A Related-Rates Problem

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You try to avoid it, but occasionally it happens. That is, you construct a test problem so that the students can use a particular principle demonstrated in class. Then, a student, having no idea how to solve the problem, puts numbers together by happenstance and produces the correct answer. Consider the following problem:

> Two roads are perpendicular. Car A is 6 miles from the intersection and is heading toward it at 39 miles per hour. Car B is 8 miles from the intersection and is heading toward it at 52 miles per hour. At this instant, how fast is the distance between them changing?

One student wrote, $\sqrt{39^2 + 52^2}$, which is 65 and happens to be correct.

This is a related-rates problem, and the standard procedure is to write $x^2 + y^2 = z^2$ and take the derivative with respect to time, that is,

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}.$$

After solving for $\frac{dz}{dt}$ and substituting values, we can easily obtain the answer.

What made the student's answer work is that $\frac{52}{39} = \frac{8}{6}$ or, in general,

$$\frac{\frac{dx}{dt}}{\frac{dy}{dt}} = \frac{x}{y}. \qquad (1)$$

The intent of this article is to show that, in this problem, the equation below is true given equation (1):

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(\frac{dz}{dt}\right)^2. \qquad (2)$$

First we establish two companion ratios. Starting with

$$x\frac{dx}{dt} + y\frac{dy}{dt} = z\frac{dz}{dt},$$

divide by

$$y\frac{dy}{dt}$$

to produce

$$\frac{x\frac{dx}{dt}}{y\frac{dy}{dt}} + 1 = \frac{z\frac{dz}{dt}}{y\frac{dy}{dt}}.$$

From equation (1), we can write

$$\frac{y\frac{dx}{dt}}{x\frac{dy}{dt}} = 1$$

and use this in place of 1 above. This gives

$$\frac{x\frac{dx}{dt}}{y\frac{dy}{dt}} + \frac{y\frac{dx}{dt}}{x\frac{dy}{dt}} = \frac{z\frac{dz}{dt}}{y\frac{dy}{dt}}.$$

Combine the left side,

$$\frac{x^2\frac{dx}{dt} + y^2\frac{dx}{dt}}{xy\frac{dy}{dt}} = \frac{z\frac{dz}{dt}}{y\frac{dy}{dt}},$$

or

$$\frac{z^2\frac{dx}{dt}}{xy\frac{dy}{dt}} = \frac{z\frac{dz}{dt}}{y\frac{dy}{dt}},$$

which easily gives the first ratio

$$\frac{z}{x} = \frac{\frac{dz}{dt}}{\frac{dx}{dt}}. \qquad (3)$$

Again start with

$$x\frac{dx}{dt} + y\frac{dy}{dt} = z\frac{dz}{dt}.$$

Divide by

$$x\frac{dx}{dt}$$

to produce

$$1+\frac{y\frac{dy}{dt}}{x\frac{dx}{dt}}=\frac{z\frac{dz}{dt}}{x\frac{dx}{dt}}.$$

From equation (1), we can also write

$$1=\frac{x\frac{dy}{dt}}{y\frac{dx}{dt}}$$

and use this in place of 1 above to get

$$\frac{x\frac{dy}{dt}}{y\frac{dx}{dt}}+\frac{y\frac{dy}{dt}}{x\frac{dx}{dt}}=\frac{z\frac{dz}{dt}}{x\frac{dx}{dt}}.$$

Combine the left side,

$$\frac{(x^2+y^2)\frac{dy}{dt}}{xy\frac{dx}{dt}}=\frac{z\frac{dz}{dt}}{x\frac{dx}{dt}}$$

or

$$\frac{z^2\frac{dy}{dt}}{xy\frac{dx}{dt}}=\frac{z}{x}\frac{\frac{dz}{dt}}{\frac{dx}{dt}}$$

which gives the second ratio

$$\frac{z}{y}=\frac{\frac{dz}{dt}}{\frac{dy}{dt}}. \tag{4}$$

Now to establish equation (2). Start again with

$$x\frac{dx}{dt}+y\frac{dy}{dt}=z\frac{dz}{dt}$$

using equation (1),

$$\left(\frac{\frac{dx}{dt}y}{\frac{dy}{dt}}\right)\frac{dx}{dt}+y\frac{dy}{dt}=z\frac{dz}{dt},$$

multiply by

$$\frac{\frac{dy}{dt}}{y},$$

$$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=\left(\frac{\frac{dy}{dt}z}{y}\right)\frac{dz}{dt}.$$

From equation (4),

$$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=\left(\frac{dz}{dt}\right)^2.$$

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