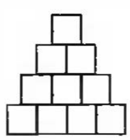
Sums of Arithmetic Sequences: Several Problems and a Manipulative

A. Craig Loewen

An extremely powerful and important link exists between manipulatives and problem solving. Through the use of manipulatives, we come to understand mathematics; through problem solving, we are challenged to apply what we have learned.

Consider the following four problems:

1. There are 50 people at a party. Each person shakes hands once with each of the other people. How many handshakes occur in total?
2. How many diagonals are in a regular hectagon (a polygon with 100 sides)?
3. Bricks are stacked to create a pyramid like the one shown below. How many bricks would be required to build a pyramid 75 rows tall?



4. As in the well-known Christmas song, on the first day of Christmas, my true love gives me one partridge in a pear tree. On the second day, my true love gives me two turtledoves. If my true love continues for a full year (365 days) to give me one gift more each day than the previous day, how many gifts will I receive in all?

Though these problems look unlike on the surface, they share at least one important quality: the solution to each requires summing a series of consecutive whole numbers.

A Historical Note

A story about a famous mathematician, Carl Friedrich Gauss, tells how Gauss, while still a schoolboy, was required to sum the numbers 1–100 as a punishment. He finished the task far ahead of his classmates. It seems that Gauss realized that by grouping the numbers he could identify a pattern and thus simplify his work:

$$\begin{aligned} \text{Sum} &= 1 + 2 + 3 + \ldots + 98 + 99 + 100 \\ &= (1 + 100) + (2 + 99) + \ldots + (50 + 51) \\ &= 101 + 101 + \ldots + 101. \end{aligned}$$

There are 50 pairs, each totalling 101. Thus,

$$\begin{aligned} \text{Sum} &= 50 \times 101 \\ &= 5{,}050. \end{aligned}$$

There is a similar way to understand and attack this problem. Below the first equation, write the equation in reverse order, then add the two equations:

$$\begin{aligned} \text{Sum} &= 1 + 2 + \ldots + 99 + 100 \\ \text{Sum} &= 100 + 99 + \ldots + 2 + 1 \\ 2 \cdot \text{Sum} &= 101 + 101 + \ldots + 101 + 101 \end{aligned}$$

Because we know that the equation has a hundred 101s, we can write

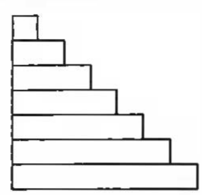
$$\begin{aligned} 2 \cdot \text{Sum} &= 100 \times 101 \\ &= 50 \times 101 \\ &= 5{,}050. \end{aligned}$$

Starting with a simpler problem, let's look at a manipulative that shows why this might work.

Solve a Simpler Problem

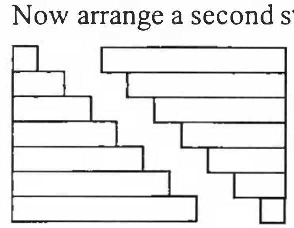
Consider the following task: Find the sum of the numbers 1–7.

A familiar manipulative such as Cuisenaire Rods can be used to model this task. Take one each of the seven shortest rods and arrange them in a staircase as shown:



The total length of all the stairs is equal to the sum we are trying to find. Also, because each rod has a width of 1 unit, the area of the staircase is equal to that sum.

Now arrange a second staircase as shown:



Slide the two staircases together to create a rectangle.

- What are the length and width of this rectangle?
- How does the area of the rectangle compare with the sum we are trying to find?
- What is the largest such staircase you could build?
- Using graph paper, draw a staircase showing the sum of the numbers 1–20. Draw the related rectangle.
- Could the same process be used regardless of the number of stairs? Why?
- Describe how you could find the sum of the counting numbers up to any given value.
- Write a formula to show how this sum could be easily calculated.

Through this manipulative and exploration, we come to see that the sum of n counting numbers starting at 1 is

$$\text{Sum} = \frac{n(n+1)}{2}.$$

Applying What We Know

Now we can easily apply what we have learned to the original four problems.

In the first problem, the first person at the party will shake hands with 49 people, the second person will shake 48 hands and so on. Thus, the solution to the first problem is

$$\text{Sum} = 49 + 48 + 47 + \ldots + 2 + 1$$

$$= \frac{49(49+1)}{2}.$$

Likewise, the solution to the third problem is

$$\text{Sum} = \frac{75(75+1)}{2}.$$

The solution to the fourth problem is

$$\text{Sum} = \frac{365(365+1)}{2}.$$

But be careful! The solution to the second problem is a bit trickier. We know that a hectagon has 100 sides, so it is tempting to think that we can just use Gauss's answer for the second problem. But the correct answer is not 5,050!

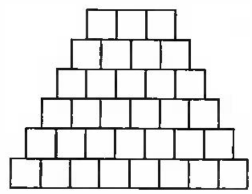
Imagine the vertices of a hectagon spread evenly around a large circle. We can begin to connect these vertices two at a time to create our diagonals. There will be 99 lines from the first vertex to each of the other 99 vertices, 98 lines from the second vertex and so on. This implies that the final answer is half of 99×100. But again, it is not! In drawing these lines, we have included the edges of the hectagon itself, and these edges are *not* diagonals (by definition, diagonals must pass through the interior of the figure). So, we must remember to subtract the 100 edges of the hectagon to reach our final answer. Thus, the number of diagonals in a regular hectagon is

$$\text{Diagonals} = \frac{99(99+1)}{2} - 100.$$

Note that, even when we have a useful formula, we must still think carefully to apply it appropriately.

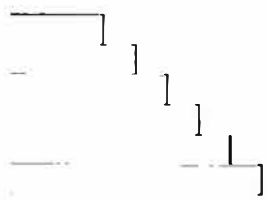
One Step Further: Extending What We Know

Now, let's return to the third problem and imagine a pyramid of bricks like the one shown:

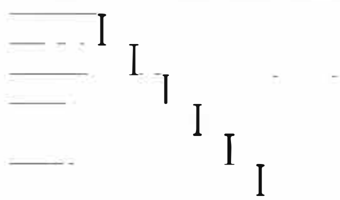


This is the same type of pyramid except the first row has something other than a single brick.

Once again, we can apply our manipulative to this sequence. First, build a staircase starting with the Cuisenaire Rod 3 units in length, and use each of the rods to a length of 8 units.



Like before, add a second staircase to form a rectangle.



- What are the length and width of this rectangle? How are the length and width determined?
- How does the area of the rectangle compare to the sum we are seeking?
- If you knew only the length of the first stair and the length of the last stair, could you predict how many stairs there are in all?
- How could we use our first formula,

  $$\frac{n(n+1)}{2},$$

  to generate a formula for this problem?

- Rewrite the Christmas problem so that my true love gives me something other than one gift on the first day. Apply your revised formula to solve this problem.

Other Variations

We can vary these problems in many other ways to increase the challenge. For example, let's assume that my true love gives me two gifts on the first day, four gifts on the second day, six gifts on the third day and so on. How many gifts would I receive in 12 days?

This problem differs from the others in that the successive values in our sequence increase by two rather than one:

Sum = $2 + 4 + 6 + \ldots + 24$.

In general terms, the first sequences we added were of the form $1, 2, 3, \ldots, n$.

Next, we considered sequences like $c, c + 1, c + 2, \ldots, c + n$.

This last problem introduces another sequence: $a, 2a, 3a, \ldots, na$.

The next logical step is to consider the sequence $c, c + a, c + 2a, c + 3a, \ldots, c + na$. In this sequence, we can begin at a value other than 1, and the difference between successive elements in the sequence can be a value other than 1. For example, in the sequence 6, 10, 14, 18, 22, . . ., $c = 6$ and $a = 4$.

- What would a pyramid of bricks that followed the fourth sequence above ($a \neq 1$ and $c \neq 1$) look like? Generate several examples.
- How are the four types of sequences related?
- Develop a formula for finding the sum of a sequence such as 7, 12, 17, . . ., 717.

Conclusion

Our most powerful learning experiences are those in which we explore and experiment in a meaningful context. Manipulatives help us to see not only how but also why something works. Also, students need opportunities to apply mathematics through problem solving. It is not necessary or even desirable to treat manipulatives and problem solving separately. When problem solving is incorporated in a manipulative activity, we can provide many dynamic learning opportunities for our students.

A. Craig Loewen, editor of delta-K, *is an associate professor of mathematics education and assistant dean of Student Program Services in the Faculty of Education at the University of Lethbridge in Lethbridge, Alberta*