Determining the Angles Between Two Lines

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In preparing a recent lecture for a course on non-Euclidean geometry, I needed a formula to determine the angles formed by two intersecting lines in Euclidean plane geometry. The relevant formulas in Proposition 6.2 of *The Poincaré Half-Plane: A Gateway to Modern Geometry* (Stahl 1993), the textbook for the course, depended on methods not needed again until the textbook's coverage of the hyperbolic version of the Pythagorean theorem (Theorem 8.3). I decided to seek alternative formulas with minimal prerequisites and the additional benefit of being easy to implement on modern calculators.

Because the task at hand would be meaningful for a precalculus class, I consulted a current leader in that market, *Precalculus: Mathematics for Calculus* (Stewart, Redlin and Watson 2002). The relevant method given in this textbook used the formula

$$\cos(\theta) = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|},$$

where $\vec{u}$ and $\vec{v}$ are vectors in directions determined by the given intersecting lines, and then used the inverse cosine function and related angles (also known as reference angles) to compute the angles in question (p. 604, Example 2). The prerequisites for this approach become available rather late in a precalculus course; for instance, the above formula for $\cos(\theta)$ is proved using the law of cosines (p. 603). Therefore, I looked further for an accessible method that could be implemented with relatively few keystrokes on a calculator.

At first glance, it seemed that *A First Year of College Mathematics* (Brink 1954, 359), a textbook of 50 years ago for the precursor of today's precalculus course, contained the answer for the angles determined by intersecting nonvertical lines having slopes m_1 and m_2, by use of the formula

$$\tan(\theta) = \frac{m_1 - m_2}{1 + m_1 m_2}.$$

Unfortunately, in this formula, θ can be negative (p. 358), contrary to our natural desire to determine angles between 0 and π. (Of course, as is appropriate for precalculus and beyond, we are measuring angles in radians.)

I modernized the formula from Brink (1954) by developing some accessible, calculator-friendly formulas (see the theorem later in the article). Part (a) of the theorem concerns the case where two nonvertical lines intersect, and part (b) addresses the situation where one of the intersecting lines is vertical. This theorem can be presented as enrichment material quite early in a precalculus course because its only prerequisites are a pair of facts from geometry (equality of corresponding angles cut from parallel lines by a transversal, and relation of an external angle of a triangle to the remote interior angles of the triangle), slope, the slope-intercept equation of a nonvertical line, angle of inclination of a line, the definitions of the tangent and inverse tangent functions, and the usual expansion formula for $\tan(u - v)$. For the sake of completeness, the next section begins with a proposition that recalls the connection between the slope and the angle of inclination of a nonvertical line. The closing remark provides an example comparing the speed of applicability of the three methods mentioned above.

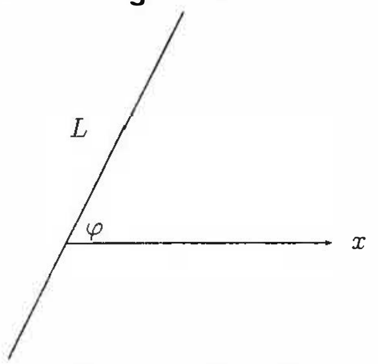
The centrality of the tangent function in trigonometry (and thus, nowadays, in precalculus) has been implicit for millennia, at least since the time of Thales. I have written a number of articles (Dobbs 1984a, 1988, 1991), suitable for use in a precalculus course, explaining how the tangent function can be used to give new proofs of various facts presented in typical high school geometry and precalculus courses. In several such notes, investigations using analytic (as opposed to synthetic) methodology have developed new results, as well (Dobbs 1984c, 1984d). This article is intended as another contribution to this program. In using it, the reader may want to consult Dobbs

(1984b) for a self-contained proof of the expansion formula for $\tan(u \pm v)$ that is more accessible than the proof in standard textbooks in that it is independent of the expansion formulas for $\sin(u \pm v)$ and $\cos(u \pm v)$. Finally, note the central role of the formula for $\tan(u \pm v)$, as it figures in a characterization of the tangent function (Dobbs 1989, Theorem 3), a result later used by the College Board and Educational Testing Service as the basis for the final question on the Advanced Placement Calculus BC examination in May 1993.

Formulas Based on the Inverse Tangent Function

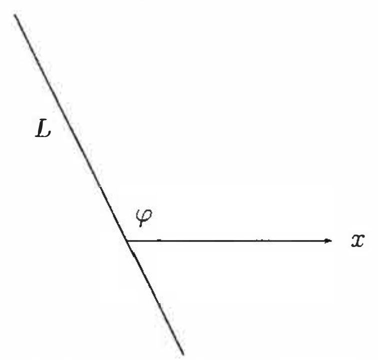
We begin by recalling the definition of what will be our key tool. If L is a line, then the angle of inclination of L is defined as the angle φ between L and the positive x-axis such that $0 \leq \varphi < \pi$. If L has positive slope, then $0 < \varphi < \pi/2$, as in Figure 1.

Figure 1
Acute Angle of Inclination



If L has negative slope, then $\pi/2 < \varphi < \pi$, as in Figure 2.

Figure 2
Obtuse Angle of Inclination



If L has slope equal to 0, then L is horizontal and it is conventional to take the angle of inclination of L to be 0. Last, if the slope of L is undefined (that is, if L is vertical), then $\varphi = \pi/2$. Part (a) of the following proposition is well known (see Brink 1954, 357); parts (b) and (c) are also essentially known and will be useful later in the proof of the theorem.

Proposition. *Let* L *be a nonvertical line having slope* m *and angle of inclination* φ. *Then,*

(a) $\tan(\varphi) = m$.
(b) *If* φ *is an acute angle, then* $\varphi = \tan^{-1}(m)$.
(c) *If* φ *is an obtuse angle, then* $\varphi = \pi - \tan^{-1}(-m) = \pi + \tan^{-1}(m)$.

Proof. (a) If φ is acute, then $\tan(\varphi)$ and m are the same ratio of two sides of a right triangle having φ as one of its angles. Suppose next that φ is obtuse, with related angle θ. Then, the preceding reasoning gives that $\tan(\theta) = -m$. Moreover, $\tan(\varphi) = -\tan(\theta)$ by the definition of the tangent function, as given in Brink (1954, 200, 233). The assertion follows easily. (The preceding argument was tailored for classes whose definition of the trigonometric functions is, like that in Brink [1954], based on angles in standard position and related angles. An alternative proof should be given to classes whose definition of the trigonometric functions is based on the unit circle.)

(b) The assertion in (b) follows from (a) and the definition of the inverse tangent function.

(c) Suppose that φ is obtuse. Let θ be the related angle of φ. Since $\varphi + \theta = \pi$, it follows that θ is an acute angle. Also, as noted in the proof of (a), $\tan(\theta) = -m$. Then $\theta = \tan^{-1}(-m) = -\tan^{-1}(m)$, the first equality holding by the definition of the inverse tangent function and the second equality holding because $\tan^{-1}$ is an odd function. Substituting these facts into the equation $\varphi = \pi - \theta$ leads to the assertions in (c), to complete the proof. □

The formulation of our main result ignores the case of perpendicular lines because this case can be handled directly. Indeed, if L_1 and L_2 are coplanar lines having slopes m_1 and m_2, respectively, every precalculus course covers the fact that L_1 and L_2 are perpendicular if and only if $1 + m_1m_2 = 0$. Moreover, a vertical line L_1 is perpendicular to a coplanar nonvertical line L_2 having slope m_2 (at their point of intersection) if and only if $m_2 = 0$.

Theorem. *Let* L_1 *and* L_2 *be two intersecting nonperpendicular lines in the Euclidean plane. Then,* (a) *suppose that* L_1 *and* L_2 *are each nonvertical, having slopes* m_1 *and* m_2, *respectively. Then, the two acute angles formed by* L_1 *and* L_2 *at their point of intersection are each given by*

$$\tan^{-1}\left(\left|\frac{m_1 - m_2}{1 + m_1m_2}\right|\right),$$

and the two obtuse angles formed by L_1 *and* L_2 *at their point of intersection are each given by*

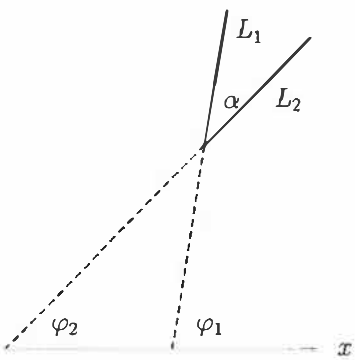
$$\pi - \tan^{-1}\left(\left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|\right).$$

(b) *Suppose that* L_1 *is vertical and that* L_2 *has slope* m_2. *If* $m_2 > 0$, *then the two acute angles formed by* L_1 *and* L_2 *at their point of intersection are each given by* $\pi/2 - \tan^{-1}(m_2)$, *and the two obtuse angles formed by* L_1 *and* L_2 *at their point of intersection are each given by* $\pi/2 + \tan^{-1}(m_2)$. *If* $m_2 < 0$, *then the two acute angles formed by* L_1 *and* L_2 *at their point of intersection are each given by* $\pi/2 + \tan^{-1}(m_2)$, *and the two obtuse angles formed by* L_1 *and* L_2 *at their point of intersection are each given by* $\pi/2 - \tan^{-1}(m_2)$.

Proof. (a) Four angles are formed at the intersection of L_1 and L_2. Since vertically opposite angles are congruent, it suffices to determine one of these angles, say α. The other three angles of intersection are then α, $\pi - \alpha$ and $\pi - \alpha$. It is convenient to distinguish six cases.

Case 1. $0 < m_2 < m_1$, with α acute. The data are depicted in Figure 3.

Figure 3
Case 1



By a basic fact about triangles in Euclidean geometry, the exterior angle φ_1 is the sum of the two remote interior angles, α and φ_2, and so $\alpha = \varphi_1 - \varphi_2$. Moreover, by part (a) of the proposition, $\tan(\varphi_1) = m_1$ and $\tan(\varphi_2) = m_2$. Therefore, by the expansion formula for $\tan(u - v)$, we have that

$$\tan(\alpha) = \tan(\varphi_1 - \varphi_2) = \frac{\tan(\varphi_1) - \tan(\varphi_2)}{1 + \tan(\varphi_1)\tan(\varphi_2)}$$

$$= \frac{m_1 - m_2}{1 + m_1 m_2} = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|.$$

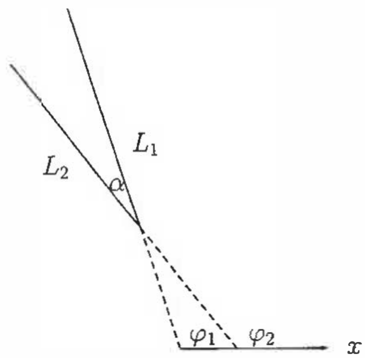
Since α is an acute angle, the definition of the inverse tangent function ensures that

$$\alpha = \tan^{-1}(\tan(\alpha))$$

in this case, so the asserted formula for α has been established.

Case 2. $m_1 < m_2 < 0$, with α acute. The data are depicted in Figure 4.

Figure 4
Case 2



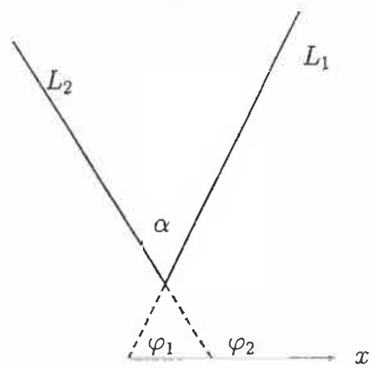
The exterior angle φ_2 is the sum of the two remote interior angles, α and φ_1, and so $\alpha = \varphi_2 - \varphi_1$. Combining part (a) of the proposition, the expansion formula for $\tan(u - v)$ and the definition of the inverse tangent function as above, we see that

$$\tan(\alpha) = \frac{m_2 - m_1}{1 + m_2 m_1} = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

and $\alpha = \tan^{-1}(\tan(\alpha))$. The asserted formula for α follows.

Case 3. $m_2 < 0 < m_1$, with α acute. The data are depicted in Figure 5.

Figure 5
Case 3



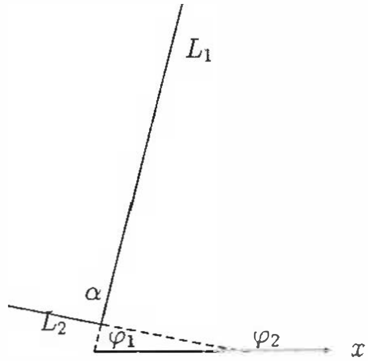
As in the analysis for Case 2, we infer that

$$\tan(\alpha) = \frac{m_2 - m_1}{1 + m_2 m_1} = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|.$$

(The last equality holds since $m_2 - m_1$ and $1 + m_2 m_1$ are both negative, but it is not really necessary to observe this, because we need only appeal to the fact that any acute angle has a positive tangent.) Case 3 can now be completed in the earlier cases by appealing to the definition of the inverse tangent function.

Case 4. $m_2 < 0 < m_1$, with α obtuse. The data are depicted in Figure 6.

Figure 6
Case 4



As in the analyses for Cases 2 and 3, we infer that

$$\tan(\alpha) = \frac{m_2 - m_1}{1 + m_2 m_1} = -\left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|,$$

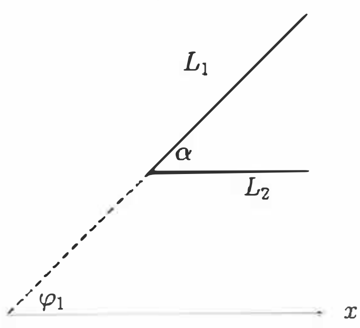
the last equality holding since obtuse angles have negative tangents. Next, note what was effectively established in part (c) of the proposition—that any obtuse angle α satisfies

$$\alpha = \pi - \tan^{-1}(|\tan(\alpha)|);$$

this can also be seen as a consequence of the basic facts about related angles (see Brink 1954, 233, Rule). Combining the assembled information leads to the asserted description of α.

Case 5. $0 = m_2 < m_1$, with α acute. The data are depicted in Figure 7.

Figure 7
Case 5



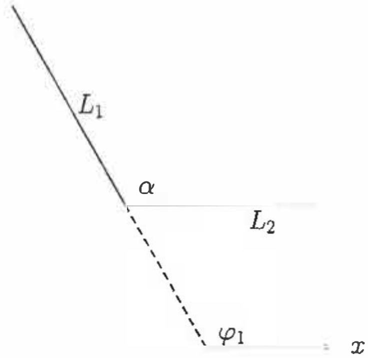
Relative to the transversal L_1, the parallel lines L_2 and the x-axis cut off exterior corresponding angles α and φ_1. Therefore, by a fundamental result in Euclidean geometry, $\alpha = \varphi_1$. Hence, by part (b) of the proposition,

$$\alpha = \tan^{-1}(m_1) = \tan^{-1}\left(\frac{m_1 - m_2}{1 + m_1 m_2}\right),$$

and the asserted description of α follows easily.

Case 6. $m_1 < m_2 = 0$, with α obtuse. The data are depicted in Figure 8.

Figure 8
Case 6



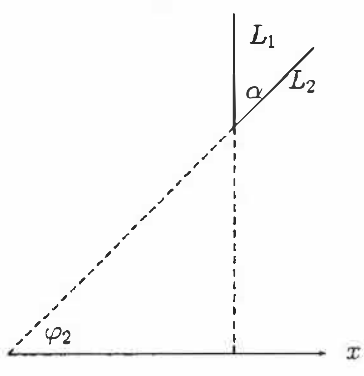
As in the analysis of Case 5, we see that α and φ_1 are exterior corresponding angles and, hence, equal (in measure). In particular, part (a) of the proposition yields that $\tan(\alpha) = m_1$. Then, since α is obtuse, a fact recalled in the analysis of Case 4—or an application of part (c) of the proposition—yields that

$$\alpha = \pi - \tan^{-1}(|\tan(\alpha)|) = \pi - \tan^{-1}(|m_1|).$$

Since $m_2 = 0$, the asserted description of α now follows easily.

(b) The opening comments in the proof of (a) are enough to prove the assertions concerning the acute angles of intersection α. Let φ_2 denote the angle of inclination of L_2. Suppose first that $m_2 > 0$, as in Figure 9.

Figure 9
L_1 Vertical and L_2 with Positive Slope

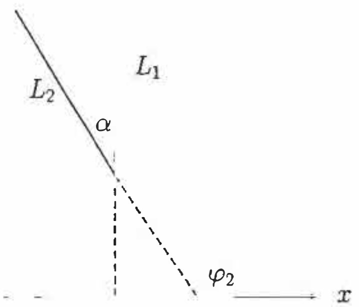


Since $m_2 > 0$, we have that φ_2 is an acute angle, so part (b) of the proposition gives $\varphi_2 = \tan^{-1}(m_2)$. Therefore, since α and φ_2 are complementary, we have $\alpha = \pi/2 - \varphi_2 = \pi/2 - \tan^{-1}(m_2)$, as asserted.

Finally, suppose that $m_2 < 0$, as in Figure 10.

Figure 10

L_1 Vertical and L_2 with Negative Slope



We have the exterior angle φ_2 equal to the sum of the two interior remote angles, α and $\pi/2$. Since φ_2 is obtuse, we could now complete the proof by using the fact recalled in the analyses of Cases 4 and 6. For variety, we argue instead through part (c) of the proposition. This gives that $\varphi_2 = \pi + \tan^{-1}(m_2)$. Therefore, $\alpha = \varphi_2 - \pi/2 = (\pi + \tan^{-1}(m_2)) - \pi/2 = \pi/2 + \tan^{-1}(m_2)$, to complete the proof. □

Remark. Consider the lines L_1 and L_2 with Cartesian equations $2x - 3y + 4 = 0$ and $5x + 6y + 7 = 0$, respectively. Solving for y, we obtain the equations in slope-intercept form as $y = \frac{2}{3}x + \frac{4}{3}$ and $y = -\frac{5}{6}x - \frac{7}{6}$, respectively. Therefore, the slopes of the given lines are the coefficients of x in slope-intercept form: $m_1 = \frac{2}{3}$ and $m_2 = -\frac{5}{6}$. Implementing part (a) of the theorem (with the aid of a TI-86 graphing calculator), we see that the radian measure of an acute angle formed by L_1 and L_2 at their point of intersection is

$$\tan^{-1}\left(\left|\frac{\frac{2}{3} - \frac{-5}{6}}{1 + \frac{2}{3}\cdot\frac{-5}{6}}\right|\right) \approx 1.28274087974,$$

and so an obtuse angle formed by L_1 and L_2 is given approximately by the supplement of the preceding value:

$$\pi - 1.28274087974 \approx 1.85885177385.$$

Notice that, in implementing the theorem, we need no diagram and there are no ambiguities. In particular, because of the absolute value symbol appearing in the formulas in part (a) of the theorem, it does not matter which line we called L_1 and which L_2. In addition, the above narrative displays the relatively few calculations and keystrokes needed in this routine application of the theorem.

Let us compare the above work with how the methods in Brink (1954, 359) and Stewart, Redlin and Watson (2002, 604) would handle the same problem.

First, we consider the method from Brink (1954, 359), using the formula

$$\tan(\theta) = \frac{m_1 - m_2}{1 + m_1 m_2}.$$

With the above values of m_1 and m_2, we find that $\tan(\theta) = \frac{27}{8} = 3.375$. One such θ is $\tan^{-1}(3.375) \approx 1.28274087974$, the acute angle that we found using the theorem; thus, by calculating the supplement of θ, one would also find the obtuse angles formed by L_1 and L_2.

However, one should not conclude that Brink's (1954, 359) method is as useful in general as that in part (a) of the theorem. What if we interchanged the labels on the lines L_1 and L_2? We would then be led to consider an angle θ such that $\tan(\theta) = -3.375$. This θ is neither the acute nor the obtuse angle that we are seeking! Moreover, a calculator cannot come to the immediate rescue, because the inverse tangent of this θ is negative. Granted, with a careful diagram and some thought about related angles, a skilled user of this method could eventually find the desired acute and obtuse angles. In contrast, a user of the theorem need never worry about such matters, because the case analyses in the proof of the theorem took care of them once and for all.

Next, we consider the currently popular vectorial method in Stewart, Redlin and Watson (2002, 604). To apply this method, we first need to find a vector $\vec{u}$ in a direction determined by L_1 and a vector $\vec{v}$ in a direction determined by L_2. To find $\vec{u}$, we first find two points on L_1, say the intercepts of L_1 on the x- and y-axes. Setting one variable equal to 0 in an equation for L_1 and solving for the other variable, we are thus led to the points $P_1(0, \frac{4}{3})$ and $P_2(-2, 0)$ on L_1. A suitable $\vec{u}$ is then the vector $\overrightarrow{P_1P_2} = \langle -2, -\frac{4}{3} \rangle$; some notational conventions would write this vector as $(-2, -\frac{4}{3})$ or $-2\vec{i} - \frac{4}{3}\vec{j}$. A similar amount of work with an equation for L_2 would find a suitable $\vec{v}$ to be the vector $\langle -\frac{7}{5}, \frac{7}{6} \rangle$. Applying the formula on page 604, we are then led to an angle θ such that

$$\cos(\theta) = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|} = \frac{-2\left(-\frac{7}{5}\right) - \frac{4}{3}\cdot\frac{7}{6}}{\sqrt{(-2)^2 + \left(-\frac{4}{3}\right)^2}\sqrt{\left(-\frac{7}{5}\right)^2 + \left(\frac{7}{6}\right)^2}}$$

$$\approx 0.284088329691.$$

One such θ is $\cos^{-1}(0.284088329691) \approx 1.28274087974$, the acute angle that we found using the theorem; thus, by calculating the supplement of θ, one would also find the obtuse angles formed by L_1 and L_2.

As one might suspect from the above discussion, this method can be made as useful in general as that in the theorem, but one would need the following additional provisos. If the calculated value of $\cos(\theta)$ is positive (resp., negative), then taking the inverse cosine of this number produces the acute (resp., obtuse) angle(s) formed at the point of intersection of L_1 and L_2.

The method in Stewart, Redlin and Watson (2002, 604) does have an advantage: it does not need to consider separately the case in which one of the intersecting lines is vertical, as we did in part (b) of the theorem. However, as the above example illustrates, the number of calculations and keystrokes needed to implement this method is considerably greater than the corresponding effort in applying the theorem.

Last, I indicate another aspect, which I view as a drawback, of this method. Notice that if we interchange the labels on the points P_1 and P_2 considered above, then $\vec{u}$ is replaced with $-\vec{u}$ and the calculated value of $\cos(\theta)$ changes to the negative of the previous value. Thus, this method cannot guarantee a priori whether the first angle θ that it finds is going to be acute or obtuse. As explained above, this ambiguity can be removed, at the possible cost of calculating a supplementary angle, after observing the sign of the calculated value of $\cos(\theta)$. By way of contrast, no such thought or supplementary calculation (pun intended) is needed in applying the theorem; once again, the point is that the case analyses in the proof of the theorem took care of such issues once and for all.

If time is short, a classroom presentation covering the main points given above could be based on the proposition; the statement of the theorem; cases 1, 4 and 6 from the proof of the theorem; and the part of the remark in which the theorem is applied.

References

Brink, R. W. *A First Year of College Mathematics*. 2nd ed. New York: Appleton-Century-Crofts, 1954.

Dobbs, D. E. "Proving Heron's Formula Tangentially." *College Mathematics Journal* 15, no. 3 (1984a): 252–53.

———. "A Classroom Note on the Tangent Expansion Formula." *Computer and Mathematics Education Journal* 18 (1984b): 119–20.

———. "A Trigonometry-Based Method for Constructing Square Roots by Straight-Edge and Compass." *International Journal of Mathematics Education in Science and Technology* 15 (1984c): 127–28.

———. "A Tangential Approach to Central Angles." *Illinois Mathematics Teacher* 36, no. 4 (1984d): 18–23.

———. "The Law of Tangents: A New Proof and an Application." *International Journal of Mathematics Education in Science and Technology* 19 (1988): 759–63.

———. "A Characterization of the Tangent Function." *Elemente der Mathematik* 44 (1989): 101–3.

———. "A Tangential Approach to Angle-Bisectors." *Illinois Mathematics Teacher* 42, no. 2 (1991): 23–25.

Stahl, S. *The Poincaré Half-Plane: A Gateway to Modern Geometry*. Boston: Jones and Bartlett, 1993.

Stewart, J., L. Redlin and S. Watson. *Precalculus: Mathematics for Calculus*. 4th ed. Pacific Grove, Calif.: Brooks/Cole, 2002.

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