Developing Three Distinct Number Patterns from a Single Diagram

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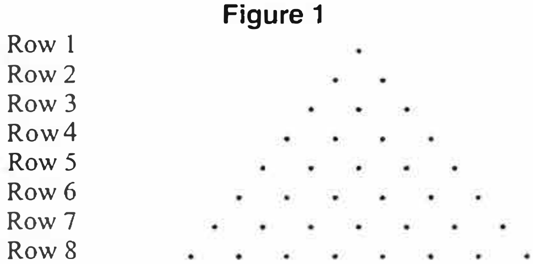
Mathematics is in large part a study of relationships and patterns. Teachers are always on the lookout for settings in which these relationships and patterns can be discovered and considered. Finding several patterns in a single setting is a serendipitous occurrence.

Consider the following set of points:

Figure 1



We will discuss three problems in this setting.

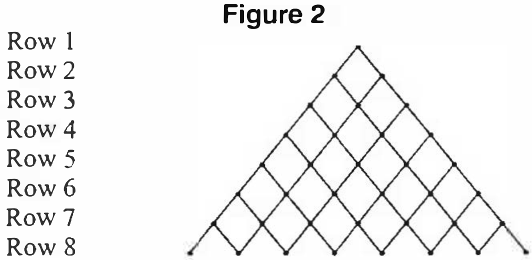
Problem 1

How many points does Figure 1 contain? Rows 1, 2, 3, . . ., 8 contain, respectively, 1, 2, 3, . . ., 8 points. Consequently, the eight rows together contain 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 points. Your students may recognize this to be the eighth triangular number.

Problem 2

Figure 2 displays the same set of points as Figure 1 but with connecting pathways superimposed.

Figure 2



In how many ways can you proceed downward from Row 1 to Row 8, following only the indicated paths? We can break this task into a series of seven consecutive tasks:

- Task 1: You can proceed from Row 1 to Row 2 using either of two paths.
- Task 2: From any point in Row 2, you can proceed to Row 3 using either of two paths.
- Task 3: From any point in Row 3, you can proceed to Row 4 using either of two paths.
- Tasks 4–7: From any point in each of Rows 4–7, you can proceed to the next row using either of two paths.

Using the fundamental principle of counting, we find that the number of ways to perform Tasks 1–7 is $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^7$. Thus, there are 2^7, or 128, distinct paths from Row 1 to Row 8.

Problem 3

Let us again consider Figure 1. In Problem 2, we were restricted to only two pathways from each point to the following row. Suppose that we abandon that condition and proceed directly from a point in one row to any point in the next row. In how many ways can you proceed from Row 1 to Row 8 under these more flexible rules?

- Task 1: You can proceed from Row 1 to Row 2 using either of two paths.
- Task 2: From any point in Row 2, you can proceed to Row 3 using any one of three paths (remember that you can go directly to any of the three points in Row 3).
- Task 3: From any point in Row 3, you can proceed to Row 4 using any one of four paths.
- Task 4: From any point in Row 4, you can proceed to Row 5 using any one of five paths.
- Task 5: From any point in Row 5, you can proceed to Row 6 using any one of six paths.
- Task 6: From any point in Row 6, you can proceed to Row 7 using any one of seven paths.
- Task 7: From any point in Row 7, you can proceed to Row 8 using any one of eight paths.

Again using the fundamental principle of counting, we find that the number of ways to perform Tasks 1–7 is $2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8$, or 8!. Thus, there are exactly 8!, or 40,320, distinct ways of proceeding from Row 1 to Row 8.

We have been using a set of eight rows of points for Problems 1–3. If Figure 1 were extended to *n* rows of points formed in the same way, the answers to the three problems would be as follows.

Problem 1

How many points does the figure contain? There are $1 + 2 + 3 + 4 + \ldots + n$ (the nth triangular number). Recall that the nth triangular number is $\frac{n(n+1)}{2}$.

Problem 2

How many paths can you take from Row 1 to Row n when each point is connected to two points in the following row? You can take $\underbrace{2 \cdot 2 \cdot 2 \ldots 2}_{n-1 \text{ factors}}$, or 2^{n-1}, such paths.

Problem 3

In how many ways can you proceed from Row 1 to Row n if the paths of Problem 2 need not be followed? You can proceed in $2 \cdot 3 \cdot 4 \cdot \ldots \cdot n$, or $n!$, such ways.

These three results represent three fundamentally different categories of mathematical formulations: summations, exponentials and factorials.

Have your students check out these formulas for specific values of n by drawing the figures and counting whatever the problem calls for. Can you and your students find other problems arising from Figure 1?

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