. In the instance of the mean, in which a simple algorithm is widely known, teachers frequently assume students' understanding of the mean in relation to a data set when students demonstrate a mastery of the algorithm. This article presents two conceptual interpretations of the mean and simple manipulative and graphing activities that can help students form a deeper understanding of this important statistic. In a world that overwhelms us with quantitative information, it is important that students be taught the mean meaningfully.

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Problem Solving: Dealing with Data in the Elementary School

Harry Bohan, Beverly Irby and Dolly Vogel

Standard 11 of the K–4 recommendations of the National Council of Teachers of Mathematic's *Curriculum and Evaluation Standards for School Mathematics* and Standard 10 of the Grades 5–8 portion of this document suggest that students be given opportunities to

- collect, organize and describe data;
- formulate and solve problems that involve collecting and organizing data; and
- develop an appreciation for statistical methods as a powerful means for decision making (1989, 54, 105).

A basic assumption of the standards document is that students will learn better through problem-solving situations that involve them doing mathematics rather than having it done to them—so that they become producers of knowledge rather than merely consumers. The Elementary Mathematics Research Model furnishes a vehicle for problem solving through real data collection and analysis.

The Elementary Mathematics Research Model

To incorporate a research component into the curriculum, two aspects must be considered. First, students need a research model that is easy to understand and apply. Second, students must have an understanding of some basic statistical tools, such as mean, median, mode and range. At higher grade levels, measures of dispersion other than the range might also be included. Rather than being taught as isolated topics, the statistical tools are used in applying the research model to real situations. This concept is supported by Moore (1990), who emphasizes the need for statistics to be couched in realistic settings.

Getting Started

The Elementary Mathematics Research Model (Irby and Bohan 1991) has students move through seven steps to produce knowledge through mathematics. See Figure 1. In step 1 students must attempt to identify a problem. For the students to become involved and have ownership in the project, the first item of business is to let them think—of things that they would like to know, of some questions that they would like to answer or of some problems that they have observed in the school or community. During this brainstorming session, establish a rule that no one is to judge the thoughts of another. Let the ideas come freely. If someone repeats an idea already on the chalkboard, go ahead and write it. Never say, "We already said that," as this type of response stifles creative thinking. The job of the teacher is to see that a risk-free environment is maintained. After brainstorming, let the students take one of the generated ideas and work through the remaining steps in the design.

Step 2 is a natural outcome of step 1. One of the issues from the brainstorming session is chosen, a problem to be solved is developed and a research question is stated. The following is a problem formulated from a brainstorming session in a sixth-grade class:

> The students were concerned with the amount of garbage produced in the school cafeteria and its impact on the environment (the problem). The research questions was, What part of the garbage in our school cafeteria is recyclable?

In step 3 students hypothesize the expected outcome of the research. The teacher might ask, "What do you think will be the outcome of your research or investigation?" Students should be accustomed to hypothesizing from science classes. With regard to the first question, the students might answer, "We believe that half of the waste is recyclable."

Step 4 will find students developing a plan for how to test the hypothesis and answer the question. The following items will need to be considered in developing the plan: (a) permission—who will give us permission, the principal, the cafeteria supervisor, the maintenance director or others? (b) courtesy—when can we conveniently discuss this project with the cafeteria management? (c) time—how much time can we spend on this investigation, when should we do this project each day, how long do we think it will take to gather all the data? (d) money—will it cost anything, how can we get the money, do we need to write a grant proposal to request the money through the principal or the PTA? and (e) safety—what measures must we take to ensure safety, for instance, gloves and masks?

The students will need to develop an exact plan to address these concerns. In the process they may discover subquestions related to the original research question, such as, Which group is more environmentally aware—fifth or sixth graders? On which day do most students bring lunches? What buying trends should be observed by the cafeteria management on the basis of analysis of food in the garbage? Each question may call for different statistical treatments.

The teacher may have different groups in the class work on each related question, so that at the end of the research all questions will have been answered. Each subquestion will prompt development of a specific plan:

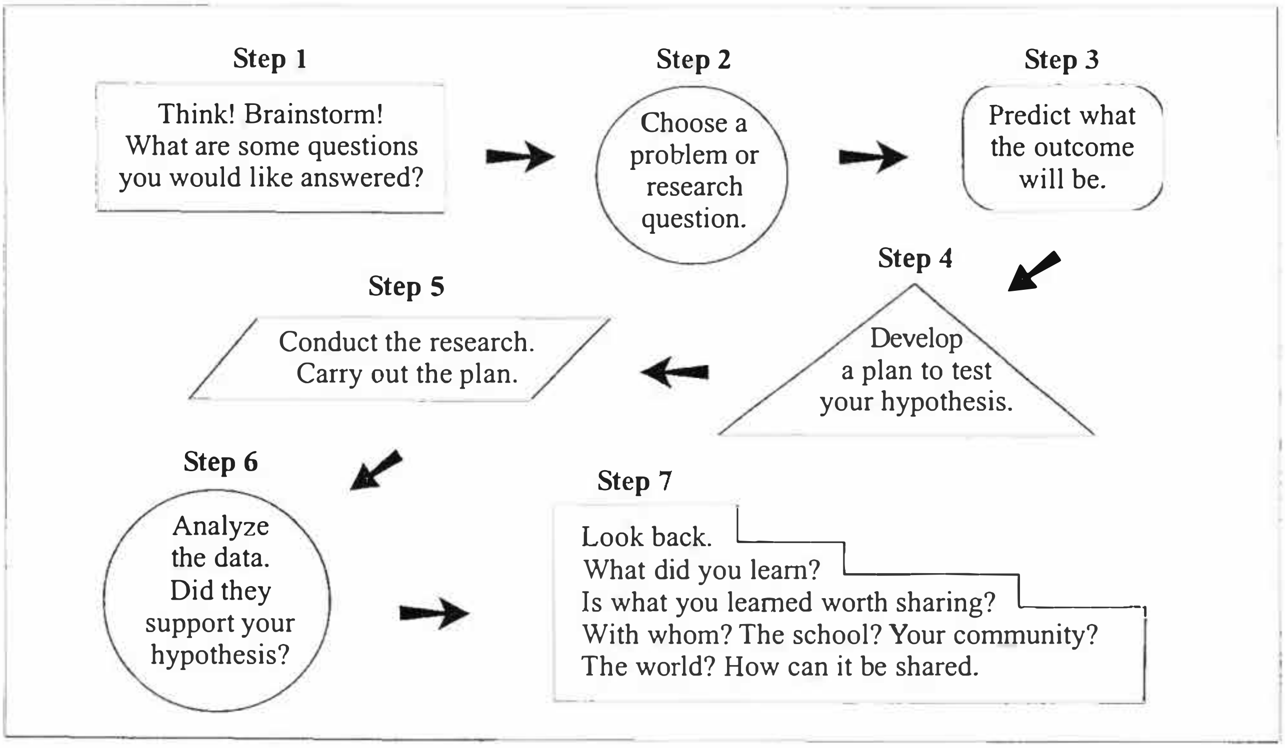
> We will have our study last for three weeks, giving us fifteen opportunities to collect data. We will check the garbage every day and request that it not be thrown out until we do so. We will request the help of our fellow students when throwing out their garbage in the cafeteria by requesting that they separate it into six different cans that are clearly marked—uneaten foods, partially eaten food, Styrofoam, paper, plastic and aluminum. We will weigh the amount of each can and keep the records each day. The number of aluminum cans will be counted.

As the students determine how they will gather the data, they need to determine what variables are involved in the research study. In this example, they might determine that the weight of each individual can be one variable, the length of the study might be another and so on.

Step 5 is "carry our the actual plan." During the time the data are being collected, discuss the ways in which the students might report the findings. Graphs should certainly be discussed as a possibility, as should types of graphs best used for various purposes (see, for example, Curcio [1981]). At this point, the need for statistical measures to describe the data becomes apparent. For example, since this study is to last 15 days, it is not probable that the same number of aluminum cans would be collected each day. How can the number of cans collected daily be described without having to list 15 numbers?

Figure 1

The Elementary Mathematics Research Model



Developing Measure of Central Tendency

The Mean

To teach the concept of mean, pose a situation for students in which 80 cans are collected one day and 60 the next. Have students use a metrestick and adding-machine tape to represent these numbers by cutting off pieces 80 and 60 centimetres in length. This tactic gives students a physical representation of their two-day collections. Have students attach the tapes end-to-end. Hold up the combined tapes and ask, "What does this paper represent?" (the total number of cans collected for two days). "Use this paper and the metrestick to decide the total number of cans you have collected."

"Suppose that on two other days you collected the total number of cans represented by the combined tapes. However, an equal number of cans was collected each day. Use the combined tape to decide what that number was." Since this paper represents two days of collecting, the combined tape can be folded into two equal parts and compared with the metrestick to find the number. Once the number 70 has been determined, define this number as the mean. Repeat this activity with different numbers of cans and days; this extension is necessary, as otherwise some students form the misconception that we always divide by 2 when finding the mean.

Present various situations in which students try to predict what would happen to the mean, if, on the next day, a greater or smaller number of cans was collected. Predictions can be investigated by using adding-machine tapes. The conceptual work done with the tape can readily be connected to the symbolic procedure for finding the mean. Connecting the tapes represents finding the sum of the numbers, and folding the combined tapes represents dividing the total into equal parts. The number of parts into which the combined tape is folded is determined by the original number of pieces of tape.

Figure 2
A Table of 15 Number Cards

				80				
	72		76	80	84	85		
61	72	74	76	80	84	85	87	91

Demonstrate the need for other measures of central tendency by pointing out the main weakness of the mean—the extent to which its value can be affected by extreme scores. This weakness can be demonstrated within the framework of activities discussed earlier by showing the effects that a day when no cans were collected would have on the mean of three days of collection averaging 80 cans per day. The median and mode can then be presented as different measures of central tendency that minimize the effect of extreme scores.

The Mode

To teach the meaning of the mode, have students write 15 numbers on index cards and place them in a box. For example, 76, 80, 84, 72, 85, 80, 74, 61, 72, 84, 76, 80, 91, 87 and 85. Have students pull a card at random from the box and place it on a chart. A second card is then extracted, and the question asked, "Is this number greater than, less than or equal to the first number?" This second card is placed on the chart to the right, to the left or above the first card depending on whether the number on it is, respectively, greater than, less than or equal to the number on the first card. (See Figure 2.)

The students continue to pull cards, asking the same question over and over until all cards are arranged on the chart in order, left to right, from smallest to largest. Looking at all the cards on the chart, ask, "What number appeared the greatest number of times?" After identifying the tallest column, define the number of cards in that column as the *mode.*

The Median

To teach the concept of median, have students use the 15 numbers they have placed on the chart. Ask, "Where have you heard the word *median* used before?" (The median of the highway is the part that divides the highway into two equal parts.) "In mathematics the word *median* is used to tell us something about a set of numbers. What do you think it tells us? (It is the point that separates a set of numbers into two equivalent subsets.) Have students work with a partner to find the median of the set of numbers represented by the cards on the chart. One way is to begin removing cards from either end of the set simultaneously, one with each hand. This process is continued until only a single card remains. Have groups share their method with the class. Identify the number on the middle card as the median.

"If we find the median by eliminating cards from each end, will we always get to a point where a single card remains?" After getting a consensus that we would not, discuss the conditions under which this outcome would or would not occur, capitalizing on the opportunity to review the concept of even and odd numbers. Next place an even number of cards in the box, place them on the chart in order as indicated previously and eliminate cards simultaneously from either end until only two cards remain. Give students an opportunity to discuss how the median might be identified. Try to get the class to agree that the best solution would be to call the point halfway between the two remaining cards the median. Introduce situations in which

- the median is not a whole number, as when the two remaining cards contain such consecutive numbers as 87 and 88 (median 87.5); and
- the median is a whole number but not a number on one of the cards; for example, the two remaining cards have such whole numbers as 84 and 88 (median 86).

In either case, the median is the mean of the two remaining numbers.

Dealing with the Data

In step 6, at the end of the three weeks, analyze the data. The question to be answered is, "Did the test support our hypothesis?" The data will be analyzed on the basis of the statistical tools previously developed.

As they look back in step 7, students should ask such questions as the following:

- What did we learn?
- Will our findings contribute to our school, our community or our world?
- How can we share our findings with others?
- If we repeated this experiment during a different three weeks, would we expect the same results? Why or why not?
- Who might be interested in our results?"

The teacher should assist the students in presenting the findings to a particular audience. In the example presented here, the students presented the information to the fifth-grade students, the cafeteria workers and the teachers. A formal presentation with charts and graphs is important in showing students that research is valuable when it can be related to the real world and put into practice. Additionally, it emphasizes the need to communicate mathematically.

A Real-Life Research Winner

Using the Elementary Mathematics Research Model, Dolly Vogel, a Grade 6 mathematics teacher at Houser Intermediate School, Conroe, Texas, presented her students with the opportunity to conduct research. She encouraged her students to focus their studies on science and mathematics. All studies had to be submitted with a written report in research format and had to include statistical data that were graphically displayed. The three studies submitted for review were as follows:

Group 1: Which pollutants are most harmful (Survey research)
Group 2: Does life exist on other planets? (Survey research)
Group 3: How much trash can the students at Houser Intermediate School eliminate by recycling aluminum and Styrofoam? (Observational research)

Significant findings were reported by each group, with the findings from group 3's research of particular interest. The students found that, by recycling only the aluminum and Styrofoam, the school's garbage could be cut in half. As it turned out, the research was award-winning, with the school receiving a set of statistical software from the American Statistical Association, which sponsored the competition. (For information on this national contest, write to the American Statistical Association, 1429 Duke Street, Alexandria, VA 22314-3402.) Mrs. Vogel and her students are to be commended for their award-winning efforts.

Conclusion

The Elementary Mathematics Research Model allows the students to begin early to collect, organize and describe data. The model is founded on problem-finding and problem-solving behaviours and is designed to support the development of higher-level thinking as students' thoughts diverge and converge throughout the research process. Additionally, the appreciation for statistical methods used in problem solving can be emphasized. Zawojewski (1988) reported that when students applied memorized algorithms for finding measures of central tendency in a role manner, they tended to make predictable errors that they did not tend to make when these measures were presented in the context of real-world situations.

The research studies may be completed as group or individual projects. As studies are developed they may be concentrated in the community or in the school. Although in some instances the teacher's assistance may be required, research topics should preferably be chosen by students. The teacher's responsibility should be to assist with the design of the study so that the students will be able to use statistical treatments, tools and terms in the analysis of the data collected in the study. The format for a report and record keeping throughout the project is open.

The final thought to leave with students is that they can be researchers and products of new information and that new knowledge can be produced and communicated through mathematics. Their findings may contribute to the knowledge base of the class, the school, the community or society as a whole. Their findings may affect their school or their world in a very positive way.

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Johannes Buteo (c. 1525)

This French mathematician wrote the following in his book *Logistica*:

If the price of 9 apples, reduced by the price of one pear, is 13 Dinar, and the price of the 15 pears, reduced by the price of one apple, is 6 Dinar, what is the price of an apple and a pear?

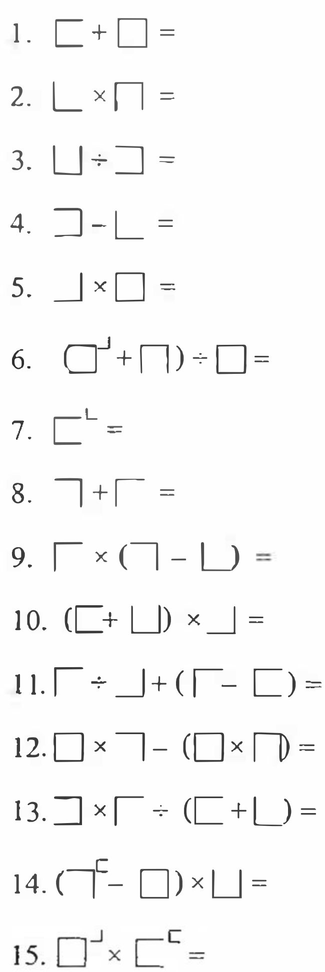
Calendar Math

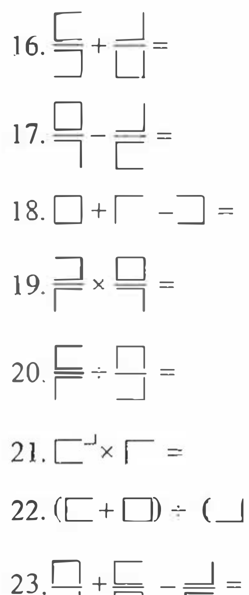
Art Jorgensen

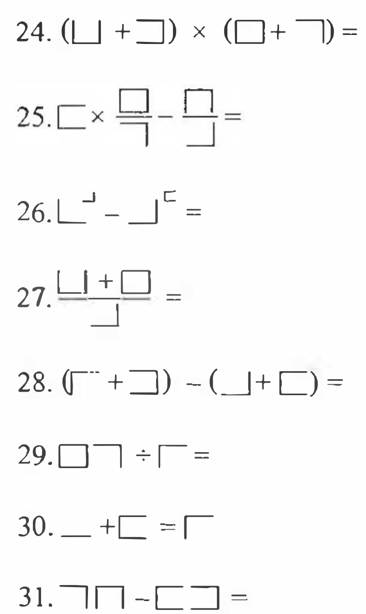
Deciphering codes is an excellent exercise in mathematical reasoning, sound thinking and solid logic. Almost any code will provide this practice. Here is a simple one that can be adopted and reused as student abilities change or increase.

Use the table below to solve the following mathematical exercises:

2	9	4
7	5	3
6	1	8







Answers

1. 8	9. 16	17. $^{1}/_{6}$	24. 176
2. 4	10. 24	18. 6	25. 2
3. $^{9}/_{7}$ or $1^{2}/_{7}$	11. 9	19. $^{35}/_{48}$	26. 8
4. 3	12. 25	20. $^{3}/_{4}$	27. 7
5. 10	13. 8	21. 72	28. 10
6. 10	14. 1,899	22. $^{8}/_{11}$	29. 7
7. 81	15. 675	23. $^{13}/_{30}$	30. 5
8. 14	16. $^{41}/_{63}$		31. 24

Activities for the Middle School Math Classroom: Games and Problem Solving

A. Craig Loewen

There is an unmistakable similarity and strong link between problem solving and mathematical games. First let's explore problem solving. Consider this definition of problem solving: Problem solving is a process that occurs when a set of conditions and a goal are given, but the solver must provide or develop the means to achieve the goal. In other words, problem solving is an active task that has both a starting and (at least one) end point.

Process

The whole notion of problem solving as process is extremely important because we must teach our students that most problems do not have an obvious or necessarily quick solution. Indeed most problems in real life are not resolved quickly at all. Wouldn't it be nice if they were? Many of our students become handicapped by the two-minute rule: they say or think, "If I can't solve it in the first two minutes, I will never solve it," and they concede defeat too quickly. Unfortunately we often encourage the two-minute rule through examples we present (always solved correctly within the time limits) and our evaluation practices (20-minute quizzes comprising 10 questions). Real problem solving takes and deserves adequate time. In fact, we may have to accept that real problem-solving abilities can not realistically be measured in a typical testing format; it simply contradicts the process dimension of the activity.

Conditions and Goal

Problems always come with a set of conditions. Sometimes the conditions are obvious and straightforward, and sometimes we have to collect our own data, sort through the information given to weed out extraneous statements and so on. The goal in a problem becomes the resolution of these conditions. When we have somehow reconciled all of the information given, collected or found (and thus met the task presented), we consider the problem to be solved. Typically the conditions in a problem are given as a series of statements, and the goal is given as a question that follows those statements. We can dramatically alter the complexity and interest of problems by altering the number, nature and format of both the conditions and goal.

Developed Means

If it is intuitively obvious to the solver how the conditions and the goal(s) may be reconciled; that is, how the problem is solved, then it is probably fair to say the given activity is merely a task (in classroom terms, a drill-and-practice exercise) and not a problem at all. In other words, in real problem solving we expect to have to wrestle a little. We expect the activity to be challenging and fraught with frustration, some disappointments and failures and, of course, some successes. The nature of problem solving is that the solver brings forward his or her own insights, skills and strategies into the process while developing new methods, thus experiencing a number of small failures and victories before the ultimate success (a solution) is finally attained.

The above elements constitute a basic description of problem solving, but let's not forget that problem solving is ultimately a complex task. For example, remember that the solver must also be motivated to solve the problem. The problem must have some relevance or inherent interest about it before the solver will be willing to engage it. Let's also not forget that another higher level of mental processing is associated with problem solving: the ability to monitor progress, the ability to evaluate the effectiveness of strategies, the flexibility to modify the chosen strategies that have proven to be ineffective and so on. While it is not the purpose of these games specifically to address these higher mental abilities, the good news is that these metacognitive talents can be encouraged and can be (should be) actively taught in our classrooms.

But what does all this have to do with games in the math classroom? Simply put, a game is in fact a type of problem. Like a problem, a significant process is involved in playing the game, and like a problem a game usually requires a strategy and a means to achieve the goal. It is fair to think of the rules of the game as the conditions of a problem, and the goal of the game is usually how the game is won or finished. Sometimes the goal is to beat your opponent and sometimes it is just to work well with your team or to make it further in the game than the last time. A game is a complex problem that often contains many smaller problems embedded within it. The advantage a game has over more typical problem-solving activities falls in the motivation element inherent within it. My students used to groan when I announced we would try some problems, yet they would cheer when I announced I had a game for them instead.

Problems come in many different forms, and games represent one motivating way to engage our students in some significant and challenging problem-solving activities.

Here are some suggestions and considerations for integrating games in your math classroom:

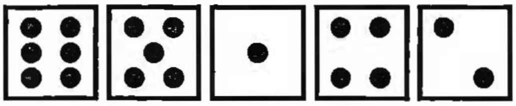
- Where possible emphasize the mathematics vocabulary used in any game. To work with others, we need to communicate clearly, and students will need to use appropriate mathematical terms to communicate effectively during all classroom activities.
- Try to collect games that can be easily adjusted to make them more or less difficult. Highly flexible games can also let you apply a familiar game structure to an alternative math concept. Games that are flexible in this way help reduce the amount of time spent introducing a new game to the class.
- Recognize that games are appropriate for all grade levels; young children and adults alike enjoy these types of activities. Students may have some suggestions for how certain games can be modified or adapted, consider getting your students to modify, develop and create their own games, too.

Five Dice

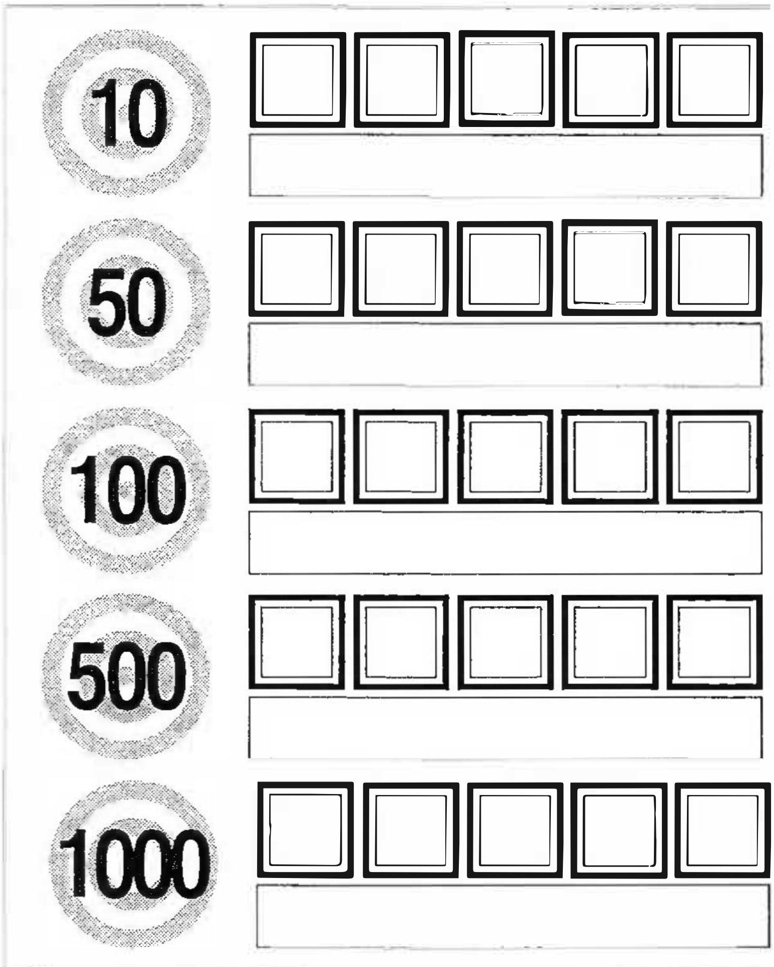
Objective: Solve problems involving multiple steps and multiple operations (Alberta mathematics program of studies, Number [Number Operations], Grade 5, Outcome 13)
Materials: Five regular six-sided dice for each team or player, pencil, paper and calculator.
Players: Two or more

Rules

1. Each player or team rolls five dice and records the values rolled.
2. These values are now used to construct an equation with the result as close to the target as possible using the rules stated below. Assume we rolled the values shown:



   a. Numbers may be used to specify place values. For example, using the digits above, we could make the values 56, 16 or even 156 or 651 and so on.
   b. Brackets may be used wherever desirable.
   c. Exponents are not permitted.
   d. The values may be used to create whole numbers only.
   e. A number may be used only once unless doubles are rolled.
   f. All numbers must be used.
3. Players roll the five dice for each new target number.
4. After completing each target, teams or players compare results to see which equation has a value closest to the target. The closest player scores one point for each target.
5. The team with the greatest point total at the end of the game wins.



Adaptations

1. Substitute 10-sided dice for the 6-sided dice.
2. Have players use the values to build equations with results between 0 and 100. The player who can build the greatest range wins.
3. Permit players to use exponents.

Factors

Objective: Distinguish among and find multiples, factors, composites and primes, using numbers 1 to 100 (Alberta mathematics program of studies, Number [Number Concepts], Grade 6, Outcome 3).
Materials: Factors game board, two crayons, calculator
Players: Two

Rules

1. The objective of this game is to shade the greatest number of spaces on the Factors game board as possible, which, when summed, have the highest total.
2. On a turn a player will select any available number (a number not already shaded) and will proceed to shade it as well as all remaining numbers that are a factor of his or her number. For example, if a player chooses the number 75, he or she could shade all of the following (assuming they have not already been shaded on an earlier turn or by another player): 75 1 3 5 15 25
3. Players alternate until each player has had three turns. At the end of the third turn, players take the calculator and add up the numbers they shaded.
4. If a player fails to shade a factor that he or she could have shaded, the opponent may point out the error and claim that number. If a player shades a number incorrectly, the opponent claims that number.
5. The player with the highest sum after three turns wins the game.

Adaptations

1. Instead of playing for the highest sum of shaded numbers, play to see who can shade the greatest number of spaces.
2. Allow players to use factors more than once. If a player selects 100, the number 100 is out of the game, but the factors (1, 2, 4, 5, 20, 25 and 50) can be used again.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Pattern Detective

Objective: Represent, visually, a pattern to clarify relationships and to verify predictions (Alberta mathematics program of studies, Patterns and Relations [Patterns], Grade 6, Outcome 1).
Materials: Pattern Detective game board (one per player)
Players: Two

Rules

1. To begin the game, each player places a series of letters arranged in a pattern in the boxes down the left side of his or her game board. Players may only use the letters A, B, C and D to construct their patterns, and letters must be arranged in a pattern. Examples:
   - A B C A B C A B C A B C A B C A B C A B
   - C A A C A A C A A C A A C A A C A A C A
   - D B D D B D D B D D B D D B D D B D D B
   - D C A B D C A B D C A B D C A B D C A B
2. Players now take turns guessing what letters are written in their opponent's boxes. Example: "Do you have a C in box 3?"
3. If a player guesses correctly, he or she gets to keep guessing until a mistake is made. After a mistake, play passes to the opponent.
4. The first player to correctly reveal his or her partner's entire pattern wins.

Adaptations

1. Include more players. Players drop out of the game as their patterns are uncovered by other players. Last player in the game is the winner.
2. Allow a greater range of letters or symbols for more complex patterns.
3. Use numbers instead of placing letters in the boxes. Players must provide to their opponent the first two numbers in their pattern at the start of the game. This allows for much more complex patterns, such as growth patterns:
   - 0, 1, 1, 2, 3, 5, 8, 13, 21, ...
   - 1, 2, 4, 7, 11, 16, 22, 29, 36, ...
4. Play cooperatively as teams rather than as individual players.

My Pattern	Correct	Incorrect Guesses
* 1 *	* 1 *	
* 2 *	* 2 *	
* 3 *	* 3 *	
* 4 *	* 4 *	
* 5 *	* 5 *	
* 6 *	* 6 *	
* 7 *	* 7 *	
* 8 *	* 8 *	
* 9 *	* 9 *	
* 10 *	* 10 *	
* 11 *	* 11 *	
* 12 *	* 12 *	
* 13 *	* 13 *	
* 14 *	* 14 *	
* 15 *	* 15 *	
* 16 *	* 16 *	
* 17 *	* 17 *	
* 18 *	* 18 *	
* 19 *	* 19 *	
* 20 *	* 20 *	

Triplet

Objective: Make a connection between the number of faces, for various dice, and the probability of a single event (Alberta mathematics program of studies, Statistics and Probability [Chance and Uncertainty], Grade 6, Outcome 11

Materials: A variety of dice with different numbers of faces (for example, 4, 6, 8, 10, 12, 20-sided dice), Triplet game board, two crayons.

Players: Two

Rules

1. On a turn, a player selects one die and rolls it. This player now colours a space in any one of the three game boards corresponding to the value rolled.
   Example: If the player selects the eight-sided die and rolls a seven, the player could cross off any one of the following:
   - The space labelled >6 (greater than 6) in the top board
   - Either the 7 or odd space in the middle board
   - Either the prime or <20 (less than 20) space in the bottom board
2. Play passes to the opponent.
3. A player may cross off only one space on a given turn. If there is no remaining space the player can claim, the player passes the turn.
4. The first player to cross off three spaces in a line (horizontal, vertical or diagonal) in any of the three game boards is the winner.

6	<4	11
16	10	>6
even	13	3

<3	20	7
12	odd	>10
9	6	4

18	1	prime
11	5	>2
<20	2	8

Adaptations

1. At the start of the game, have players select one game board as their own. Players now take turns rolling and racing to cross off all of the spaces in their respective game boards.
2. Use plastic bingo markers of two colours. Instead of shading spaces, use the bingo markers to show chosen spaces. Your opponent can claim that space back from you on a later roll by removing your marker and adding his or her own.
3. Imagine that the game boards are stacked one on top of the other and allow three in a row in three dimensions. For example, the even, odd and prime spaces would be three in a row.
4. Allow players to select two dice and add or subtract the values rolled.

Four in a Row

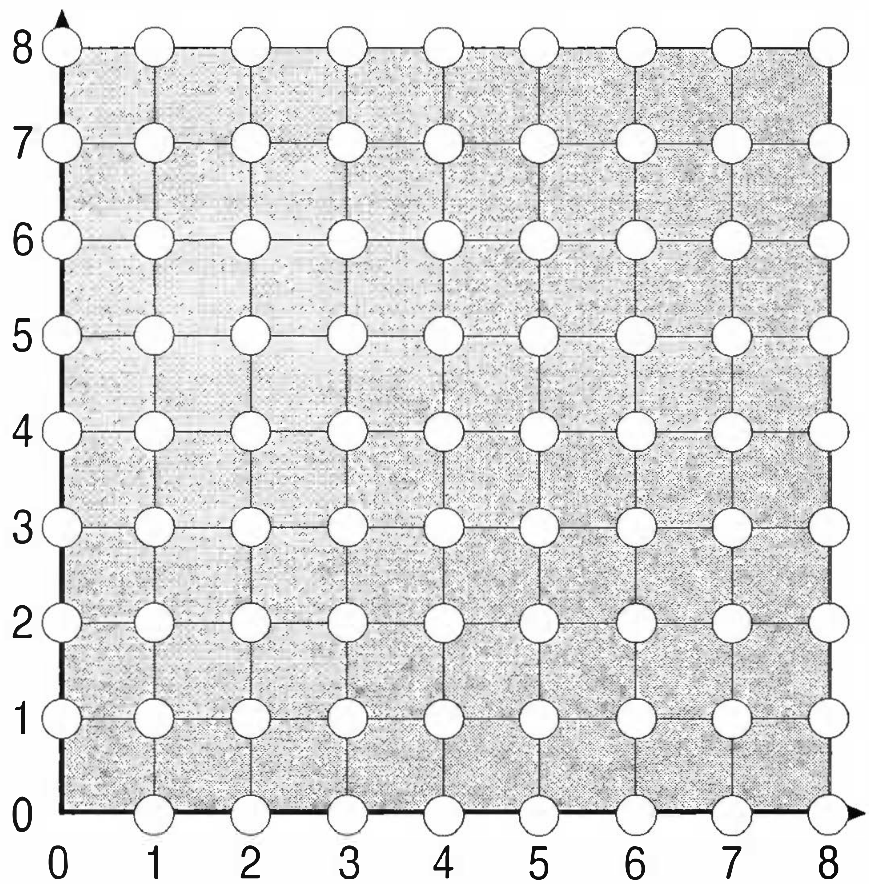
Objective: Graph relations, analyze the result and draw a conclusion from a pattern (Alberta mathematics program of studies, Patterns and Relations [Patterns], Grade 7, Outcome 3)
Materials: Four in a Row game board, two eight-sided dice (blue and red), two crayons
Players: Two

Rules

1. The objective of this game is to plot four points that fall in a straight line on a Cartesian coordinate system.
2. On a turn a player rolls both dice. The blue die represents the *x*-coordinate of a point, the red die the *y*-coordinate. The player may opt to plot the point as rolled, or may opt to alter either the *x*- or *y*-coordinate, but not both.
3. Either coordinate may be altered using the rules shown.
   Example: If the player rolls a three on a die, it could be used as a one (by selecting the first box: "decrease the coordinate by two"). Likewise, it could be used as a four or as a six.
4. The player now plots his or her point in his or her colour crayon. Players continue taking turns plotting points according to these rules until either player has positioned four points in a row horizontally, vertically or diagonally. This player is the winner.

Adaptations

1. Substitute or add other rules that may be used to alter coordinates.
2. Establish two sets of formulas for altering coordinates. One set applies to the *x*-coordinates only, the other to the *y*-coordinates. A player may select from either list, neither list or both lists on a turn.
3. Include two 4-, two 6- and two 8-sided dice. A player may select any combination of two dice to roll on a turn.
4. Players work cooperatively to try to colour all spaces in the fewest number of rolls.
5. Allow players to apply more than one rule on a turn.



INCREASE A COORDINATE BY 1	DECREASE A COORDINATE BY 2	DOUBLE A COORDINATE

Percentage Snap

Objective: Estimate and calculate percentages (Alberta mathematics program of studies, Number [Number Operations], Grade 7, Outcome 18)
Materials: Deck of cards (jack, queen, king removed, A=1), calculator
Players: Two

Rules

1. To begin the game, shuffle the cards and split them into two equal-sized piles. Each player takes one-half the deck and holds the cards face down in his or her hand.
2. At the same time each player turns over one card and places it on the top of the table.
3. Each player now estimates the percentage equivalent to the fraction formed by the two cards (to the nearest 10 per cent). For example, if the cards were a two and a six, the players would estimate a value as close as possible to 33.33 per cent (that is, 30 per cent).
4. The first player to call a value wins the two cards if there are no closer estimates. The second player must provide a closer estimate or concede the cards.
5. The closest estimate to the actual per cent (use a calculator to calculate actual percentages if necessary) wins those two cards.
6. Play continues until both players have exhausted their decks. The player with the most cards at the end of the game wins.

Adaptations

1. Play the same way, but let the fractions created all be improper fractions, that is, the larger number over the smaller.
2. Play the same way, but instead of calling out percentages, call *snap* every time the fraction built has a percentage equivalent to any one of the following: 25 per cent, 33 per cent, 50 per cent, 67 per cent, 75 per cent or 100 per cent
3. Estimate to the nearest 5 per cent.

Target Game

Objective: Estimate, compute and verify the sum, difference, product and quotient of rational numbers (Alberta mathematics program of studies, Number [Number Operations], Grade 8, Outcome 10)
Materials: Calculator
Players: Two or more

Rules

1. To begin the game, have one player type into the calculator a random four-digit number, no two digits alike. For example, start with 2753 in your calculator.
2. The players will now take turns multiplying the value in the calculator by any decimal they wish, trying to obtain a product that falls in the range specified below:
3. $499 \leq x \leq 501$
4. The calculator is not cleared between players, but instead, each player starts with the value left behind by the previous player.
5. The first player to hit the target is the winner.

Adaptations

1. Play again, but this time you may only divide (no multiplying) to hit the same target.
2. Start with a larger (5- or 6-digit number) or with a value between 0 and 1.
3. On a turn, before you may multiply or divide, you must hit the square root key.
4. Play as a solitaire game: what is the fewest number of turns necessary to hit the target?
5. Play against a friend. Play the game as above except each player has his or her own calculator. Each player may take four turns. Whoever is the closest after four turns is the winner.

The Right Stuff

Objective: Use the Pythagorean relationship to calculate the measure of the third side of a right triangle, given the other two sides in two-dimensional applications (Alberta mathematics program of studies, Shape and Space [Measurement], Grade 8, Outcome 2)
Materials: The Right Stuff game board, ruler, pencil, calculator, one six-sided die, one ten-sided die
Players: Two

Rules

1. On a turn, a player rolls both dice (that is, both the six- and ten-sided dice). These values represent the base and height of a right-angled triangle.
2. The player now draws this triangle on the grid provided and, using the Pythagorean relationship, calculates the length of the third side. The accuracy of the calculation can be checked by his or her opponent by measuring the length of the third side (in centimetres).
3. Using the table provided, the player now determines his or her score for this roll, based on the length of the third side.
4. Players both take three turns, rolling the dice, drawing the triangles, calculating the length of the hypotenuse and scoring points. The player with the highest score after three turns wins.

Example

Assume a player rolls a five (on the 6-sided die) and a seven (on the 10-sided die). The player would draw a triangle with a base of 7 cm and a height of 5 cm.

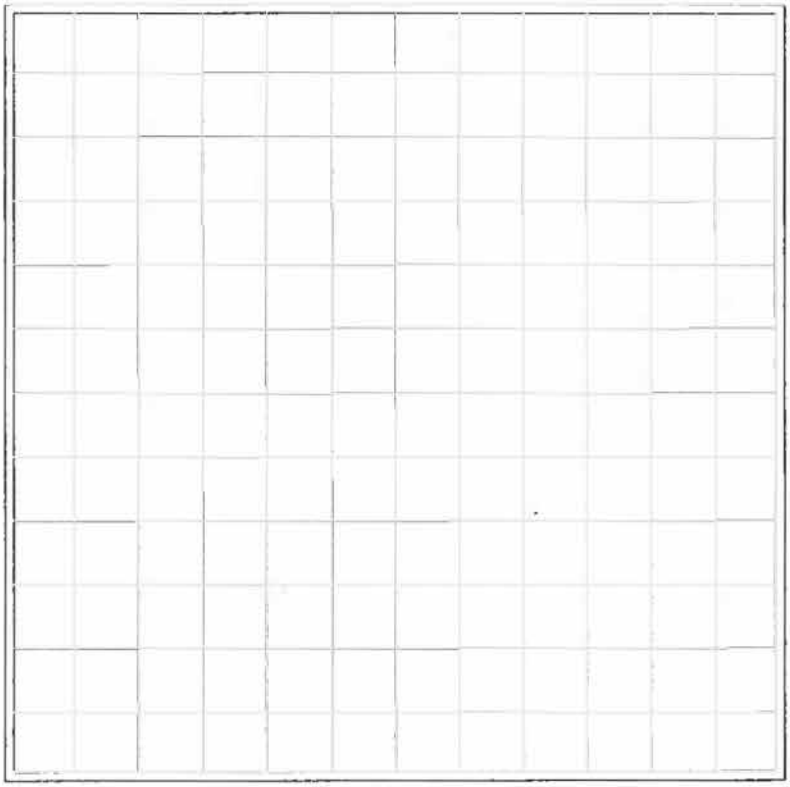
The third side would have a measure of 8.60 cm, and the player would score two points.

Adaptations

Change the rules so that the player scores the same number of points as the length of the third side.

For a solitaire game, allow a player to take as many turns as she or he can up until she or he can not fit the triangle on the grid. Triangles may not overlap, cross, be contained within another triangle and so on.

Score double points if the third side of your triangle has a whole number value.



1.00 – 3.00 cm	4 points
3.00 – 5.00 cm	2 points
5.00 – 8.00 cm	1 point
8.00 – 10.00 cm	2 points
10.00 – 12.00 cm	3 points

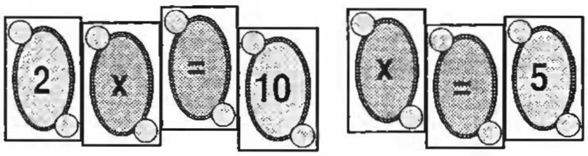
Your score:

Equation Rummy

Objective: Solve and verify one- and two-step first-degree equations (Alberta mathematics program of studies, Patterns and Relations [Variables and Equations], Grade 8, Outcome 5)
Materials: Equation Rummy cards
Players: Two or more

Rules

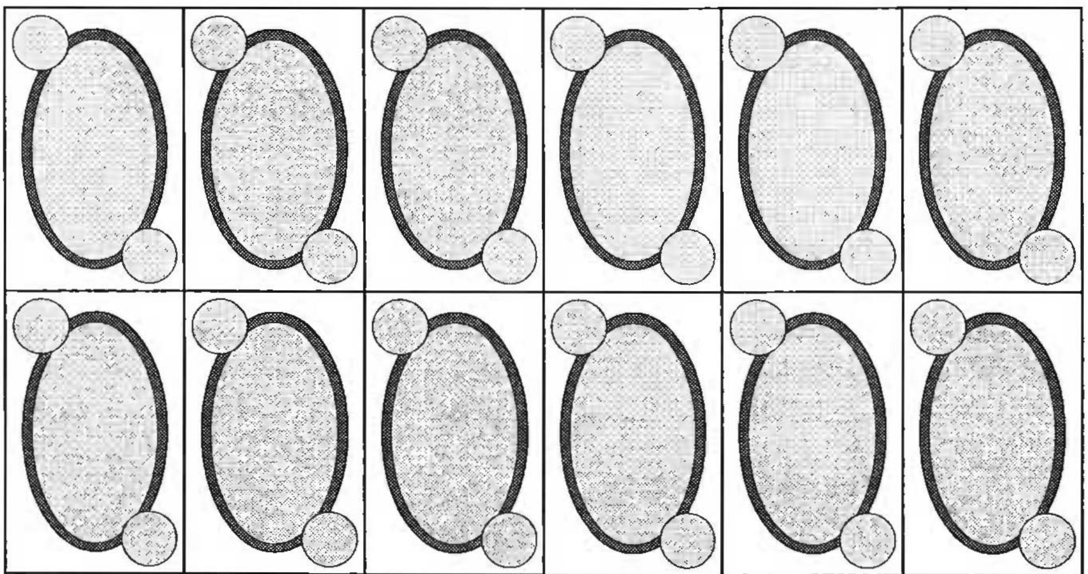
1. Shuffle the cards well and deal each player seven cards, which will serve as that player's hand. Place the remaining cards face down in a pile. Turn over the first card to start the discard pile.
2. On a turn, a player may draw either the top card from the deck or the top card from the discard pile. This card may either be discarded or exchanged for one card in the hand. Discarded cards are placed face up on the discard pile.
3. Players continue exchanging cards until they construct a collection of cards that build two equations of the form $ax=b$ and $x = b/a$. An example:

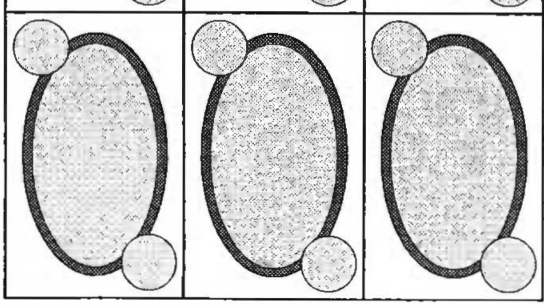


4. The first player to construct the pair of equations is the winner.

Adaptations

1. Add some "+" and "–" cards to the deck and start each player with eight cards. Build equations of the form $x + a = b$ and $x = (b - a)$.
2. Add the "+" and "–" cards and build equations of the form $ax + b = c$ and the root $x = c - b/a$.
3. Add two wild cards.





To make Equation Rummy cards:

2: make two	**3:** two	**4:** three	**5:** two
6: three	**7:** one	**8:** two	**9:** two
10: two	**12:** three	**14:** one	**15:** one
16: two	**18:** two	**20:** two	**21:** one
24: two	**27:** one	**28:** one	**30:** one
32: one	**36:** one	**40:** one	

10 variable (x) cards 10 equals (=) cards

About Integration in Teaching College Mathematics and Computer Programming

Yuri Yatsenko

This article deals with some challenges in teaching university mathematics and computer science courses. Colleges, textbooks and teaching techniques separate these areas, but students often take these courses simultaneously. Modern learning theories recommend that the teaching should be integrated in such cases. It does not mean that the basic subjects should be changed, but teachers should be aware of potential problems and how to avoid them.

Engineering colleges offer a degree in computer science (CS), and business colleges offer a degree in computer information systems (CIS). These majors require advanced mathematical skills and knowledge. Both CIS and CS majors first take an introductory course in computer science, which may be named differently (computer concepts, foundations of programming, introduction into programming logic and so on) but has the same basic curriculum: programming logic (PL) without using a computer and a specific programming language. Although the students' background (and prerequisites) for the PL course includes College Algebra and sometimes Calculus I or Finite Mathematics, they experience many difficulties in understanding the logic concepts. The author has extensive experience in helping students to overcome such problems.

As a general field, the programming logic is closely related to mathematics (see more in the last section below). However, this subject is taught in the context of engineering (CS) or business colleges (CIS); therefore, the practical skills are of primary importance. During the PL course, students are expected to understand and be able to develop several basic patterns of PL (like decision and iteration structures). More precisely, they learn how to read and write computer algorithms in the simplest formalized language (so-called pseudo-code). The PL course is a prerequisite for all further and more advanced CS courses, which may include specific programming languages (Visual Basic, C/C++, Java, Ada and others), software design and development techniques, database development and others. If a student has trouble passing this first course, he or she would not succeed in more sophisticated subjects that follow.

There are some essential differences between mathematical reasoning and PL that students often experience for the first time in a CS course. Students must understand these differences and learn how to use them correctly.

1. Equalities

Even the common equality formula,

$$\textbf{Expression 1} = \textbf{Expression 2} \qquad (1)$$

has different basic meanings in mathematics and PL. In mathematics, (1) sets a relationship between variables (that is, A = B) or represents an equation with respect to some unknown variables, for example $3x + 5 = 0$.

Expression (1) is an assignment statement in PL, which means that the right side of the expression is assigned to its left side at this step of the algorithm. A common form of the assignment statement (1),

$$X = X + 1 \qquad (2)$$

at first glance seems senseless for some PL students, especially for those with a strong mathematical background. Indeed, there is no solution to the equation (2). The alternative two-step form of the same statement,

$$Y = X + 1, X = Y \qquad (3)$$

is useful in the explanation of how the assignment statement works. However, it's worth to mention that the formula (3) is less efficient than (2) from the viewpoint of PL.

The Order of Operations and Parentheses

Another important issue students must understand to succeed in PL is the order of operations and the use of parentheses. Students usually know the basic rules for the arithmetic operations, but they often lack practice. The exercises are important because the students soon learn about additional operations (logical, unary and binary).

As a CIS professor, I would expect mathematics teachers to practise more with the use of parentheses. Simple and useful examples with parentheses follow:

Problem 1

Evaluate the following expressions (that differ only by the number of parentheses):

No parentheses	With parentheses	Multiple parentheses
3 + 5 × 2 – 8 / 4 =?	(3 + 5) × 2 – 8 / 4	((3 + 5) × 2 – 8)/ 4
3 + 10 – 8 / 4	8 × 2 – 8 / 4	(8 × 2 – 8) / 4
3 + 10 – 2	16 – 8 / 4	(16 – 8) / 4
13 – 2	16 – 2	8 / 4
11	14	2

Students usually have more fun with the following problem:

Problem 2

Obtain all numbers from 0 to 10 using exactly five 2s and four arithmetic operations +, –, ×, ÷ (fill in the question marks with the operation signs). Do this (a) without parentheses, (b) with parentheses:

2 – 2 / 2 – 2 / 2 = 0
2 ? 2 ? 2 ? 2? 2 = 1
....
2 + 2 + 2 + 2 + 2 = 10
....

A challenging (bonus) version of the problem is to obtain as many integers as you can. We shall notice that the solution to problem 2 is not unique even without using parentheses; that is, the same results can be achieved in different ways. Generally, the existence of several solutions is much more common in PL than in traditional mathematics. Sometimes the best solution (the shortest, the most efficient and so on) can be found among several acceptable solutions, but it requires changes in the statement of a problem under study and some advanced techniques.

Vectors and Arrays

Arrays are a common technique in PL. From a mathematical viewpoint, the arrays correspond to vectors and matrices. However, even after completing a College Algebra course, students often have problems understanding and using even one-dimensional arrays. The main difficulties are in differentiating between an array and its index (subscript), more generally, between independent and dependent variables. A possible reason lies in the shifted focus; namely, the PL students need to define, fill in and handle arrays, while in mathematical courses these tasks are supposed to be solved before a problem starts.

As compared with college mathematics, a new concept is storing data (information) in arrays. In PL, a key feature of arrays is that they can keep (store) data values instead of immediately processing them one-by-one and forgetting them. To clarify this point, the following problem is useful. It also illustrates a version of the pseudo-code for mathematics teachers.

Problem 3

The sum or the average (mean) value of an arbitrary number of input data values may be calculated without saving the values. A possible algorithm includes three variables *X*, SUM and COUNTER and one iteration loop to accumulate the sum:

```
COUNTER = 0
SUM = 0
LOOP WHILE COUNTER < MAX
   INPUT X
   SUM = SUM + X
   COUNTER = COUNTER + 1
END LOOP
AVERAGE = SUM / COUNTER
```

However, to calculate and output the individual deviations of the data values from the mean value, all input data values should be stored; that is, the above algorithm needs to use an array *X*() of the dimension MAX instead of the simple variable *X* and at least two loops (to accumulate the sum at first and to calculate the deviations next):

```
COUNTER = 0
SUM = 0
LOOP WHILE COUNTER < MAX
   INPUT X (COUNTER)
   SUM = SUM + X (COUNTER)
   COUNTER = COUNTER + 1
END LOOP
AVERAGE = SUM / COUNTER
COUNTER2 = 0
LOOP WHILE COUNTER < MAX
   COUNTER = COUNTER + 1
   DEV = X (COUNTER) – AVERAGE
   OUTPUT DEV
END LOOP
```

The first code is more efficient; it consumes less computer memory and does not depend on the length of the input data file. However, this type of processing is possible for very restricted and simple tasks only. The code with an array is much more general.

We shall notice that the second code calculates the individual deviations rather than the standard deviation. Using basic statistical rules, the latter may be calculated without an array.

Similar calculations are very common in college mathematics and statistics, so it would be worth to stress the above algorithmic problems while studying statistics.

Decimal and Binary Numbers

During the PL course, students often gain knowledge of binary numbers for the first time. Computers use binary rather than decimal numbers to store data in memory. Binary numbers are the numbers "in base 2" and use only two digits (0 and 1). Examples follow:

- Decimal number 9 is binary number 1001:
  $(9)_{10} = 8 \times 1 + 4 \times 0 + 2 \times 0 + 1 = (1001)_2$
- Decimal number 15 is binary number 1111:
  $(15)_{10} = 8 \times 1 + 4 \times 1 + 2 \times 1 + 1 = (1111)_2$
- $(10001)_2 = (23)_{10}$, $(100011)_2 = (35)_{10}$ and so on.

The exercises like the problem below are used in the PL course but they can be also mentioned in the college mathematics.

Problem 4

Do simple conversions from decimal numbers to binary numbers and vice versa:

$(7)_{10} = (111)_2$

$(23)_{10} = (1101)_2$

$(101)_2 = (5)_{10}$

$(1101)_2 = (13)_{10}$ and so on.

Doing simple arithmetic with binary numbers helps students better understand the common base-10 numeration system as well. For example, the sum of two binary units

$(1)_2 + (1)_2 = (10)_2$

makes the rule of moving the unit to the decimal location of the next higher order more clear. It also demonstrates the case when "one plus one is not two" in mathematics.

Generally speaking, the programming logic stands out as the first consumer of abstract mathematical knowledge that students have learned in a business or engineering college. First, the students begin to feel and understand the applied character and practical importance of mathematical theories. On the other hand, as the above examples show, the programming logic can enrich the students' understanding of mathematics itself.

The section that follows is more about advanced relations between mathematical theory and computer algorithms practice, and they are worth dealing with in both mathematics and CS courses.

Advanced Relations Between Mathematics and Programming Logic

As a scientific discipline, the programming logic is closely linked to several fields of advanced university mathematics, such as computational mathematics, numeric methods, optimization and complexity of algorithms. Generally speaking, these mathematical fields investigate what operations are necessary to solve a specific mathematical problem and the programming logic shows how to implement them in a form suitable for computer implementation.

After a new applied problem arises in science or engineering practice, it usually passes through the following stages in its computer investigation.

Stage 1. Developing an algorithm for solving the problem (any algorithm)

This stage uses a combination of numeric methods and PL. The first developed algorithm is used for a while to investigate the problem for different initial data. It normally reveals some flaws and needs an improvement. The next stage is also the subject of numeric methods and PL.

Stage 2. Developing various algorithms for different situations

At this stage, a set of different algorithms is obtained and some experience in solving the problem numerically is gained. Then one can formulate a problem of finding the best possible algorithm under certain assumptions. Mathematically correct definitions of "the best possible algorithm" include the most accurate algorithm under given restrictions on initial data errors, the fastest algorithm under given restrictions on the accuracy of results, the most robust algorithm and so on. Such problems are investigated by the theory of algorithms optimization and complexity. They may be used for the next step.

Stage 3. Finding the best (optimal) algorithm under the given specific assumptions

Even after finding an optimal algorithm, the analysis may not be finished. Different optimal algorithms can be obtained under various assumptions about initial data and tested on real data and real problems. This leads to more experience in numerically solving the problem, which results in the next step.

Stage 4. Obtaining a set of efficient algorithms optimal in different situations

Starting with the first step, an applied outcome of the above mathematical theories is standard software programs for solving specific problems. The exposed scheme has been used in the development of many efficient software packages for statistics, sorting and filtering data, solving differential equations, optimization and so on.

Of course, not all real-life problems are undergoing such theoretical treatment. It depends on their importance and specifics. In some cases, the first algorithms appear to be good enough for practical purposes. Other problems never become standard and are occasionally solved using custom algorithms and software.

Isaac Newton (1642–1727)

This British mathematician wrote in his book *Arithmetica Universalis*: "In my studies I discovered that the actual problems are often of more value than the rules." He posed the following problem:

Three pastures have an area of 3 1/3 ha, 10 ha and 24 ha. The growth conditions are identical in all three pastures. The grass density and yield per unit area are the same. On the first pasture, 12 oxen graze for the duration of 4 weeks and on the second pasture 21 oxen graze for the duration of 9 weeks, at which time these pastures have been completely grazed. How many oxen could graze on the third pasture for the duration of 18 weeks?

Student Projects in the Educational Process

Natali Hritonenko

Learning and teaching are links of the same chain. They represent an interactive, constructive and continuous process. Many modelling resources and tools are available in teaching and used in curriculum from introductory to advanced courses and across disciplines. Learners are unique. They have, among a number of things, different skills, experience, background and attitudes. Hence, whatever works well for one group may not work for another. Therefore, different strategies should be used in teaching that are aimed at encouraging students and making their learning informative, effective, and at the same time, pleasant, enjoyable and relaxing. The best learning happens when all accessible teaching tools are properly used.

This article focuses on the use of student projects as a challenging teaching technique. Individual and team projects play an incredibly important role in the learning process in that they add a challenging dimension to the teaching/learning process. They represent an important method of integrated learning, which is the mainstream of modern teaching philosophy. Here, students do not listen to lectures (which they forget by the next class) but are challenged to work independently. The major problems with this approach are to convince students to remain task-oriented and to provide them with appropriate challenges. You cannot force students to do this unless they are interested in the project itself and understand that the project is a self-portrait of those who do it. Thus, the first task to ensure success is to encourage students to put some magic into the chosen topics.

Projects can be on various topics, depending on the goals of the course. To avoid repetition of the same projects year after year, it is necessary to change the titles or the theme for each term. For example, one algebra class may be devoted to scientists who founded algebra; namely, "Al-Khowarizmi and his manuscripts," "Viète and Cardano" or "Pascal and his discoveries." Next term's theme may be on the basic steps and discoveries of algebra with projects like "The Birth of Algebra," "Quadratic and Polynomial Equations" or "Magic of the Pascal's Triangle." As one can see, there are corresponding topics in these two groups. They cover approximately the same material: it is impossible to look at the first steps of algebra without talking about Al-Khowarizmi and so on. Thus, different classes will discuss the same topics without noticing this.

The same idea can be used in assigning projects to any class. Examples of some related projects in Calculus 1 could be "Newton and Leibniz," "Derivatives and Limits," "L'Hospital and His Rule" or "Applications of Derivatives," and in statistics, "Gauss and de Moivre" and "The Normal Distribution."

When assigning a project, the instructor should be clear about the goals, expectations, completion dates and examples to be used. Discussing mathematicians' discoveries and presenting interesting, nontrivial details of their life are very beneficial. One's brain always pays attention to anything out of the ordinary. For example, in the case of Euler, it is important to give dates of his life (1707–1783) and mention that he discovered definitions of modern trigonometry and analytic geometry. He was a pioneer in differential and integral analysis, was a creator of variation analysis, worked on popularization of mathematics and published more then 500 papers, textbooks and monographs. Students will be even more impressed if one adds that Euler was completely blind for the last 17 years of his life during which he continued his research. Dictated books provided mathematical proofs in his mind using his extraordinary abilities. Euler also had 13 children. Such information always keeps the students focused and generates more interest. It can often awaken even the most unwilling and uninvolved students.

It is essential to encourage students to show their best, but even if a project is below the expectations, one should try to find at least something useful in the project to develop and discuss. Show that the project is worth the time and effort students employ, make them proud of the work they have done and the next project will be better. Students need this support and a student who needs the teacher's support the most is often the one who appears to deserve it the least.

Working in a team is a very important educational process and experience for the students. Students may create teams by themselves or an instructor may assign them to a team. Each team of students is required to prepare a poster, make a presentation and submit a report. A poster reflects basic ideas and steps of a presentation. Working on a poster, students learn how to choose the most important facts and organize it on the poster, making it visible, understandable and helpful. A presentation teaches students how to express themselves, organize their thoughts, provide vocal delivery, speak mathematically, answer questions and defend their opinion. Frequently, students gather much more material than they are able to outline. This is a reflection of their lack of presentation experience. In preparing a report, students learn how to work with various sources, such as scientific and popular journals, monographs and the web. They have to provide collaborative research and interactive investigations. Preparation of a project, as well as explaining the content to each other will help students understand the material better.

Evaluation of projects consists of three criteria: peer evaluation, evaluation within a group and instructor evaluation. The peer evaluation is provided by all students of a class. They grade each project, express their opinion, write their comments and demonstrate what they have learned. In so doing, students will refresh the project again. The peer evaluation can be turned into fun if the instructor decides to play a lotto game with a class, which requires the student to arrive at the same grades as the instructor. The student will then receive a bonus.

Unfortunately, not all members of a team make the same contributions. Evaluation within a group, where each student evaluates his or her co-members, helps determine each student's contribution to the team project.

Information tends to be forgotten quickly when students are passive or do not review at least some basic concepts or information. A presentation of a project alone does not guarantee that learning takes place. The more students are involved and engaged in discussions, the more they are likely to learn and remember. There are many ways to enhance students' learning, increase their attention and encourage them to participate in presentations. Two have already been mentioned: peer evaluations and adding some memorable facts of a biography. Other ways may include granting bonus points for each question to speakers, additions to a presentation topic and an open note quiz on projects at the end of all presentations. The quiz helps rehearse basic ideas of projects. If students expect a quiz, they will often take notes, especially if the instructor points out all the questions that might be on the quiz.

As one can see, a project itself is an effective integration of research and learning. It stimulates learning through the excitement of discovery and broadens the participation of students into an educational process. Projects also increase the students' mathematical culture and literacy, as well as develop analytical and communication skills. Discussing the lives and discoveries of great mathematicians sets up positive examples for students and helps them find and establish their lifetime goals because the human brain naturally searches for patterns as a way to succeed.

I have successfully used projects in all my courses. My students enjoy working on projects. Some of them continue to use their projects in their own teaching and organize weeks devoted to a certain page in mathematical history.

Edmonton Junior High Mathematics Contest 2001

Andy Liu

The Edmonton Junior High Mathematics Contest is designed to challenge the top 5 per cent of Grade 9 math students in Edmonton. The annual contest is run by a group of mathematics teachers and is written by Andy Liu. The main sponsors are the Association of Professional Engineers, Geologists and Geophysicists of Alberta (APEGGA), IBM, MCATA, Edmonton Public Schools and Edmonton Catholic Schools. After the contest, the top 50 students are recognized at a dinner banquet also attended by parents and teachers.

Part 1: Multiple Choice

1. The last digits of 49^{2001} are

   (a) 01 (b) 49 (c) 69 (d) 81

   Answer: (b)

2. The sum $7^7 + 7^7 + 7^7 + 7^7 + 7^7 + 7^7 + 7^7$ is equal to

   (a) 8^7 (b) 7^8 (c) 49^7 (d) 7^{49}

   Answer: (b)

3. For any numbers x and y, define $x \oplus y = x + y + xy - 1$ and $x \otimes y = a^2 + b^2 - ab$.
   The value of $3 \oplus (2 \otimes 4)$ is

   (a) 36 (b) 42 (c) 48 (d) 50

   Answer: (d)

4. The price is first increased by r% and then reduced by r%. If the final price is divided by the original price, the quotient is

   (a) 1 (b) $1 - \frac{r}{10000}$ (c) $1 + \frac{r2}{10000}$ (d) $1 - \frac{r2}{10000}$

   Answer: (d)

5. There is enough cabbage to last the goat x days and the rabbit y days. The number of days the cabbage will last both the goat and the rabbit is

   (a) $\frac{1}{x+y}$ (b) $\frac{1}{x} + \frac{1}{y}$ (c) $\frac{xy}{x+y}$ (d) $\frac{1}{xy}$

   Answer: (c)

6. When Ace was as old as Bea is now, Bea was 10 years old. When Bea is as old as Ace is now, Ace will be 25 years old. Ace is older than Bea by

   (a) 5 years (b) 10 years (c) 15 years (d) none of these

   Answer: (a)

7. The length of each side of a triangle is a positive integer and the sum of the these three integers is odd. If the difference between two of them is 5, the smallest possible value of the third is

   (a) 4 (b) 6 (c) 7 (d) 8

   Answer: (b)

8. The sum of the angles of a polygon is less than 2001°. The largest possible number of sides of this polygon is

   (a) 11 (b) 12 (c) 13 (d) 14

   Answer: (c)

9. In triangle ABC, $AB = AC$. E is the point on AC such that BE is perpendicular to AC. F is the midpoint of AB. If $BE = EF$, then the measure of $\angle C$ is

   (a) 65° (b) 70° (c) 75° (d) 80°

   Answer: (c)

10. If the number x satisfies $\frac{2}{x} - |x| = 1$, what is the value of $\frac{2}{x} + |x|$.
    Hint: $|2| = 2$, and $|-2| = 2$.

    (a) –3 (b) –1 (c) 1 (d) 3

    Answer: (d)

Part 2: Numeric Response

1. The value of $\dfrac{-2 \div (-2)^{-2} + (-2)^3}{\left(-\frac{1}{2}\right)^{-1} + \frac{1}{2}}$ in fractional form.

   Answer: 10 $^2/_3$ or $^{32}/_3$

2. In a student union election, 1,500 votes are cast. Of the first 1,000, Ace receives 350, Bea 370 and Cec 280. Of the remaining 500 votes, at least how many must Ace receive in order for him to have more votes than either Bea or Cec?
Answer: 261

3. The ratio of Peter's and Steven's running speed is 5:3. They start from the same point on a circular track at the same time. After some time, they meet again at the starting point, and Peter has run 4 more laps than Steven. How many laps has Steven run?
Answer: 6

4. What is the smallest positive integer n such that $n + 13$ is a multiple of 5 and $n - 13$ is a positive multiple of 6?
Answer: 37

5. The positive integers w, x, y and z are such that $\frac{w}{x} = \frac{x}{y} = \frac{y}{z} = \frac{5}{8}$. What is the smallest possible value of $w + x + y + z$?
Answer: 1,157

6. D is a point on the side BC of triangle ABC. If $AC = 5$, $AD = 6$, $BD = 10$ and $CD = 5$, what is the area of triangle ABC?
Answer: 36

7. What is the unit digit of $1^3 - 2^3 + 3^3 - 4^3 + ... + 1{,}999^3 - 2{,}000^3 + 2{,}001^3$?
Answer: 1

8. How many four-digit multiples of 11 are there in which each of the digits 1, 2, 3 and 4 appears?
Answer: 8

9. Three thousand three hundred and seventy-five 1-cm cubes are used to form a larger cube. The outer surfaces of the newly assembled cube are painted. After drying the paint, the cubes are knocked apart. Find the total surface area of the unpainted surfaces on the 3,375 cubes.
Answer: 18,900

10. $ABCD$ is a rectangle of area 24. E, F and G are points on AB, BC and CD, respectively such that $BE = 3AE$, $CF = 2BF$ and $DG = CG$. What is the area of triangle EFG?
Answer: 8

Christian Goldbach (1690–1764)

This mathematician was born in Prussia, lived in various Western European countries and settled in Russia. In the Goldbach conjecture, it is claimed that every even number (except 2) is equal to the sum of two prime numbers. This conjecture has been verified up to 2×10^{10}. Check this conjecture yourself for numbers up to 50.

(In 1742 Goldbach wrote about this conjecture to the Swiss mathematician Leonard Euler. This conjecture remains unproved until today.)

Edmonton Junior High Mathematics Invitational 2001

Andy Liu

The Edmonton Junior High Mathematics Invitational is a follow-up exam to the Edmonton Junior High Mathematics Contest. The top 50 Grade 9 students on the Edmonton Junior High Mathematics Contest are invited back to participate on the Edmonton Junior High Mathematics Invitational.

The annual invitational exam is run by a group of mathematics teachers and is written be Andy Liu. The main sponsors are the Association of Professional Engineers, Geologists and Geophysicists of Alberta (APEGGA), IBM, MCATA, Edmonton Public Schools and Edmonton Catholic Schools.

2001 Solution

In what follows, we give a detailed discussion on the approach to each problem. Thus the write-up is much longer than a formal solution. We hope that reading through this additional material will help you improve your problem-solving skill.

1. Find 100 different positive integers such that the product of any five of them is divisible by the sum of these five numbers.

Solution

Some problems look hard because the numbers involved are large. As a preliminary analysis, cut them down to size. For instance, we may replace 100 with a smaller number. The smallest number which makes sense is 5. If we simply take 1, 2, 3, 4 and 5, $1 \times 2 \times 3 \times 4 \times 5 = 120$ is indeed divisible by $1 + 2 + 3 + 4 + 5 = 15$. If we want 6 numbers, will 1, 2, 3, 4, 5 and 6 do? Let us see.

(a) If we leave out 1, $2 \times 3 \times 4 \times 5 \times 6 = 720$ is divisible by $2 + 3 + 4 + 5 + 6 = 20$.
(b) If we leave out 2, then $1 \times 3 \times 4 \times 5 \times 6 = 360$ is not divisible by $1 + 3 + 4 + 5 + 6 = 19$.
(c) If we leave out 3, then $1 \times 2 \times 4 \times 5 \times 6 = 240$ is not divisible by $1 + 2 + 4 + 5 + 6 = 18$.
(d) If we leave out 4, then $1 \times 2 \times 3 \times 5 \times 6 = 180$ is not divisible by $1 + 2 + 3 + 5 + 6 = 17$.
(e) If we leave out 5, $1 \times 2 \times 3 \times 4 \times 6 = 144$ is divisible by $1 + 2 + 3 + 4 + 6 = 16$.
(f) If we leave out 6, $1 \times 2 \times 3 \times 4 \times 5 = 120$ is divisible by $1 + 2 + 3 + 4 + 5 = 15$ as we have already observed.

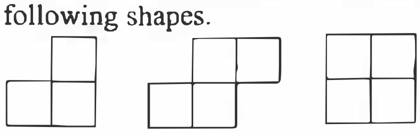
Let us try to fix the problem in case (c). Here we need an extra factor of 3. Had we started with 3, 6, 9, 12, 15 and 18 instead, then $3 \times 6 \times 12 \times 15 \times 18 = 240 \times 3^5$ will be divisible by $3 + 6 + 12 + 15 + 18 = 18 \times 3$. However, neither case (b) nor case (d) is fixed this way. We would start instead with 19, 38, 57, 76, 95 and 114 in the former, and 17, 34, 51, 68, 85 and 102 in the latter.

Nevertheless, this analysis suggests that we should start with M, $2M$, $3M$, $4M$, $5M$ and $6M$ for a suitably chosen M. Certainly, $M = 3 \times 17 \times 19 = 969$ will work here. Hence 969, 1,938, 2,907, 3,876, 4,845 and 5,814 solve the problem with 100 replaced by 6.

We are now ready to tackle the original problem. Our numbers are M, $2M$, $3M$, ..., $100M$. All we have to do is to choose a suitable M. There are too many cases for us to repeat the earlier analysis. Fortunately, it is not necessary to do so. Let aM, bM, cM, dM and eM be any five of the numbers. Then their product is $P = abcdeM^5$ while their sum is $S = (a+b+c+d+e)M$. Since $15 = 1+2+3+4+5 \leq a+b+c+d+e \leq 96+97+98+99+100 = 490$, we can choose $M = 490 \times 489 \times 488 \times \cdots \times 17 \times 16 \times 15$. Then M is a multiple of $a+b+c+d+e$, so that P is indeed divisible by S.

The value of M is larger than is necessary, but we do not have to pay for it. It saves us a lot of work. Going back to the earlier example with 6 numbers, we could have avoided the case analysis by taking $M = 20 \times 19 \times 18 \times 17 \times 16 \times 15$. However, that much work was valuable because it points us in the right direction.

2. You have a large number of pieces of each of the following shapes.

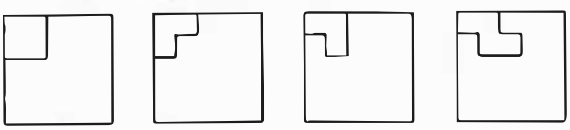


Each square of a piece covers a square of the chessboard. No overlapping within or protrusions beyond are allowed. You do not have to use all three shapes. Is it possible to cover every square of 5 × 5 chessboard using any combination of these pieces? Either give such a covering or give a short proof why no such coverings exist.

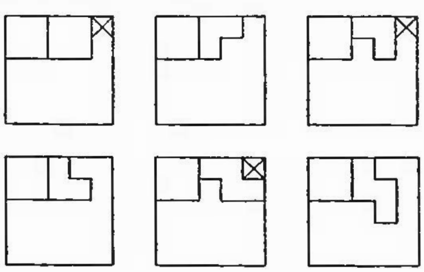
Solution

The 5 × 5 board is not that big, and it does not take long to convince ourselves that the desired covering does not exist. However, convincing other people that we know what we are talking about takes more doing. It is not enough to just show a few failed attempts. Perhaps we have overlooked a solution.

One approach, favoured by computing scientists, is known as backtracking. It is an exhaustive analysis of all cases by brute force. We number the square from 1 to 25 row by row from left to right and top to bottom. At each point, we examine all possible ways of covering the uncovered square with the lowest number. Thus at the start, we attempt to cover square #1. There are six cases, but the two not shown are equivalent to the third and fourth ones below.



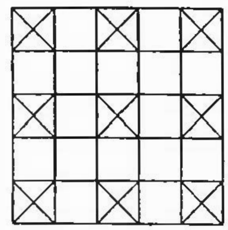
From the first case, six subcases are generated according to how square #3 is to be covered. Three of these lead immediately to impossible situations, but further analysis are needed for the remaining subcases. Thus this is not an attractive approach for mortal souls without the backing of immense computational power.



So what can we do? Let us try an algebraic approach. Suppose we use x, y and z copies of the three shapes, respectively. Then $3x + 4(y + z) = 25$. We can have $x = 7$ and $y + z = 1$, or $x = 3$ and $y + z = 4$. So the total number of pieces used in any successful covering is 8 or 7. This gives us more information, but the road ahead is still unclear.

Perhaps we should consider a smaller board, say the 3 × 3. Now $3x + 4(y + z) = 9$ and $x = 3$, $y = 0$ and $z = 0$ is the only solution. It is easy enough to use backtracking here to show that 3 pieces of the first shape cannot cover the board, but we need an argument which can be extended to the 5 × 5 board.

Backtracking has focused our attention to the corner squares of the board. The 3 × 3 board has four, and the first shape can cover at most one of them. Hence we need at least 4 pieces, and there is simply not enough room for them.



The nine squares marked × in the 5 × 5 board above place the roles of the comer squares in the 3 × 3 board. Each of the three shapes can cover at most one of them. Hence at least 9 pieces are needed. This is a contradiction since we have already determined that at most 8 pieces are to be used. Hence the desired covering cannot exist.

3. Let a, b, c and d be any numbers. Prove that $(1 + ab)^2 + (1 + cd)^2 + (ab)^2 + (cd)^2 \geq 1$.

Solution

First, we can simplify the problem by replacing ab with x and cd with y. We want to prove that $(1 + x)^2 + (1 + y)^2 + x^2 + y^2 \geq 1$ for any numbers x and y. Note that x or y may be negative, as otherwise the expression is certainly at least 2.

Since the expression contains a lot of squares, it may be a good idea to examine squares more closely. The square of 0 is 0. The square of a positive number is positive. The square of a negative number is also positive. Hence a square is never negative.

Expanding the squares, we obtain $1 + 2x + x^2 + 1 + 2y + y^2 + x^2 + y^2$. If we set aside a 1 and can combine $1 + 2x + 2x^2 + 2y + 2y^2$ into squares, we will have the desired result. By separating x from y and splitting the 1 between them, we have $\frac{1}{2} + 2x + 2x^2 = \frac{1}{2}(1 + 4x + 4x^2) = \frac{1}{2}(1 + 2x)^2$. An analogous expression exists for y. It follows that

$$(1 + x)^2 + (1 + y)^2 + x^2 + y^2 = 1 + \tfrac{1}{2}(1 + 2x)^2 + \tfrac{1}{2}(1 + 2y)^2 \geq 1.$$

Another way is to observe that $(1 + x + y)^2 = 1 + 2x + x^2 + 2y + y^2 + 2xy$, so that

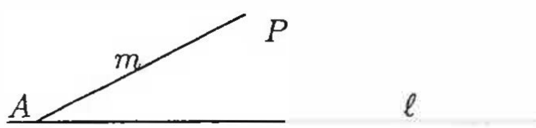
$$\begin{aligned}(1 + x)^2 + (1 + y)^2 + x^2 + y^2 &= 1 + (1 + x + y)^2 + x^2 + y^2 - 2xy \\ &= 1 + (1 + x + y)^2 + (x - y)^2 \\ &\geq 1.\end{aligned}$$

Note that equality holds if and only if each of $1 + 2x$, $1 + 2y$, $1 + x + y$ and $x - y$ is 0. In other words, the minimum value of 1 of the given expression is attained if and only if $x = y = -1$(over)2.

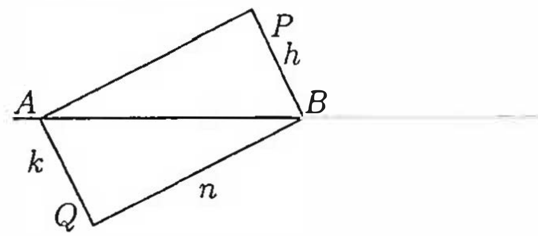
4. You have an instrument that allows you to draw the line segment connecting two given points, and to draw through a point on the line perpendicular to the given line. Given a point P not on a given line l, describe a construction using your instrument only the line through P parallel to l. You do not have to justify your construction.

Solution

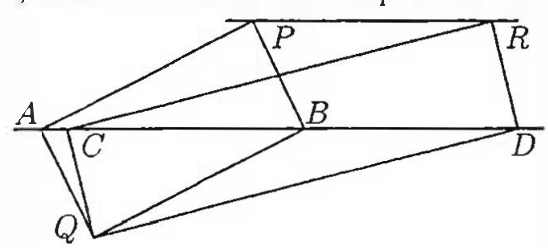
At the start, we have a line l and a point P not on it. Thus there is no immediate way of applying either of the allowable operations. So we should choose a second point A somewhere, and it makes sense for this point to lie on l. Now we can join A to P, or draw a line through A perpendicular to l. The latter does not lead us anywhere, so we connect A and P by a line m.



What can we do now? Still rejecting the drawing of a perpendicular to l through A, we can instead draw perpendiculars h and k to m at P and A respectively. Now h will intersect l at a point B. Drawing a perpendicular n to h at B, it will intersect k at a point Q.



Note that $PAQB$ is a rectangle with A and B on l. In particular, P and Q are equidistant from l. Repeating the above process with Q playing the role of P, we can construct a rectangle $QCRD$ with C and D on l. Then Q and R are equidistant from l, so that PR is the desired line parallel to l.



Fun with Mathematics—Challenging the Reader

Andy Liu

Each issue of delta-K *will contain problem sets, which will also be posted on the MCATA website (www.mathteachers.ab.ca).*

The spring issue will contain a set of problems for January, March and May, and the fall issue will contain a set of problems for September and November. Teachers and students are invited to participate by submitting the full solution to each problem by the deadline stated on the website.

Note that the solutions to the problem sets will be published in delta-K *only, with fall issue containing the solutions for the January, March and May problems and the spring issue containing the solutions for the September and November problems.*

Submit your full solutions to Andy Liu, Department of Mathematics, University of Alberta, Edmonton, Alberta, T6G 2G1; fax (780) 492-6826, e-mail aliumath@telus.net.

The solutions to the January, March and May problems are shown below, followed by the September and November problem sets.

Solutions to January 2003 Problems

Problem 1

The numbers 1, 2, . . ., 16 are placed in the cells of a 4 × 4 table as shown below:

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

One may add 1 to all numbers of any row or subtract 1 from all numbers of any column. How can one obtain, using these operations, the table shown below?

1	5	9	13
2	6	10	14
3	7	11	15
4	8	12	16

Solution

Denote the number of addition operations applied to the rows by a_1, a_2, a_3, a_4 and the number of subtraction operations applied to the columns by b_1, b_2, b_3, b_4. Comparing the initial and required tables, we see the necessity of the following relations: $a_1 = b_1$, $a_2 = b_2$, $a_3 = b_3$, $a_4 = b_4$, $a_1 - b_2 = 3$, $a_1 - b_3 = 6$, $a_1 - b_4 = 9$. Thus, letting a_4 be an arbitrary non-negative integer, we solve the problem; the order of performing the operations does not matter. One of the solutions is $a_1 = 9$, $a_2 = 6$, $a_3 = 3$, $a_4 = 0$, $b_1 = 9$, $b_2 = 6$, $b_3 = 3$, $b_4 = 0$.

Problem 2

There are four kinds of bills: \$1, \$10, \$100 and \$1,000. Can one have exactly half a million bills worth exactly \$1 million?

Solution

Assume there exists a set of notes as described in the problem. Let a, b, c and d be the numbers of notes in the set whose values are \$1, \$10, \$100 and \$1,000, respectively. Then we have two equations:

$$a + b + c + d = 500{,}000$$
$$a + 10b + 100c + 1{,}000d = 1{,}000{,}000$$

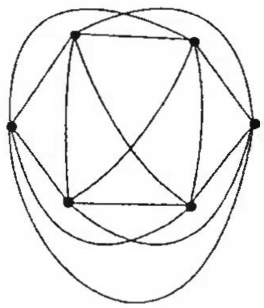
Subtracting the first equation from the second, we get $9b + 99c + 999d = 500{,}000$, which is impossible because 500,000 is not divisible by 9 whereas the left side is clearly a multiple of 9.

Problem 3

The king intends to build six fortresses in his realm and to connect each pair with a road. Draw a diagram of the fortresses and roads so that there are exactly three intersections and exactly two roads crossing at each intersection.

Solution

The appropriate diagram is shown



Problem 4

If each boy purchases a pencil and each girl purchases a pen, they will spend a total of 1¢ more than if each boy purchases a pen and each girl purchases a pencil. There are more boys than girls. What is the difference between the number of boys and the number of girls?

Solution

Denote the number of boys and the number of girls by B and G, respectively, and the prices of a pencil and a pen by x and y, respectively. Then we have the equation $Bx + Gy = By + Gx + 1$, that is, $(B - G)(x - y) = 1$. But the product of two integers can be equal to 1 only if both are equal to either 1 or –1. Since we know that the difference of $B - G$ is positive, we conclude that it is equal to 1.

Problem 5

A six-digit number from 000000 to 999999 is called lucky if the sum of its first three digits is equal to the sum of its last three digits. How many consecutive numbers must we have to be sure of including a lucky number if the first number is chosen at random?

Solution

The answer is 1,001. Note that if the first ticket we buy happens to be 000001, then the first lucky ticket we can get is 001001; that is, there are cases when to purchase fewer than 1,001 tickets is not sufficient.

Now we have to show that 1,001 is a sufficient number of tickets for our aim. Write the six-digit number of the first bought ticket as AB, where A represents the number formed by the first three digits and B the number formed by the last three. If $A \geq B$, we can buy $A - B \leq 1{,}000$ tickets and obtain lucky ticket AA. If $A < B$, the purchase of $1{,}001 - B$ tickets leads us to ticket $A'B'$, with $A' = A + 1$ and $B' = 0$. Then we buy an additional $A + 1$ tickets and obtain the lucky ticket. So we have reached our aim this time having bought $1{,}002 - (B - A)$ tickets, and since $B - A \geq 1$, we conclude that 1,001 is indeed a sufficient number of tickets to buy.

Problem 6

Two players play the following game on a 9 × 9 chessboard. They write in succession one of two signs in any empty cell of the board: the player making the first move writes a plus sign (+) and the other player writes a minus sign (–). When all the squares of the board are filled, the scores of the players are tabulated. The number of rows and columns containing more plus signs than minus signs is the score of the first player, and the number of all other rows and columns is the score of the second player. What is the highest number of points the first player can gain in a perfectly played game?

Solution

It is easy to devise a strategy providing 10 points for the first player. She has to make her first move in the central square of the board and then write a plus sign each time in the square symmetric (with respect to the centre of the board) to the square the second player has filled in at the previous move. This strategy guarantees that the central row and column bring two points to the first player. Further, all other rows can be split into pairs of symmetric rows, and we see that each pair shares the points among both players equally. The same is true for the columns, and thus the first player gets exactly 10 points.

Now we have to prove that the second player is able to play so that he earns at least eight points (the total number of the rows and columns on the board is 18). The main idea is that the second player also can achieve the symmetric filling of the board, which, as we have seen, leaves him with the desired eight points. If the first player follows the preceding strategy, the actions of the second player are of no importance, but if the first player makes a nonsymmetric move, her opponent should begin to support symmetry. If, at the very beginning, the first player makes her first move in a square, other than the central square, the second player can still support the necessary symmetry, and since the last move is made by the first player, she will be compelled to complete the symmetric filling of the board. Thus, we have proved that the answer is 10.

Solutions to March 2003 Problems

Problem 1

Initially, there is a 0 in each cell of a 3 × 3 table. One may choose any 2 × 2 subtable and add 1 to all numbers in it. Can one obtain, using this operation a number of times, the table shown below?

4	9	5
10	18	12
6	13	7

Solution

Each 2 × 2 subtable contains the central box and exactly one of the comer boxes. Thus, the number in the central box must be equal to the sum of the numbers in the corners. But this relation does not hold true for the pictured table, and thus the solution follows.

Problem 2

A teacher plays a game with 30 students. Each writes the numbers 1, 2, . . ., 30 in any order. Then the teacher compares the sequences. A student earns a point each time the same number appears in the same place in the sequences of that student and of the teacher. It turns out that each student earns a different number of points. Prove that at least one student's sequence is the same as the teacher's.

Solution

Each player could gain from 0 to 30 aces. However, he cannot earn exactly 29 aces. Consequently, using the pigeonhole principle, we conclude that one of the players gained exactly 30 aces, and so his sequence coincided with that of the leader.

Problem 3

Is it possible to write the numbers 1, 2, . . ., 100 in a row so that the difference between any two adjacent numbers is not less than 50?

Solution

Yes. One can write the following sequence: 51, 1, 52, 2, 53, 3, . . . , 49, 100, 50.

Problem 4

Do there exist two nonzero integers such that one is divisible by their sum and the other is divisible by their difference?

Solution

No, such numbers A and B do not exist. If A and B are nonzero integers, then either $A + B$ or $A - B$ has an absolute value greater than the absolute values of A and B. This can be checked from the fact that either sign (A) = sign (B), or sign (A) = – sign (B). It is now sufficient to recall that, if a nonzero integer X is divisible by Y, then $|X| \geq |Y|$.

Problem 5

A game starts with a pile of 1,001 stones. In each move, choose any pile containing at least two stones and remove one of them, and then split any pile containing at least two stones into two nonempty piles, which need not be of equal size. Is it possible for all remaining piles to have exactly three stones after a sequence of moves?

Solution

The answer is no. The basic idea of the solution is the very popular idea (in mathematics and science) of invariance. Define the quantity S to be the sum of the number of piles, and the number of stones. Under the conditions described in the problem, S is invariant. The initial value of S is 1,002, and if it is possible to obtain n piles each containing exactly three stones, then S should be equal to n (the number of piles) + $3n$ (the number of stones) = $4n$, which gives a contradiction, since 1,002 is not divisible by 4.

Problem 6

A square castle is divided into 64 rooms in an 8 × 8 configuration. Each room has a door on each wall and a white floor. Each day, a painter walks through the castle, repainting the floors of all the rooms he passes, so that white is changed to black and vice versa. Can he do this so that, after several days, the floors in the castle will be coloured like a chessboard?

Solution

Yes, the painter can do it. Assume he walks from any room to another room A and then returns the same way; then the color of A changes while all other rooms in the castle retain their colors. Surely, using these operations, the painter can arrange the chess pattern on the castle floor.

Solutions to May 2003 Problems

Problem 1

A jury makes up problems for an Olympiad, with a paper for each of Grades 7–12. The jury decides that each paper should consist of seven problems, with exactly four of them not appearing on any other paper of the Olympiad. What is the greatest number of distinct problems that could be included in the Olympiad?

Solution

The answer is 33 problems. There are exactly 24 problems such that each of them is included in exactly one list, and since each of k other problems belongs to more than one list, the total number of problems for all grades is no less than $24 + 2k$. But this amount is equal to $6 \times 7 = 42$, which implies that $k \leq 9$ On the other hand, the jury is clearly able to make up the lists containing 33 distinct problems—it is sufficient to include three common problems in the lists of the fifth and sixth grades, the seventh and eighth grades, and the ninth and tenth grades, respectively.

Problem 2

A six-digit number (from 000000 to 999999) is called lucky if the sum of its first three digits is equal to the sum of its last three digits. Prove that the number of lucky numbers is equal to the number of six-digit numbers with a digit sum of 27.

Solution

Clearly, if we change the first three digits a, b, c in a lucky number to $9 - a$, $9 - b$, $9 - c$, we obtain a number whose digits' sum is equal to 27, and vice versa. (For instance, 273390 is a lucky ticket and $(9-2)+(9-7)+(9-3)+3+9+0=27$). This one-to-one correspondence between tickets of the considered types implies that the number of tickets in one group is equal to the number of tickets in the other.

Problem 3

Given 32 stones of distinct weights, prove that 35 weightings on a balance are sufficient to determine which are the heaviest and the second heaviest.

Solution

First, dividing 32 stones into pairs and using 16 weighings, we extract a set of 16 stones that contains the heaviest stone. Further, performing an analogous operation for the extracted set, we reduce the number of candidates to eight stones and so on. So, using $16 + 8 + 4 + 2 + 1$ weighings in five stages, we can determine the heaviest stone. Now we should notice that the second heaviest stone is necessarily one of those five that were in the same pair with the heaviest one. Thus, to complete the solution, we must find the heaviest of five stones with four weighings. We leave this very simple exercise for the reader.

Problem 4

Find two six-digit numbers such that the number obtained by writing them one after another is divisible by their product.

Solution

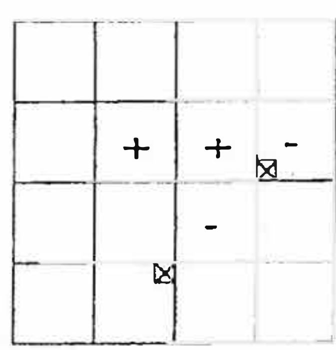
The answer is 166667 and 333334. Check that $166667333334 = 3 \times 166667 \times 333334$. By the way, it is the only answer.

Problem 5

Two players play a game of wild tic-tac-toe on a 10×10 board. They take turns putting either an X or an O in any empty cell on the board. Both players can use X or O, and not necessarily consistently. A player wins the game by making three identical symbols appear in consecutive cells horizontally, vertically or diagonally. Can either player have a winning strategy? If so, is it the player who moves first or the one who moves second?

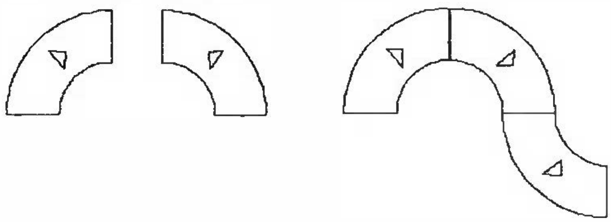
Solution

Let us call the player who makes the first move in the game "First" and his opponent "Second." We show that Second wins in an errorless game. She has to use the following strategy. If her current move may be a winning one (that is, she can complete a chain of three identical signs) she undoubtedly should do it. Otherwise, she must "reverse" the last move of her opponent; that is, place an opposite sign in the square symmetric to that occupied by the previous move with respect to the centre of the board. The described strategy ensures that First can never win this game (check it), and we have only to show that eventually Second will gain a victory. Consider the central 4×4 fragment of the board just after the second move of First in the central 2×2 square—it looks like the one in the figure below (modulo the reversion of all the signs). If the square with the shaded bottom corner were occupied by a plus, then First would be able to win, but we know that this is impossible. On the other hand, if this square is empty, Second wins by writing a plus in it. So the unique interesting case is when the square with the shaded bottom corner contains a minus. In this case, if the square with the shaded top corner is empty or contains a plus, then Second can terminate the game by writing minus or plus into the relevant square. All that is left is to notice that the square with the shaded top corner cannot contain a minus because otherwise the game would already be finished. These arguments complete the solution.



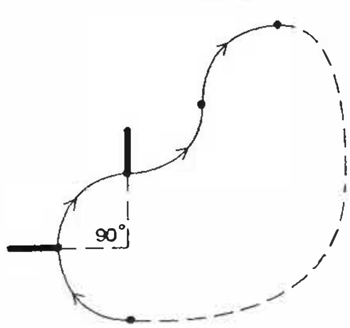
Problem 6

Each section of tracks in a model railway is a quadrant of a circle directed either clockwise or counterclockwise, as shown below in the diagram on the left. One may only assemble the track in such a way that the directions of the sections are consistent along the whole track, as illustrated below in the diagram on the right. If such a closed track can be assembled using given sections, prove that this is no longer the case if one clockwise section is replaced by a counterclockwise section.



Solution

Note that if we have assembled a legitimate closed track using k sections of type 1 and m of type 2, then $k - m$ must be divisible by 4. This can be clarified using the following reasoning. Choose the beginning of some section and travel along the track in the direction indicated by the section arrows until we return to the chosen point. Imagine that while we move along the track, we drag a small arrow that is perpendicular to the track (more strictly, perpendicular to the tangent line of the track) and that points to the outside of the track (see figure). The small arrow turns by an angle of 90 degrees in a clockwise direction when we move along a section of type 1 and in a counterclockwise direction when we move along a section of type 2. Therefore, it makes k turns in a clockwise direction and m turns in a counterclockwise direction. Assume (without loss of generality) that $k \geq m$. Since the initial and final positions of the arrow coincide, we conclude that $k - m$ turns in a clockwise direction must mean a complete turn by an angle whose measure is a multiple of 360 degrees. But this means that $k - m$ is divisible by 4. We are done now, because we see at once that $(k - 1) - (m + 1) = (k - m) - 2$ cannot be a multiple of 4.



September 2003 Problems

1. Paula bought a notebook with 96 sheets and numbered its pages in sequence from 1 to 192. Paula pulled out 25 sheets at random and added together all 50 numbers written on them. Prove that this sum cannot be equal to 1,990.
2. Exactly one of 101 coins is counterfeit. The 100 genuine coins have the same weight, but a different weight from that of the counterfeit coin. It is not known whether the counterfeit coin is heavier or lighter than a genuine coin. How can this question be resolved by two weighings on a balance scale? It is not necessary to identify the counterfeit coin.
3. Is it possible to dissect a 39×55 rectangle into 5×11 rectangles?
4. A two-player game starts with the number 1,234. Tom moves first, and he and Jerry make alternate moves thereafter. In each move, the player subtracts from the current number one of its nonzero digits. The player who obtains zero wins. Who should win in a perfectly played game?
5. Three students together solved 100 problems from the textbook. Each of them solved exactly 60 problems individually. We call a problem difficult if it is solved by only one of them, and easy if it was solved by all of them. Prove that the number of difficult problems exceeds the number of easy problems by 20.
6. For every boy in a certain village, all his female acquaintances know one another. Among the acquaintances of any girl, the number of boys is greater than the number of girls. Prove that, in this village, the number of girls is not greater than the number of boys.

November 2003 Problems

1. Each of 40 students at a technical institute has several nails, screws and bolts. Exactly 15 of them have unequal numbers of nails and bolts, and exactly 10 of them have equal numbers of screws and nails. Prove that at least 15 students have unequal numbers of screws and bolts.
2. In the stock exchange of Funny City, one can exchange any two shares for three others and vice versa. Can John exchange 100 shares of Fun Oil for 100 shares of Auto Fun, giving exactly 1,991 shares in the process?
3. Cars A, B, C and D start simultaneously from the same point on a circular racetrack. The first two cars move in a clockwise direction while the other

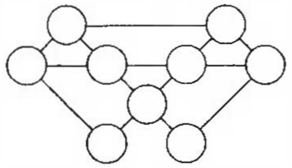
two move in a counterclockwise direction. Each moves at a constant speed, and A is faster than B. If A meets C for the first time at the same moment as B meets D for the first time, prove that A catches up with B for the first time at the same moment as C catches up with D for the first time.

4. Beginning on August 1, 1991, Baron Munchhausen tells his cook, "Today I will bring home more ducks than two days ago but fewer than one week ago." What is the greatest number of days that the baron can repeat this and not be caught in a lie?
5. A red stick, a white stick and a blue stick have the same length. Julie breaks the red stick into three parts; then Ben does the same with the white stick; and then Julie breaks the blue stick into three parts. Can Julie break the sticks so that, no matter what Ben does, it would be possible to assemble from the nine parts three triangles such that each has sides of different colours?
6. Nine teams took part in a tournament in which every two teams played against each other exactly once. Does there necessarily exist two teams such that every other team has lost to at least one of them in the tournament?

Albert Einstein (1879–1955)

This German-American theoretical physicist and philosopher, despite his fame, continued to pose mathematical problems in *Frankfurter Zeitung*.

In the diagram below, the nine spheres represent vertices of four small and three larger equilateral triangles. Place the numbers from 1 to 9 in the spheres in such a way so that the sum in each of the seven triangles is always the same.



Corner triangles are equilateral

Infinite Sets and Georg Cantor

Sandra M. Pulver

Infinite quantities have perplexed mankind for thousands of years. To the caveman, the distance to the horizon was infinite. To the modern man, infinite is the size of the universe. Throughout man's existence, the concept of infinity has become so increasingly complex that some men have protested against the use of infinite magnitude in mathematics.

In the late 19th century, Georg Cantor shed an enormous light on the concept of infinity. His work has given the subject an acceptability in mathematics.

Georg Ferdinand Ludwig Philip Cantor was born in St. Petersburg, Russia, on March 3, 1845. By the time of his death 73 years later, Cantor had permanently changed the world of mathematics with his ideas on set theory and infinite sets. He was a revolutionary who held to his theories and the notion of the completed infinite in the face of strong opposition. Mathematicians argued against Cantor, yet in the end, Cantor prevailed. Cantor's theories were shown to be logically sound and consistent under a certain set of axioms.

The concepts Cantor introduced into set theory with his idea of transfinite sets changed the outlook of mathematicians. Students of mathematics must understand the ideas of Cantor to truly understand their discipline.

At the very heart of Cantor's theories lies the concept of cardinality or set equivalence. This concept rests on the fact that two groups need not be counted to be proven to have the same number of elements. One can attempt to establish a one-to-one correspondence between the two groups. If a one-to-one correspondence between the two groups can be set up, then they have the same number of elements. Thus Cantor defined set equivalence as follows: Two sets M and N are equivalent if it is possible to put them, by some law, in such a relation to one another that to every element of each one of them corresponds one and only one element of the other. Mathematicians say that two equivalent sets have the same cardinality.

Cantor defined those sets that could be put into a one-to-one correspondence with the natural numbers as a denumerable or countably infinite set. Thus any set with a cardinality equal to that of the natural numbers was called denumerable.

In 1893, after Cantor established the different cardinality of various infinite sets, he needed a notation to represent the different cardinalities. Because of his Jewish background, he chose the Hebrew letter aleph. Cantor defined aleph-null, $\aleph_0$, as the cardinality of the natural numbers or positive integers. The aleph notation is the one used today to describe infinite sets.

It is easy to establish that the set of integers is denumerable using elementary algebra. A one-to-one correspondence with the set of integers can be shown by simply rearranging them so that there is a definite first element of a set such as 0, 1, −1, 2, −2, 3, −3, 4, −4, 5, −5, ... We see that the set of positive and negative integers is as large or has the same cardinality as the set of natural numbers.

$$\begin{array}{cccccccc} 0, & 1, & -1, & 2, & -2, & 3 & \dots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ 1, & 2, & 3, & 4, & 5, & 6 & \dots \end{array}$$

Even though the set of even numbers is a proper subset of the natural numbers, it is equivalent to it in cardinality and is denumerable. We can set up a one-to-one correspondence between these two sets.

$$\begin{array}{cccccc} 1, & 2, & 3, & \dots, & n, & \dots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ 2, & 4, & 6, & \dots, & 2n, & \dots \end{array}$$

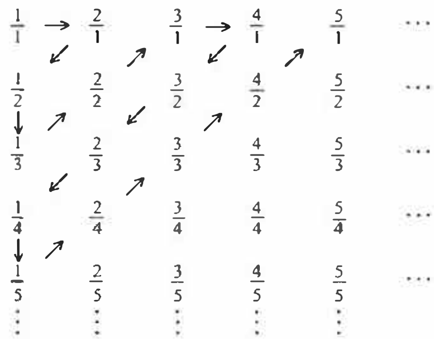
The set of odd numbers is also denumerable.

$$\begin{array}{cccccc} 1, & 2, & 3, & \dots, & n, & \dots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ 1, & 3, & 5, & \dots, & 2n-1, & \dots \end{array}$$

Cantor used the existence of this equivalence as his definition of infinity, where he stated that an infinite set is one that can be put into a one-to-one correspondence with a subset of itself.

Finding the cardinality of the set of rational numbers required a different approach. We see that an infinity of rational numbers can be "packed in" between any two rational numbers. This means that the set of rational numbers is a dense set because no rational number has an immediate successor. The set of positive integers is discrete because every element of the set has an immediate successor. The question is whether the set of rational numbers, which is dense, has the same number of elements ($\aleph_0$) as the set of positive integers, which is discrete. How can anyone put the rational numbers in a one-to-one correspondence when an infinity of rational numbers can be "placed in" between any two?

For this proof Cantor constructed a two-dimensional array of rational numbers. When zero is placed above the array, a list of all the rational numbers is formed. Cantor then proceeded to count the rational numbers. He drew arrows up and down the diagonals of this array, effectively counting the rational numbers. This set up a one-to-one correspondence between the rationals and the natural numbers. Through this simple method Cantor showed the set of rational numbers to be denumerable.



These fractions can be written as the set of two integers and then put into a one-to-one correspondence with the natural numbers as follows:

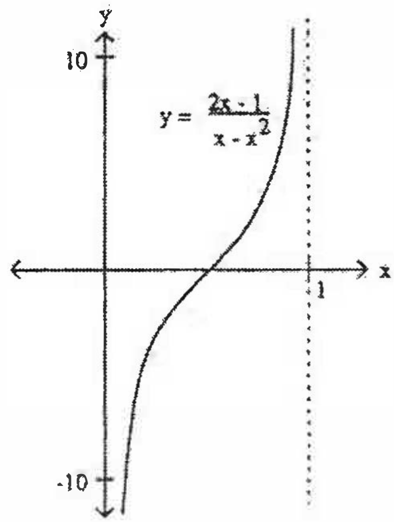
1,	2,	3,	4,	5	...
↕	↕	↕	↕	↕	
(1, 1)	(2, 1)	(1, 2)	(1, 3)	(2, 2)	...

Therefore $\aleph_0$ is also the cardinal or transfinite number for the set of rational numbers.

After these proofs, it seemed to the mathematical community that every infinite set was denumerable and that no set had a higher transfinite cardinal number than $\aleph_0$. In 1874 with his paper "On a Property of the Collection of all Algebraic Numbers," Cantor revealed that a nondenumerably infinite set existed. The paper dealt with the cardinality of the set of real numbers. Cantor proved that the set of real numbers is not denumerable.

Cantor began the proof by establishing that a one-to-one correspondence existed between the open interval (0, 1) and the set of real numbers using the function:

$$\frac{(2x-1)}{x - x^2}$$



Cantor then assumed that the set of all reals in (0, 1) was denumerably infinite. This leads to the conclusion that a list can be formed pairing each natural number with one real number. Cantor established a hypothetical list pairing the set of real numbers between zero and 1 and the set of natural numbers. He then listed a set of infinite decimals:

$1 \rightarrow 0.\ a_1\ a_2\ a_3\ a_4\ \ldots$
$2 \rightarrow 0.\ b_1\ b_2\ b_3\ b_4\ \ldots$
$3 \rightarrow 0.\ c_1\ c_2\ c_3\ c_4\ \ldots$
⋮

This set should contain all the real numbers in the given interval, (0, 1).

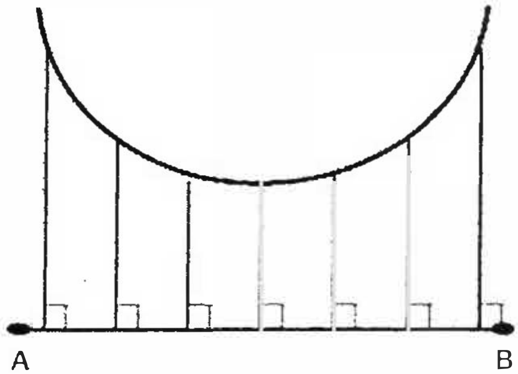
In his proof, usually known as the diagonal proof, Cantor found that he could define a real number z in (0, 1) not on the list. He constructed a number that had as its n-th decimal place a number different from the n-th decimal place of the n-th number on the list and not equal to zero or nine,

$0.\ a'_1 b'_2 c'_3 d'_4 \ldots$

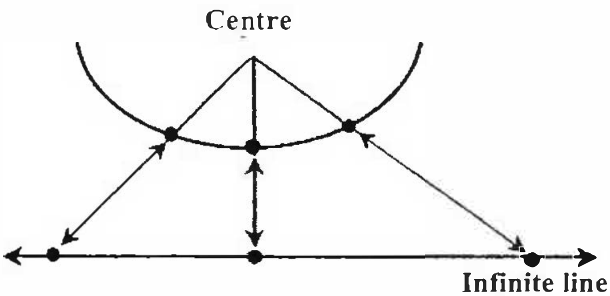
This created a number z that could not be on the list. The number z differed in at least one decimal place with every number on the list. The elimination of zero or nine as choices for a decimal place negated the possibility of infinitely repeating decimals equal to zero or one. Thus, the open interval (0, 1) could not be denumerable, and because the open interval (0, 1) has the same cardinality as the set of real numbers, the reals could not be denumerable.

Cantor named this cardinality aleph-one, $\aleph_1$. Modern mathematicians usually call this cardinality C instead of aleph-one (C for continuum, an open interval of real numbers).

To show geometrically that the reals are nondenumerable, Cantor established an astonishing fact: there are as many points along an infinite (straight) line as there are on a finite segment of it.



Each vertical line segment is perpendicular to the segment AB, thus ensuring not only that each vertical line segment will pass through the segment AB itself, but that it will only pass through one point on the semicircle. So we can match the points of the segment AB in a one-to-one correspondence with the points on the semicircle, thus proving that the segment and the semicircle have the same number of points.



Now each line segment is drawn from the centre of the same semicircle to the line, again ensuring that each segment passes through only one point on the circle and only one point on the line. Thus, the points of the semicircle are now matched one-to-one with the points of the entire line. Therefore, since the finite line segment and infinite line have both been put into a one-to-one correspondence with points on a semicircle, we can conclude that a finite line segment and an infinite line have exactly the same number of points.

But was there a set with cardinality greater than aleph-one? Where could an infinite set with a higher cardinality lie?

With his proof of the nondenumerability of the continuum, Cantor created, in effect, a hierarchy of infinites. There is no infinite set with a cardinality that is less than that of the natural numbers ($\aleph_0$), and all sets that are not denumerable (have the same cardinality as the real numbers) have a higher level of infinity than all the countable sets. At this point one may be wondering if any sets exist with a cardinality that is greater than the real numbers. It may seem reasonable to presume that C (the cardinality of the real numbers) is the greatest possible cardinality.

However, as Cantor himself soon discovered, it turns out that there are sets that are greater in cardinality than the set of real numbers.

These sets are Power Sets, the set of all subsets of a given set. For example, the power set of the set $\{a, b, c\}$ consists of the subsets $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{a, c\}$ and $\{b, c\}$, to which we must add the "empty" or "null" set $\{\ \}$ and the set $\{a, b, c\}$ itself. Thus, from the original set of three elements we get a new set of eight ($= 2^3$) elements. In fact, Cantor concluded that the power set of any given set always has more elements than the original set. Cantor's theorem (as it is called today) shows that, given any set, we can always construct a set that has a greater cardinality.

Cantor's theorem was proven, once again, by contradiction, for he first assumed that there is a largest infinite set, and then demonstrated that there must be one even larger (that is, the power set).

Let X be an arbitrary infinite set (of any cardinality), which we can represent as $X = \{a, b, c, d, e, \ldots\}$. Now let us assume that the members of X can be put into a one-to-one correspondence with its power set, which can be represented by

$$P(X) = \{\{e\}, \{a, b\}, \{b, c, e\}, \{a, c\}, \{a\}, \ldots\}.$$

Such an arbitrary matchup would look something like:

X		$P(X)$
a	⇔	$\{c, d\}$
b	⇔	$\{a\}$
c	⇔	$\{a, b, c, d\}$
d	⇔	$\{b, e\}$
e	⇔	$\{a, c, e\}$
⋮	⋮	⋮

Now let us consider the different ways that a member of X could be paired with the subsets of X. Some of the elements of X are matched with subsets that contain them. For example, here the element e is matched with the subset $\{a, c, e\}$, of which it is a member. Also notice that some of the elements of X are matched with subsets that do not contain them, such as the element d, which is matched with the subset $\{b, e\}$, of which it is not a member.

Let us consider the set of elements of X that are not matched up with subsets that contain them. This set, which we'll call S, is clearly a subset of X; thus, it must appear somewhere in our matchup listing above. However, what could the element X be that matches with S? It cannot be a member of S, because S was specifically constructed to contain only those elements of X that do not match up to the sets containing them. What happens if the element of X that matches up with S is not contained in S? Well, then it must be contained in S, again by the definition of S. Clearly, this is a contradiction. (The existence of this contradiction forces us to understand that no element of X can be matched with this subset S.) This means that X and $P(X)$ cannot be put into a one-to-one correspondence, thus indicating that they cannot have the same cardinality. Therefore, we can conclude that one set must be larger than the other. Because X cannot be put into a one-to-one correspondence with a proper subset of $P(X)$, we can conclude that the cardinality of $P(X)$ must therefore be larger than that of X. Hence, Cantor's theorem is indeed true and, as a result, there can be no "largest infinity" and the kinds of infinity are therefore infinite.

It would seem to the observer that Cantor's set theory was an incredible success. But no one knew better than Cantor the imperfections in his theory. The problem with the theory that troubled Cantor most was his inability to prove that no aleph value existed between aleph-null and C. He searched his entire life for a proof, but died without ever formulating one.

It is unsurprising that Cantor never created a successful proof for his theorem. In 1938, Kurt Godel demonstrated that Cantor's "Continuum Hypothesis" (that is, that $C=\aleph_0$) could not be disproved within the confines of set theory, (that is, that the Continuum Hypothesis was relatively consistent with and did not contradict the axioms of set theory). In 1963 Paul Cohen demonstrated that the Continuum Hypothesis could not be proved from (and was independent of) the axioms of set theory. In other words, he proved that the negation of the Continuum Hypothesis ($C > \aleph_1$) would also be consistent with the axioms of set theory. This means that two different systems can be set up, both valid, using the continuum hypothesis and its negation.

Georg Cantor opened up whole new vistas in the world of mathematics. He engaged the minds of a whole generation with his concept of the infinite.

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On Vectorial Proofs, Circumradii and Equilateral Triangles

David E. Dobbs

Introduction

In a recent note, Norman Schaumberger (2002) has given a new proof of the following fact. Let A, B and C be noncollinear points (in, say, the Euclidean plane or three-space) and let G be the centroid of the triangle $\Delta = \Delta ABC$; then, as P varies over all points, $|PB|^2 + |PB|^2 + |PC|^2$, the sum of the squares of the distances from P to the vertices of Δ is as small as possible if and only if $P = G$. The preceding assertion is an immediate consequence of what Schaumberger actually proves, namely, that for any point P, $|PA|^2 + |PB|^2 + |PC|^2 = 3|PG|^2 + |GA|^2 + |GB|^2 + |GC|^2$. The preceding equation is not new, although earlier proofs of it were not especially illuminating either: see, for instance, the proof via seemingly unmotivated calculations in analytic geometry published about 50 years ago in ([Altshiller-Court 1952], Theorem 109, 70). Nevertheless, Schaumberger's proof has considerable merit, for it does illuminate the situation, prompting Schaumberger (2002) to precede his proof with the declaration, "Here is why." Indeed, Schaumberger's proof is able to address the "why" of the situation because it uses vectorial methods to simultaneously discover and prove the underlying facts. In this way, Schaumberger's argument fits well into the "investigative, quantitative" pedagogic philosophy described in Dobbs (2001, 28), and thus could be used as enrichment material for precalculus students who are familiar with the basic properties of the dot product of vectors.

In fact, the argument of Schaumberger (2002) leads to additional ways of providing enrichment material for a precalculus course. Suppose, in the above notation, that P is taken to be the circumcentre K of Δ (that is, the centre of the circum[scribed]circle of Δ). Then the equation that was established by Schaumberger (and already known in [Altshiller-Court 1952]) leads at once to a formula for the circumradius R of Δ; that is, the radius of the circumcircle of Δ. Although this formula for R is already known [(Altshiller-Court 1952), Corollary 110, 71], we believe that there would be much pedagogic value in making available a proof of it that is based on Schaumberger's vectorial methods. Doing so is the first main purpose of the present note. While this note could be used in conjunction with other enrichment material involving circumcircles, such as (Dobbs, to appear), we have arranged it to be essentially self-contained. The presentation of the first four results would be appropriate for classes acquainted with dot product and the material on centroids typically covered in the prerequisite course on geometry.

Now, let k denote the incentre of Δ (that is, the centre of the in(scribed)circle of Δ), and let r be the inradius of Δ. According to a theorem of Euler (see [(Coxeter and Greitzer 1967), Theorem 2.12]), $|Kk|^2 = R^2 - 2Rr$. As noted in [Exercise 5, 31], it is an easy consequence that $R \geq 2r$. Accordingly, it seems natural to ask which triangles Δ satisfy $R = 2r$. By the above-cited result of Euler, it is equivalent to ask which triangles Δ satisfy $K = k$. The second main purpose of this article is to answer this and related questions: see Theorem 2.5. While it may be possible to prove Theorem 2.5 using some arcane, sophisticated methods, the proof given for it in this paper can be presented in the typical Precalculus class. In fact, Theorem 2.5 may be read/presented independently of the first four results in the paper. Among the topics reinforced by the proof of Theorem 2.5 are the following: the midpoint formula, slope, equations of lines, absolute value, the distance formula (between two points), the formula for the distance from a point to a line and criteria for congruence of triangles (as covered in the typical geometry course that is a prerequisite for precalculus).

We next describe the organization of this note. Lemma 2.1 gives the well-known formula for the position vector of the centroid G in terms of the position vectors of the vertices A, B and C. Proposition 2.2 (a) develops useful expressions for the vectors $\overrightarrow{GA}$, $\overrightarrow{GB}$ and $\overrightarrow{GC}$; then Proposition 2.2 (b) uses these expressions to prove a fact that was stated without proof in Schaumberger (2002), namely, that $\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC}$ is the zero vector. Proposition

2.3 (a) uses vectorial methods to prove a new formula for the sum $|GA|^2 + |GB|^2 + |GC|^2$, while, for the sake of completeness, Proposition 2.3 (b) includes the proof of Schaumberger. Then Corollary 2.4 gives the promised formula for the circumradius R. Finally, Theorem 2.5 shows that the equality of the circumcentre, centroid and incentre characterizes equilateral triangles.

Results

First, we recall the notion of a position vector. If a fixed point P is taken as the origin for a vectorial representation of the points of the Euclidean plane, then the *position vector of a point Q (relative to P)* is $\vec{PQ}$.

Lemma 2.1. *If the vertices of $\Delta = \Delta ABC$ have position vectors* $u = \vec{PA}$, $v = \vec{PB}$ *and* $w = \vec{PC}$ *and if G is the centroid of Δ, then the position vector of G is* $\vec{PG} = (1/3)(u + v + w)$.

Proof. Let D be the midpoint of the segment AB. Observe that the position vector of D is

$$\vec{PD} = \vec{PA} + \vec{AD} = u + \tfrac{1}{2}\vec{AB} = u + \tfrac{1}{2}(\vec{PB} - \vec{PA}) = u + \tfrac{1}{2}(v - u) = \tfrac{1}{2}(u + v).$$

As $\vec{PG} = \vec{PD} + \vec{DG}$, we proceed next to describe $\vec{DG}$. It is well known that G is located two-thirds of the way along the median from the vertex C to the midpoint D, and so

$$\vec{DG} = \tfrac{1}{3}\vec{DC} = \tfrac{1}{3}(\vec{PC} - \vec{PD}) = \tfrac{1}{3}(w - \tfrac{1}{2}(u + v)) = \tfrac{1}{3}w - \tfrac{1}{6}(u + v)$$

Therefore,

$$\vec{PG} = \vec{PD} + \vec{DG} = \tfrac{1}{2}(u + v) + \tfrac{1}{3}w - \tfrac{1}{6}(u + v),$$

which simplifies to $(\tfrac{1}{3})(u + v + w)$, to complete the proof.

Proposition 2.2 (b) isolates a well-known fact that forms the starting point in the proof of Schaumberger (2002).

Proposition 2.2. *Let G be the centroid of $\Delta = \Delta ABC$. Then:*

(a) If P is any point and the vertices of Δ have the position vectors $u = \vec{PA}$, $v = \vec{PB}$ *and* $w = \vec{PC}$, *then* $\vec{GA} = (\tfrac{1}{3})(2u - v - w)$, $\vec{GB} = (\tfrac{1}{3})(2v - u - w)$, *and* $\vec{GC} = (\tfrac{1}{3})(2w - u - v)$.

(b) $\vec{GA} + \vec{GB} + \vec{GC} = 0$, *the zero vector.*

Proof. (a) Using Lemma 2.1, observe that

$$\vec{GA} = \vec{PA} - \vec{PG} = u - 1/3(u + v + w),$$

which simplifies to $(\tfrac{1}{3})(2u - v - w)$, as asserted. The proofs for $\vec{GB}$ and $\vec{GC}$ are similar and, hence, left for the reader.

(b) Using the results of (a), we find, after algebraic simplification, that $\vec{GA} + \vec{GB} + \vec{GC}$ can be rewritten as $0u + 0v + 0w = 0 + 0 + 0 = 0$. The proof is complete.

The above proofs have made use of addition and subtraction of vectors, as well as scalar multiplication of vectors. The next proof also uses the properties of the dot product of vectors. Recall, in particular, that if v is a vector, then the square of its length is given by the dot product $v \times v$.

Proposition 2.3. *Let G be the centroid of $\Delta = \Delta ABC$. Then:*

(a) $|GA|^2 + |GB|^2 + |GC|^2 = (\tfrac{1}{3})(|AB|^2 + |BC|^2 + |CA|^2)$.

(b) If P is any point, then $|PA|^2 + |PB|^2 + |PC|^2 = 3|PG|^2 + |GA|^2 + |GB|^2 + |GC|^2$.

Proof. (a) Let u, v and w be as in the statement of Proposition 2.2 (a). Then, by the above remark and Proposition 2.2 (a), we can rewrite $|GA|^2 + |GB|^2 + |GC|^2$ as

$$\tfrac{1}{3}(2u - v - w) \times \tfrac{1}{3}(2u - v - w) + \tfrac{1}{3}(2v - u - w) \times \tfrac{1}{3}(2v - u - w) + \tfrac{1}{3}(2w - u - v) \times \tfrac{1}{3}(2w - u - v),$$

which simplifies algebraically to

$$\tfrac{2}{3}(u \times u + v \times v + w \times w - u \times v - u \times w - v \times w).$$

On the other hand, $|AB|^2 + |BC|^2 + |CA|^2$ can be rewritten as

$$(\vec{PB} - \vec{PA}) \times (\vec{PB} - \vec{PA}) + (\vec{PC} - \vec{PB}) \times (\vec{PC} - \vec{PB}) + (\vec{PA} - \vec{PC}) \times (\vec{PA} - \vec{PC}),$$

that is, as

$$(v - u) \times (v - u) + (w - v) \times (w - v) + (u - w) \times (u - w),$$

which simplifies algebraically to

$$2(u \times u + v \times v + w \times w - u \times v - u \times w - v \times w).$$

The assertion follows immediately.

(b) (Schaumberger 2002) Just as in the proof of (a), the proof of (b) involves a vectorial reformulation of the assertion. Once again, we let u, v and w denote the position vectors of A, B and C, respectively. Observe that $|PA|^2 + |PB|^2 + |PC|^2 = u \times u + v \times v + w \times w$. Rewriting u as $\vec{PG} + \vec{GA}$, with analogous renderings of v and w, we see, after algebraic simplification, that the above sum of dot products differs from the sum

$$3\vec{PG} \times \vec{PG} + \vec{GA} \times \vec{GA} + \vec{GB} \times \vec{GB} + \vec{GC} \times \vec{GC} = 3|PG|^2 + |GA|^2 + |GB|^2 + |GC|^2$$

by

$$2\vec{PG} \times (\vec{PA} + \vec{PB} + \vec{PC}).$$

According to Proposition 2.2 (b), this difference is $2\vec{PG} \times 0 = 0$. Since the difference is 0, we have that $|PA|^2 + |PB|^2 + |PC|^2 = 3|PG|^2 + |GA|^2 + |GB|^2 + |GC|^2$, to complete the proof.

We can now complete an essentially vectorial proof of a formula for the circumradius of a triangle that was given in [(Altshiller-Court 1952), Corollary 110, p. 71].

Corollary 2.4. *If G is the centroid and R is the circumradius of* $\Delta = \Delta ABC$, *then* $R = \sqrt{|KG|^2 + (1/9)(|AB|^2 + |BC|^2 = |CA|^2)}$.

Proof. In Proposition 2.3, take the "origin" P to be K, the circumcentre of Δ. Then $|PA| = |PB| = |PC| = R$, and so, combining parts (b) and (a) of Proposition 2.3, we have that

$3R2 = |KA|^2 + |KB|^2 + |KC|^2 = 3|KG|^2 + 1/3(|AB|^2 + |BC|^2 + |CA|^2)$.

By straightforward algebra, we can solve for R^2 and then for R, thus yielding the asserted formula. The proof is complete.

The formula for R in Corollary 2.4 takes a particularly simple form in case $K = G$. Accordingly, it seems natural to ask which triangles have the property that their circumcentre coincides with their centroid. One could also ask which triangles have the property that their circumcentre coincides with their incentre. The answers to these questions are given in Theorem 2.5, which includes several characterizations of equilateral triangles.

First, it is convenient to recall the following consequences of congruence criteria: the circumcentre K (respectively, the incentre k) of Δ is the intersection of the perpendicular bisectors of the sides (respectively, the intersection of the bisectors of the interior angles) of Δ.

Theorem 2.5. *Let G be the centroid, K the circumcentre, k the incentre, R the circumradius, and r the inradius of* $\Delta = \Delta ABC$. *Then the following six statements are equivalent:*

(1) $K = G$;
(2) $K = k$;
(3) $k = G$;
(4) $K = k = G$;
(5) $r = R/2$;
(6) Δ *is an equilateral triangle.*

Proof. (6) ⇒ (4): Besides the material recalled above, we mention two additional facts that should be familiar. The first of these is the following consequence of the side–angle–side congruence criterion: the bisector of the vertical angle of an isosceles triangle is the perpendicular bisector of the base of the triangle. The second fact is actually a definition: the centroid of a triangle is the intersection of the medians of the triangle. Taking all the "recalled" information into account, we see at once that if Δ is equilateral and a line L is the bisector of an interior angle of Δ, then K, k and G all lie on L, whence (by intersecting two such angle bisectors) $K = k = G$. Therefore, (6) ⇒ (4).

(2) ⇒ (6): Assume (2); that is, $K = k$. To prove (6), it suffices to show that any two sides of Δ have the same length. We shall show that $|CA| = |CB|$. To show this, it is enough to prove that $\angle CAB \cong \angle CBA$. (Indeed, given this congruence of angles, if the altitude from C meets the side AB at the point P, then $\Delta CAP \cong \Delta CBP$ by the angle–angle–side congruence criterion, whence $|CA| = |CB|$.) Observe that the bisectors of angles $\angle CAB$ and $\angle CBA$ meet at $k = K$. As K is on the perpendicular bisector of the side AB, it now follows easily (via side–angle–side) that K is equidistant from A and B; that is, $|KA| = |KB|$. Then, since the base angles of an isosceles triangle are congruent (another consequence of the side–angle–side criterion), $\angle KAB \cong \angle KBA$. Therefore, $\angle CAB = 2\angle KAB \cong 2\angle KBA = \angle CBA$, thus proving (6).

(1) ⇒ (2): Assume (1): that is, $K = G$. Let the line CK meet the side AB at D; the line AK meet the side BC at E; and the line BK meet the side CA at F. Since K is the centroid of Δ and two points determine a line, it follows that D, E and F are the midpoints of the sides AB, BC and CA, respectively, and, moreover, that the lines CD, AE and BF are the perpendicular bisectors of the sides AB, BC and CA, respectively. Therefore, the distances $|KA|$, $|KB|$ and $|KC|$ are equal. As the base angles of an isosceles triangle are congruent, if follows that $\angle KAD \cong \angle KBD$, $\angle KBE \cong \angle KCE$ and $\angle KCF \cong \angle KAF$. Then $\Delta KAD \cong \Delta KBD$ (by either the angle–angle–side criterion or the hypotenuse–side criterion), whence $\angle AKD \cong \angle BKD$ as corresponding parts of congruent triangles. Similarly, $\angle BKE \cong \angle CKE$ and $\angle CKF \cong \angle AKF$. Using the fact that vertically opposite angles are congruent, we now see that all the above six angles with vertex at K are congruent. (For instance, considering the vertically opposite angles relative to the lines CD and FB leads to the conclusion that $\angle BKD \cong \angle CKF$.) Then the angle–angle–side criterion yields that $\Delta KFA \cong \Delta KDA$, whence $\angle FAK \cong \angle DAK$. In other words, K is on the bisector of the angle $\angle CAB$. Similarly, K is also on the bisectors of $\angle ABC$ and $\angle BCA$, and so $K = k$, completing the proof of (2).

(3) ⇒ (6): Assume (3); that is, $k = G$. Orient matters so that Δ is ΔABC, with vertices $A(0, 0)$, $B(c, 0)$ and $C(d, e)$, for some real numbers c, d and e such that $c > 0$ and $e > 0$. Our task is to prove (6), namely, that the sides of Δ all have the same length. By the distance formula, this means that our task is to prove that $c = \sqrt{(d - c)^2 + e^2} = \sqrt{d^2 + e^2}$.

Observe that the coordinates of $k = G$ are $((d + c)/3, e/3)$. One way to see this is to find the coordinates of the point of intersection of the medians $y = (e/(d + c))x$ and $y = (e/(2d - c))(2x - c)$; another way is to use Lemma 2.1. The details are straightforward and, hence, left to the reader.

We next find the (perpendicular) distances from k to the sides of Δ. Of course, the distance from k to the side AB is $e/3$. Next, observe that the line CA has equation $y = (e/d)x$, or equivalently, $ex - dy = 0$. Therefore, the distance from k to the side CA is

$$\frac{\left| e\left(\frac{d+c}{3}\right) - d\left(\frac{e}{3}\right) \right|}{\sqrt{e^2 + d^2}} = \frac{\frac{ec}{3}}{\sqrt{e^2 + d)^2}}.$$

Similarly, the line BC has equation $y = (e/(d - c))(x - c)$, or, equivalently, $ex + (c - d)y - ec = 0$. Therefore, the distance from k to the side BC is

$$\frac{\left| e\left(\frac{d+c}{3}\right) + (c - d)\left(\frac{e}{3}\right) - ec \right|}{\sqrt{e^2 + (c - d)^2}} = \frac{\frac{ec}{3}}{\sqrt{e^2 + (c - d)^2}}.$$

As the incentre k is equidistant from the sides of Δ, we now have that

$$\frac{e}{3} = \frac{\frac{ec}{3}}{\sqrt{e^2 + d^2}} = \frac{\frac{ec}{3}}{\sqrt{e^2 + (c - d)^2}}.$$

Cancelling the positive quantity $e/3$ and cross-multiplying, we find that $c = \sqrt{e^2 + d^2} = \sqrt{e^2 + (c - d)^2}$, thus completing the proof of (6).

Next, notice that (4) trivially implies each of (1), (2) and (3). In view of the implications that were proved above, this completes the proof that conditions (1), (2), (3), (4) and (6) are equivalent.

Finally, it suffices to prove that (2) ⇒ (5). According to the theorem of Euler [(Coxeter and Greitzer 1967) Theorem 2.12] that was recalled in the introduction, $|Kk|^2 = R^2 - 2Rr$. Hence, (2) ⇒ $|Kk|^2 = 0 \Rightarrow R(R - 2r) = R^2 - 2Rr = 0 \Rightarrow R = 2r$ (since $R > 0$) ⇒ (5). The proof is complete.

We close with two points. First, the proof given above for the equivalence of conditions (1), (2), (3), (4) and (6) in Theorem 2.5 is self-contained, making no reference to the theorem of Euler [(Coxeter and Greitzer 1967) Theorem 2.12] that was mentioned in the Introduction. Second, the proof given above that (2) ⇒ (6) in Theorem 2.5 was shown to the author by his colleague, Pavlos Tzermias. This proof replaces the author's original analytic proof, which was considerably longer. The proof given here that (2) ⇒ (6) was seen by Tzermias as a high school student in Greece approximately 20 years ago, thus providing additional anecdotal evidence of a significant difference between the typical North American and European high school curricula in geometry.

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Srinivasa Ramanujan (1887–1920)

One day this Indian mathematician drove with the British mathematician Godfrey Harold Hardy (1877–1947) in a taxi, which was marked with the number 1729. "A very boring number," commented Hardy. "But quite the opposite!" responded Ramanujan immediately. "It is a very interesting number in that it is the smallest number which can be written as the sum of two cubic numbers in two different ways." What are the two different ways?

Fractions Are Much More Than Pies

Jerry Ameis

Fraction fun begins as early as Kindergarten where students have been known to say, "Cut the cookie in half and give me the bigger half." It continues as students encounter fraction notation, fraction equivalency and operations with fractions. Why do fractions seem to be a difficult notion for students?

Our notation system can be a source of difficulty. When students encounter, for example, 2/3, they must understand that writing down two whole numbers does not represent a whole number. This can be magical from their point of view (Moss and Case 1999). A glimpse of early fraction notation (Cajori 1993) may help the reader appreciate students' struggles with fraction notation (see Table 1).

Table 1
Ancient Symbols for Fractions

Ancient Culture	Culture's Symbol	Our Symbol
Egypt		1/5
Greece	⊖′	1/9
Mesopotamia		1,132/100 (11.32)
Rome		7/12
India		2/3
China		48,125/1,000 (48.125)

Ancient cultures invented a variety of notation systems for fractions. The Mesopotamian and Chinese cultures did not invent special symbols for them, but they extended their place value system to name fractional amounts. Most ancient cultures invented a notation system for fractions that involved some form of fraction indicator. For example, the early Greeks placed an apostrophe-like mark above a counting number symbol. Some of the fraction notation systems were limited in the kinds of fractions they named. The early Egyptians, with a few exceptions, had symbols only for unit fractions. The Romans had symbols for a restricted set of fractions whose denominators were 12, 24, 48 and 72.

While a notation system can partly explain why students have difficulty with fractions, the complexity of fraction meanings is another reason. The literature indicates that students are unlikely to address problem situations involving fractions well or understand fraction arithmetic unless they clearly understand the full range of meanings of fractions (Tzur 1999). Five meanings of fractions seem important for teaching middle-year students.

Five Meanings of Fractions

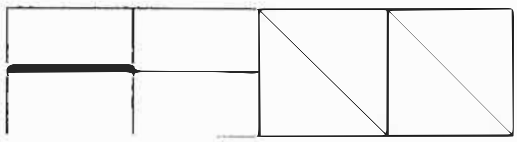
The five meanings recommended here serve as conceptual models or tools for thinking about and working with fractions. They also serve as a framework for designing teaching activities that engage students in sense making as they construct knowledge about fractions. A discussion of these meanings follows. The order of presentation does not imply an order of teaching.

The Part of a Whole Meaning

When a five-year-old child says, "I ate half of the cookie," he or she is expressing a part-whole relationship. The child uses "half" not in the sense of a number but in the sense of an actual or imagined action that involves cutting a whole physical object in the middle. The imagined or actual action of cutting a whole object into n equivalent/equal parts underlies the part of whole meaning of fraction. We represent each part symbolically by the fraction notation $1/n$.

Equivalent/equal may have different implications in different contexts. When we cut a rectangular granola bar into eight equivalent pieces and call each piece 1/8, we mean equivalent in the sense of area. The pieces normally look the same but they need not be. Figure 1 illustrates one way of cutting a granola bar into eight pieces equal in area but where not all of the pieces are the same shape. When we fold a rope into eight equal sections by making a succession of half folds, we mean equal in the sense of length.

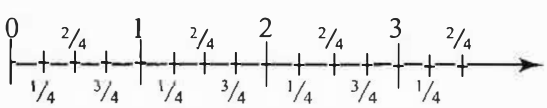
Figure 1
Cutting a Granola Bar into Eight Equivalent Pieces



The Name for a Point Meaning

This meaning best illustrates fractions as numbers distinct from counting numbers. The name for a point meaning grows out of the part of a whole meaning—the latter is useful for identifying and naming the points that lie between the points on a line that name whole numbers. These in-between points are obtained by applying the part of a whole meaning to a succession of unit segments of the number line (see Figure 2). This results in unit segments that are subdivided in the same way into equal lengths (into quarters in Figure 2). The segments differ in that each one begins with a different whole number name. In this way, a different number name is attached to each identified point on the number line (for example, 1/4, 1 1/4, 2 ¼ and so on). A number line that contains fraction-named points can serve as a ruler for measuring length using fractional amounts of units of length.

Figure 2
A Number Line Naming Quarters



The Part of a Group/Set Meaning

The part of a group/set meaning of a fraction involves a subgroup/subset of a collection of discrete things. For example, if there are 23 books of varying size and content on a shelf and 14 of them are novels, we can represent this situation by the fraction 14/23. In this case, by 14/23, we mean that 14 out of the 23 books are novels. This meaning is significantly different from the "part of the whole" and "name for a point" meanings. The part of a group/set meaning involves mentally placing discrete things into categories (for example, red, prime)—a different enterprise than cutting up a whole into equivalent parts or naming a point on a number line.

The quantification of probability involves the part of a group/set meaning of a fraction. For example, suppose a bag contains 20 marbles of various sizes and types. Seven of the marbles are crystals (big ones, medium ones and small ones). You reach into the bag and pull out one marble. The probability of that marble being a crystal is 7/20.

Fractions that are derived from the part of a group/set meaning have an interesting property. They are not additive in the linear measurement sense. Consider the following situation.

A football team played 8 away games and won 4 of them. It played 10 home games and won 9 of them. We can use the part of a group/set meaning to translate this situation into fraction form. The team won 4/8 of the away games and 9/10 of the home games. We can ask the question, "How many games did it win altogether, expressed as a fraction?" The answer is 13/18, and by this we mean that the team won 13 out of the 18 games it played. In obtaining the fraction 13/18, we added the away wins and home wins and added the away games and home games. If we represent what we did using fraction notation, we are compelled to write $4/8 + 9/10 = (4 + 9)/(8 + 10) = 13/18$. In other words, we added two fractions by adding numerators to numerators and denominators to denominators. This unorthodox algorithm produces the answer that makes sense (13/18 in this case). The standard fraction addition algorithm would not produce the answer that makes sense.

This situation is not restricted to football games. It occurs whenever the part of a group/set meaning is the conceptual basis for using fractions. I leave it to the reader to work out the details for a probability example. Imagine two bags containing dimes. Bag #1 has 5 dimes, 3 of which were minted in 1990. Bag #2 has 10 dimes, 4 of which were minted in 1990. What is the probability of pulling out a dime minted in 1990 from bag #1? From bag #2? Now imagine combining both bags into one bag (thus adding the contents). What is the probability of pulling out a dime minted in 1990 now? Which fraction addition algorithm produces that result? The standard one or the unorthodox one suggested above?

The intent of the above is not to encourage the teaching of an unorthodox fraction addition algorithm. Rather, it is to call attention to a fundamental quality of mathematics and its applications. Within formal mathematics, the rules are clear. But applications of mathematics (involving meanings) are always at our risk. When a mathematical notation is given more than one meaning, this can lead to confusing and even conflicting uses of the notation.

The Ratio Meaning

Ratio is a statement about a numerical relationship between two quantities that may or may not involve different constructs. Suppose the ratio between flour and butter in a recipe is 5 cups flour to 3 tablespoons butter. In this situation, the relationship is between the same construct (volume) but it does involve different units. Suppose that in an ecosystem, the optimal ratio between deer and forage is 1 deer to 2 tons of forage per square mile. This ratio involves different constructs, a count of discrete entities and a measurement of weight and area. Ratio may be explicit as in the examples above or it may be implicit. If I have two-thirds as many notebooks as Harry, this implies a ratio of 2 to 3 (I have 2 notebooks for each 3 that Harry has).

We can name ratios using a variety of notations (for example, 5 : 3, 5/3). Fraction notation seems to have an advantage for working with ratios. For example, we can equate two ratios when we are solving problems about similar triangles (for example, $2/3 = x/12$). The use of fraction notation for ratio leads to a comfortable algorithm for solving such problems. The danger is that students may think of 2/3 (for example) as having to do with pies (the part of a whole meaning of fraction) or some other meaning. That kind of thinking easily leads to confusion about solving ratio problems that use fraction notation to express ratio.

The Indicated Division Meaning

Another name for the indicated division meaning is the quotient meaning. It concerns the arithmetic of division where the symbol "/" is used to call for the division of two numbers. One application of this meaning is converting fraction notation to decimal notation, in which we divide the numerator of a fraction by its denominator (for example, 1/2 is $1 \div 2$ or .5).

There does not seem to be an effective way to build the indicated division meaning on concrete experience, yet this meaning is, increasingly, one of the most commonly used meanings (for example, in formulas such as density = mass/volume). It is therefore too often introduced by rote, if introduced at all.

The Meanings and Working with Students

How can we help students gain the power that ensues from understanding the above five meanings of fractions and that allows them to move flexibly among the various meanings? How do we help them see fraction notation as something that makes personal sense? A partial response to these questions is implicit in the following description of my work with three Grade 5 students on two Saturday mornings (part of a long-term study).

By the end of Grade 4, the students had had some fraction instruction in school about the part of a whole meaning. They were proficient at naming pieces of pies/pizzas using fractions but they did not see 1/3 (for example) as a number distinct from a counting number. They considered a fraction to be two whole numbers that describe what happens when you cut up things like pies. They also had some difficulty understanding wholes and parts of wholes. For example, when asked which is larger, 3 or 1/3, one of the students responded, "They are the same. If you used a whole pie there would be three thirds plus 3 on the bottom." As well, the students had little sense of why fractions were invented. In short, their knowledge of fractions was limited and confused.

I decided to redevelop fractions by first considering a reason for their invention. I used a measurement context for this, one that also made it possible to develop the name for a point meaning of a fraction. We imagined we were people from long ago that used a stick for measuring length. The students were asked to use the stick to measure the length of a table as accurately as possible. When asked how we could come up with a number for a part of a stick, the students suggested making equal marks on it and numbering them 1, 2, 3 and so on. The measurement for the part of a stick would be the number closest to the actual length of the part. They were still thinking in terms of whole numbers even though they were subdividing the stick into smaller units of length.

When asked if there was another kind of number that could be used for naming a part of a stick, the students initially were unable to make use of their school-acquired fraction knowledge. The area model (pies/pizzas) that had been used for teaching fractions to them had not empowered them to carry fractions to other situations that might involve a different meaning. After we discussed sticks as objects that could be cut into pieces equal in length where each piece could be given a fraction name, the students realized that sticks were just like pies. They decided to split the stick into 8 pieces by making a succession of half marks on it and mentally attaching a fraction name to each mark (1/8, 2/8 and so on). We generalized measuring with a stick to constructing a number line (a ruler) that was then used to measure whole and fractional lengths.

We revisited the part of a whole meaning of a fraction, using it to attach fraction names in a variety of contexts (for example, a loaf of bread). We discussed how we had made use of the part of a whole idea to make the marks on the ruler and how the fraction name for each mark was a different name for a point on the ruler. Labels for the two meanings were part of the discussion.

The students now had a good sense of fraction as a part of whole and as a name for a point on a number line and an emerging understanding of the relationship between the two meanings. They realized that fractions were useful when measuring and when describing parts of things. They did not yet realize that people did arithmetic with fractions to solve problems. This would be the next step in the development of their fraction numeracy, a step that would utilize the name for a point meaning of fraction (measurement) as the vehicle.

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Adam Riese (1492–1559)

He was Germany's best-known Rechenmeister of that century. He was the most influential of the German writers in the movement to replace the old computation by means of counters by the more modern written computation. He posed the following problem:

Three bachelors wanted to buy a house for 204 Gulden. The first bachelor contributed three times as much as the second bachelor, who contributed four times as much as the third bachelor. How many Gulden did each of the bachelors pay?

Considering Correlation Coefficients: The Meaning of Zero Correlation

Bonnie H. Litwiller and David R. Duncan

Teachers often search for situations in which correlation coefficients can be computed and interpreted. We will discuss three examples.

All students who have studied statistical concepts are familiar with the formula for correlation coefficients. If X and Y are two paired variables, the correlation coefficient describing the strength of the relationship between X and Y is:

$$r = \frac{n\sum(XY) - (\sum X)(\sum Y)}{\sqrt{[n\sum(X^2) - (\sum X)^2][n\sum(Y^2) - (\sum Y)^2]}}$$

Most students are aware that $-1 \leq r \leq 1$ and that if r is close to either -1 or $+1$, the two variables are strongly related. If r is close to $+1$, the relationship is direct (large values of X are typically associated with large values of Y and small values of X with small values of Y); if r is close to -1, the relationship is inverse (large Xs are typically associated with small Ys and small Xs with large Ys). What happens if the correlation coefficient is at or near zero? What does a coefficient of zero mean?

Example 1

Suppose that these are pairs of test scores for five students. The X value is the student's score on test 1 and the Y value is the student's score on test 2. Using the data of Table 1, the correlation coefficient can be computed.

Data Table 1

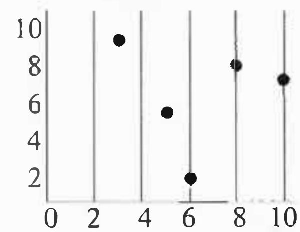
X	Y	X^2	XY	Y^2
3	10	9	30	100
5	6	25	30	36
6	2	36	12	4
8	9	64	72	81
10	8	100	80	64
32	35	234	224	285

$$r = \frac{5(224) - 32(35)}{\sqrt{[5(234) - (32)^2][5(285) - (35)^2]}}$$

$$= \frac{0}{\sqrt{(146)(200)}}$$

$$= 0$$

Graph 1



The zero coefficient suggests the absence of a relationship between the X and Y variables. In the graph, the five pairs are depicted; on inspection, no obvious relationship appears.

Example 2

Again five (X, Y) pairs of test scores are reported and the cocfficient is computed.

Data Table 2

X	Y	X^2	XY	Y^2
0	4	0	0	16
1	1	1	1	1
2	0	4	0	0
3	1	9	3	1
4	4	16	16	16
10	10	30	20	34

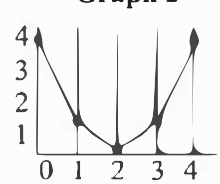
$$r = \frac{5(20) - 10(10)}{\sqrt{[5(30) - (10)^2]\cdot[5(34) - (10)^2]}}$$

$$= \frac{0}{\sqrt{(50)(70)}}$$

$$= 0$$

The zero correlation again suggests the absence of a relationship between the (X, Y) pairs; however, an examination of Graph 2 reveals that the parabola $Y = (X-2)^2$ perfectly describes the relationship between X and Y.

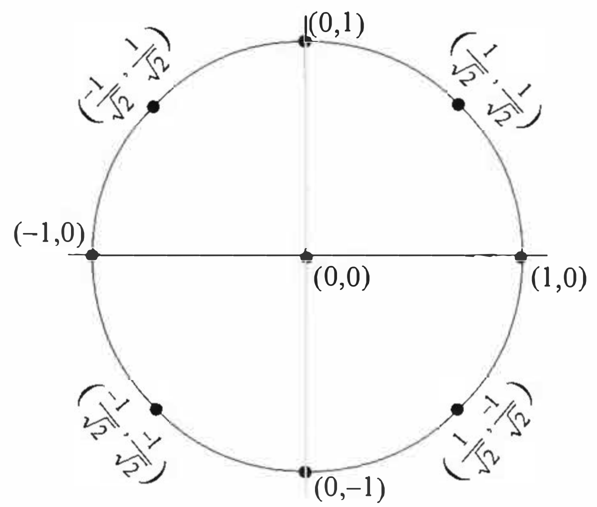
Graph 2



Why was the correlation coefficient unaware of this relationship? This anomaly occurs because the coefficient is looking only for a linear relationship between X and Y. The absence of a linear relationship on Graph 2 causes the coefficient to be 0, even though a quadratic relationship occurred.

Example 3

Graph 3



Graph 3 displays eight (X, Y) pairs on the unit circle $(X^2 + Y^2 = 1)$. We note that there is a geometric relationship among these (X, Y) pairs. Is the correlation coefficient aware of this relationship?

Data Set 3

X	Y	X^2	XY	Y^2
1	0	1	0	0
$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
0	1	0	0	1
$\frac{-1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	$\frac{-1}{2}$	$\frac{1}{2}$
−1	0	1	0	0
$\frac{-1}{\sqrt{2}}$	$\frac{-1}{\sqrt{2}}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
0	−1	0	0	1
$\frac{1}{\sqrt{2}}$	$\frac{-1}{\sqrt{2}}$	$\frac{1}{2}$	$\frac{-1}{2}$	$\frac{1}{2}$
0	0	4	0	4

$$r = \frac{8(0) - 0(0)}{\sqrt{[8(4) - (0)^2] \cdot [8(4) - (0)^2]}}$$

$= 0$

The correlation coefficient was totally unaware of the circular relationship that is evident when Graph 3 is inspected.

The key idea behind these examples is that the correlation coefficient measures the strength of the linear relationship between X and Y. Our original wording preceding the formula at the beginning of this article was, thus, incomplete.

Challenge for the Readers and Their Students

Display and calculate the correlation coefficient for other sets of (X, Y) pairs in which a nonlinear relationship is present.

Lessons Learned from *Looking Inside the Classroom*

Nancy Drickey

Horizon Research, Inc. (HRI) of Chapel Hill, North Carolina, conducted a two-year national study of K–12 mathematics and science education in the United States through funding from the National Science Foundation. The study entitled *Looking Inside the Classroom* provided the education research and policy communities with snapshots of mathematics and science education as they exist in classrooms in various contexts in the United States (Weiss et al. 2003). To examine the state of teaching inside the classroom, trained consultants observed 364 mathematics and science lessons from 40 school districts. A statistical method called systematic sampling with implicit stratification was used to obtain school districts as representative of the nation as possible. For each school district in the study, a math and a science consultant observed two classes each at the elementary, middle and secondary levels. In addition to classroom observations, data collection included individual teacher interviews. (Note: Although the study evaluated mathematics and science education, this article focuses on the mathematics portion.)

As a consultant for this study, I received extensive training in the processes used to determine effectiveness of mathematics instruction. Through hours of watching classroom videos, recording field notes and practising scoring, I learned to use the classroom observation and teacher–interview protocols. I debated with my peers about the interpretation of the lessons we observed and our differences in the levels of rating the quality of the classroom instruction. We continued watching classroom videos and teacher interviews for two days until our ratings were consistent, and we were trained in writing thorough and accurate reports. Throughout this training process and the actual data collection in the schools, I contemplated the complex art of teaching and all the factors that influence our decisions in trying to meet each student's needs. I found it extremely valuable as an educator to be exposed to a comprehensive set of guidelines suggesting multiple components that combine together to constitute effective instruction.

Components of Effective Mathematics Instruction

Information gathered from classroom observations and teacher interviews was used to assess each lesson's effectiveness. Lesson quality was determined by assessing four main components: the lesson design, lesson implementation, the mathematical content being addressed and the classroom culture. Key indicators for lesson design included careful planning and organization, tasks consistent with investigative mathematics, attention to students' prior knowledge and preparedness, a collaborative approach to learning among students, attention to issues of equity and diversity, and adequate time for sense-making and wrap-up. Lesson implementation was rated on instructional strategies used and the teacher's confidence, classroom-management style, pace of the lesson, questioning strategies and ability to gauge the students' level of understanding. The mathematical content was judged on significance, appropriateness for the students' developmental level, accuracy of teacher-provided content, students' intellectual engagement, and connections to other areas and/or real-world contexts. Classroom culture was rated on a climate of respect for students' ideas, collegial working relationships, intellectual rigour, constructive criticism, challenging of ideas and the encouragement of active participation by all.

Lesson activities were timed to determine the amount of time spent on instructional activities versus housekeeping items, interruptions and other noninstructional activities. The amount of time spent in whole-class instruction, pairs or small groups, and individual work was also tracked. Each lesson was also judged on the likely impact of instruction on the students' self-confidence, understanding of the content, interest in mathematics, ability to generalize skills to other areas and capacity to carry out their own inquiries.

Scores in all of these areas combined were used to determine the overall capsule rating, ranging from ineffective to exemplary instruction. Table 1 contains a detailed explanation of the rating scale.

Table 1
Lesson Quality Ratings

(Source: *Looking Inside the Classroom*, Appendix A, page 13)

Level 1: Ineffective Instruction

There is little or no evidence of student thinking or engagement with important ideas of mathematics/science. Instruction is highly unlikely to enhance students' understanding of the discipline or to develop their capacity to successfully "do" mathematics/science. Lesson was characterized by either

a. Passive Learning: Instruction is pedantic and uninspiring. Students are passive recipients of information from the teacher or textbook; material is presented in a way that is inaccessible to many of the students.

b. Activity for Activity's Sake: Students are involved in hands-on activities or other individual or group work, but it appears to be activity for activity's sake. Lesson lacks a clear sense of purpose and/or a clear link to conceptual development.

Level 2: Elements of Effective Instruction

Instruction contains some elements of effective practice, but there are serious problems in the design, implementation, content and/or appropriateness for many students in the class. For example, the content may lack importance and/or appropriateness; instruction may not successfully address the difficulties that many students are experiencing and so on. Overall, the lesson is very limited in its likelihood to enhance students' understanding of the discipline or to develop their capacity to successfully do mathematics/science.

Level 3: Beginning Stages of Effective Instruction

Instruction is purposeful and characterized by quite a few elements of effective practice. Students are, at times, engaged in meaningful work, but there are weaknesses, ranging from substantial to fairly minor, in the design, implementation or content of instruction. For example, the teacher may short-circuit a planned exploration by telling students what they "should have found"; instruction may not adequately address the needs of a number of students; or the classroom culture may limit the accessibility or effectiveness of the lesson. Overall, the lesson is somewhat limited in its likelihood to enhance students' understanding of the discipline or to develop their capacity to successfully do mathematics/science.

Level 4: Accomplished, Effective Instruction

Instruction is purposeful and engaging for most students. Students actively participate in meaningful work (for example, investigations, teacher presentations, discussions with each other or the teacher, or reading). The lesson is well designed and the teacher implements it well, but adaptation of content or pedagogy in response to student needs and interests is limited. Instruction is quite likely to enhance most students' understanding of the discipline and develop their capacity to successfully do mathematics/science.

Level 5: Exemplary Instruction

Instruction is purposeful and all students are highly engaged most or all of the time in meaningful work (for example, investigation, teacher presentations, discussions with each other or the teacher, or reading). The lesson is well designed and artfully implemented, with flexibility and responsiveness to students' needs and interests. Instruction is highly likely to enhance most students' understanding of the discipline and to develop their capacity to successfully do mathematics/science.

Findings

Table 2 shows a breakdown of the percentage of mathematics lessons within the K–5, 6–8 and 9–12 grade bands for each of the five capsule ratings, plus a column showing percentages for Grades K–12 combined. Overall, 60 per cent of the observed mathematics lessons received low-quality capsule ratings of 1 or 2, and only 9 per cent were rated as high-quality effective/exemplary instruction. Capsule ratings of lessons in Grades K–5 classrooms were slightly better than those in the 6–8 or 9–12 grade levels.

Mathematics lessons in the United States were found to be relatively strong in several areas. A majority of the observed lessons incorporated content that was both significant and worthwhile and had teachers who provided accurate content information and appeared confident in their ability to teach mathematics. However, fewer than 20 per cent of the mathematics lessons were strong in intellectual rigour, included effective teacher questioning strategies or provided sense-making appropriate for the needs of the students and the purposes of the lesson.

The factors that seemed to distinguish effective lessons from ineffective ones were their ability to

- engage students with the mathematics content;
- create an environment conducive to learning;
- ensure access for all students;
- use questioning to monitor and promote understanding; and
- help students make sense of the mathematics content. (Weiss et al. 2003)

With the release of the *Principles and Standards for School Mathematics* (NCTM 2000) at the time the study began, it was surprising that only 1 per cent of the teachers in this study mentioned national standards as having an influence on the selection of the content for their lessons. None of the teachers attributed the selection of their instructional strategies to the national standards. Given the current emphasis on national standards, this was an unexpected finding.

Table 2

Percentage of Mathematics Lessons by Capsule Rating and Grade Level

(Source: *Looking Inside the Clasroom*, pages C24, D24, E24)

Capsule Ratings Level	K–5	6–8	9–12	K–12
	Per Cent			
1: Ineffective Instruction	18	26	39	28
2: Elements of Effective Instruction	27	38	30	32
3: Beginning Stages of Effective Instruction	43	25	26	31
4: Accomplished, Effective Instruction	8	11	5	8
5: Exemplary Instruction	4	0	0	1

A Challenge for All Mathematics Teachers

My personal motto has always been to engage all students in meaningful mathematics. As we continue learning and growing in our quest to become better teachers, I hope that we will thoughtfully consider the effectiveness of our mathematics lessons. Perhaps it would be helpful for each of us to read the capsule rating descriptions in Table 1 while thinking of a recent lesson we taught and determine where we currently are on the rating scale. I find it most helpful to concentrate on the aspects of exemplary instruction:

- Purposeful instruction
- Highly engaged students
- Meaningful work (for example, investigation, teacher presentations, discussions or reading)
- Well-designed and artfully implemented lessons
- Flexibility and responsiveness to students' needs and interests
- Instruction that is highly likely to enhance most students' understanding of the discipline and to develop their capacity to successfully do mathematics/science

Through self-reflection and an understanding of the components and key indicators of quality instruction, we can choose from a variety of areas to work on for improvement.

Based on findings from the *Looking Inside the Classroom*, the United States is very far from the ideal of providing high quality mathematics instruction for all students. Teachers need a vision of what effective instruction is in K–12 mathematics. With this understanding, they need to critically compare a variety of lessons with the key elements of high-quality instruction.

For a full report of the study and the classroom observation and teacher interview protocols, go to www.horizon-research.com/insidetheclassroom.

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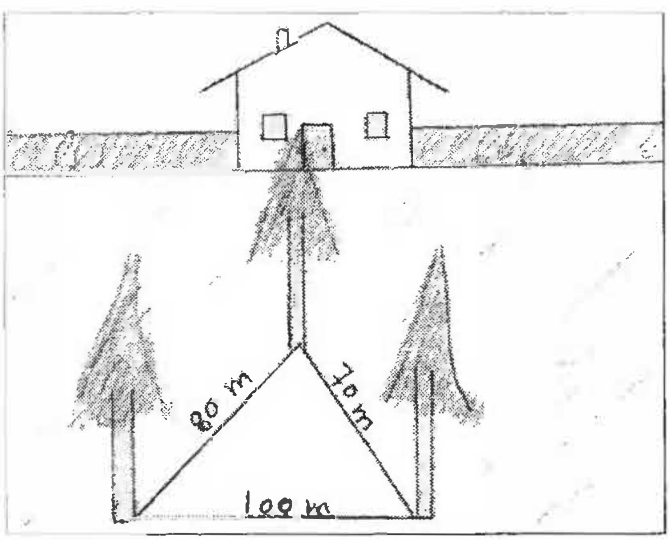
Geometry in Landscaping—What's the Diameter?

Klaus Puhlmann

Mr. Jones, the owner of a large country estate, called upon George from the local landscaping company to create a new look in his large front yard. Before he accepted, George visited Mr. Jones to see what he had in mind.

Mr. Jones showed George a sketch of what he wanted done. Mr. Jones said, "Can you see the three trees, the pine, the spruce and the tamarack? I planted these on the occasion of the birth of my three children. These three trees shall become the focus of attention in my yard. Therefore, I want you to create a circular area of lawn around each tree, with the three circles touching one another and each tree being the centre of its respective circular lawn area. In the rest of the yard, you can plant flowers or shrubs."



George scratched his head as he looked at the sketch. The three trees were 70 m, 80 m and 100 m apart from another in a triangular arrangement. As geometry was not George's strong suit, he decided to take the sketch home and think about it. At home, George worked for hours without success. He then took his old compass and tried to sketch the problem through trial and error. Again, he had no success. At that point, Ryan, George's work-experience student from the local high school, walked in to see George in total frustration. As George explained his problem to Ryan, he responded by saying, "No problem, George!"

Ryan sat down at the desk and in a few minutes gave George a piece of paper with three numbers on it. "Here is the solution to your problem," exclaimed Ryan. George was relieved and felt considerably better. He phoned Mr. Jones and accepted the job.

Are you able to solve this problem? What is the diameter of the largest of the three lawn circles around the trees?

Gauss and the Regular Heptadecagon

William Dunham

In 1796, the German journal *Allgemeine Literaturzeitung* carried the announcement of a remarkable mathematical discovery. The discoverer was an unknown adolescent named Carl Friedrich Gauss, and his achievement was the construction, with compass and straightedge only, of a regular heptadecagon; that is, a regular polygon of 17 sides. At the conclusion of this announcement, Professor E. A. W. Zimmermann added these words of endorsement:

> It deserves mentioning that Mr. Gauss is now in his 18th year, and devoted himself here in Brunswick with equal success to philosophy and classical literature as well as higher mathematics.

Why was Professor Zimmermann so enthusiastic, and what made this so significant a moment in mathematics history?

Constructability problems date back to the Greek geometers. It was they who established the compass and straightedge as the allowable tools of geometry and who thereby determined the sorts of constructions that were and were not possible.

It is intriguing to speculate as to why the compass and straightedge were adopted to the exclusion of other tools. Two reasons suggest themselves. First, the compass and straightedge existed as easy-to-use mechanical devices that could be employed by the geometer to draw figures on papyrus or in the sand. Second, they produced, respectively, circles and straight lines—the perfect two- and one-dimensional figures that so appealed to the Greeks' sense of beauty. By combining ease of use with a beautiful output, these geometrical tools won the hearts of the Greek mathematicians.

Although the compass and straightedge tradition predates Euclid, it is in his *Elements* that one finds its most perfect manifestation. Euclid enshrined them in his first three postulates and thereafter restricted himself to constructions involving them and them only. Such constructions included simple operations like bisecting lines and erecting perpendiculars, but by Book IV of the *Elements*, Euclid had demonstrated the more sophisticated constructions of regular pentagons and pentadecagons (15-gons). One wonders how many math teachers today could pull out a compass and straightedge and construct a regular 15-sided polygon. Euclid could.

Interestingly, nowhere in the *Elements* did Euclid explicitly state that a regular n-gon is constructible if and only if a regular $2n$-gon is, even though he clearly recognized this principle. Thus, among polygons with fewer than 50 sides, Euclid could construct regular 3, 4, 5, 6, 8, 10, 12, 15, 16, 20, 24, 30, 32, 40 and 48-gons. Of course he could continue *ad infinitum* by repeated doubling of the number of sides.

There was a general sense that this list included all the constructible regular polygons. Centuries passed—and mathematicians like Archimedes, Al-Khowârizmî, Newton, Leibniz and Euler came and went—but no one augmented the collection. No one, that is, until the young Carl Friedrich Gauss, 2100 years after Euclid, conquered the 17-gon.

Before addressing his discovery, we might mention other constructability challenges bequeathed to posterity by the Greeks. The three best known were the trisection of the angle; the duplication of the cube (that is, given the side of a cube, construct the side of a cube with double the volume); and the quadrature of the circle (given a circle, construct a square of equal area). These remained unresolved in classical times and intrigued a host of mathematicians well into the modern era.

All three were still open questions when Gauss announced his heptadecagon breakthrough in 1796. No one had suspected that such a construction was possible, and if less effort had been expended trying to construct 17-gons than trying to trisect angles, it was because the former seemed much more improbable than the latter. Gauss's amazing construction must have given renewed hope to the angle trisectors and circle squarers, who could reasonably argue that if a regular 17-gon were constructible, then a dose of Gaussian ingenuity might lead to a successful trisection or circle squaring as well.

Of course, things did not turn out that way. In 1837, the French mathematician Pierre Laurent Wantzel proved that both angle trisection and cube duplication were beyond the power of compass and straightedge. As a matter of fact, he proved both of these on the same page of a groundbreaking paper in Liouville's journal!

Squaring the circle held out a bit longer, until an 1882 theorem by the German Ferdinand Lindemann established that π was a transcendental number and thus not constructible. From this it followed that circles could not be squared with compass and straightedge.

We now return to the 17-gon. Needless to say, Gauss's reasoning cannot be squeezed into a few paragraphs; if it were that simple, someone would have stumbled upon it during the previous 21 centuries. Yet the difficulty lies more in its intricate interconnection of ideas rather than in the profundity of any idea in particular. What follows is the basic plot, a sort of "Cliff Notes" version of his proof:

The argument rested upon the postulates of Euclid that allowed us to construct a straight line between two points (Postulate 1), to extend a straight line segment as far as we wish (Postulate 2), and to construct a circle with a given point as centre and given length as radius (Postulate 3). Thereafter, Gauss's reasoning assumed an algebraic character—following the insights of Descartes—by noting that, if a magnitude can be expressed using only integers and a finite number of operations of addition, subtraction, multiplication, division and the extraction of square roots, then that magnitude can be constructed with compass and straightedge. For instance, beginning with a given unit length, one could construct a segment of length

$$\frac{2 + \sqrt{3 + \sqrt{5} + \sqrt{7}}}{4 + \sqrt{1} + \sqrt{11}}$$

(although I wouldn't necessarily want to).

Next, Gauss invoked trigonometry by proving the following constructability condition:

> A regular n-gon is constructible if and only if $\cos(2p/n)$ is constructible.

In the context of regular heptadecagons, this naturally led him to consider $\cos(2\pi/17)$.

It is at this point that things became truly amazing, for young Gauss veered off into the world of complex numbers. This may seem bizarre—given that all geometric constructions occur in the real world—but the key to the problem lay in DeMoivre's theorem and the roots of unity. Recall that DeMoivre had proved that

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

Letting $\theta = 2\pi/17$, we begin to sense a link between DeMoivre's theorem and the constructability condition above.

With inspired algebraic insights and deft manipulations, Gauss managed to show the hitherto unexpected fact that

$$\cos(2\pi/17) = -\frac{1}{16} + \frac{1}{16}\sqrt{17} + \frac{1}{16}\sqrt{34 - 2\sqrt{17}} + \frac{1}{8}\sqrt{17 + 3\sqrt{17} - \sqrt{34 - 2\sqrt{17}} - 2\sqrt{34 + 2\sqrt{17}}}.$$

Because the expression on the right is built of integers that are added, subtracted, multiplied, divided and square-rooted, it is constructible. Thus $\cos(2\pi/17)$ is constructible, and it follows from the constructability criterion that the regular 17-gon is as well. QED.

Of course, besides the 17-gon, Gauss could construct 34-gons, 68-gons and others by repeated doubling. In addition, from the regular 17-gon he could construct a regular $3 \times 17 = 51$-gon (and all of its doubles) and a regular $5 \times 17 = 87$-gon (and all of its doubles). His general theory showed that,

> If $2^{2^n} + 1$ is a prime, then a $2^{2^n} + 1$-gon is constructible with compass and straightedge.

For $n = 0$, 1 and 2, this yields regular 3-gons, 5-gons and 17-gons. For $n = 3$, we get $2^8 + 1 = 257$, a prime, and so a regular 257-gon is constructible (and hence a regular 514-gon and so on). For $n = 4$, $2^{16} + 1 = 65{,}537$, another prime, so along comes another batch of constructible polygons.

Unfortunately for polygon constructors,

$$2^{2^5} + 1 = 2^{32} + 1 = 4{,}294{,}967{,}297$$

is not prime, for it has a factor of 641 as the incomparable Euler had observed 50 years earlier. No further constructible polygons have been found since Gauss laid down his pen two centuries ago.

Legend says that an aging Gauss wished to have the regular 17-gon inscribed on his tombstone. The validity of the legend notwithstanding, it certainly would have been a fitting memorial to so great a scholar. Unfortunately, no such monument was executed. For what it's worth, however, the German government has seen fit to put Gauss's countenance upon its ten mark note.

Of course, the practical importance of this construction is nil. The regular heptadecagon won't help anyone balance a checkbook, build a quieter refrigerator or find a quick way through the Lincoln Tunnel at rush hour. Sad to say, it is essentially useless. But . . . it is as beautiful and unexpected a piece of mathematical reasoning as there is. If mathematical theorems can ever be termed breathtaking, this one qualifies.

The construction has a final claim to fame. In later years, when Gauss had achieved international repute as the Prince of Mathematicians, he observed that it was this discovery that propelled him into mathematics. His observation raises the unsettling possibility that, had it not been for the regular heptadecagon, Carl Friedrich Gauss might have become a cobbler or wine taster. Clearly, the mathematical world owes this polygon a huge debt of gratitude.

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Brahmagupta (c. 628)

He is the most prominent of the Hindu mathematicians of the 7th century. In his book *Cutta ca*, he writes:

If one reduces the number of days by 1, divides the difference by 6 and adds 3, the result is always one-fifth of the original number of days. What is the original number of days?

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